



# Super Pair Sum Labeling of Graphs

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## ABSTRACT

Let  $G$  be a graph with  $p$  vertices and  $q$  edges. The graph  $G$  is said to be a super pair sum labeling if there exists a bijection  $f$  from  $V(G) \cup E(G)$  to  $\{0, \pm 1, \pm 2, \dots, \pm(\frac{p+q-1}{2})\}$  when  $p+q$  is odd and from  $V(G) \cup E(G)$  to  $\{\pm 1, \pm 2, \dots, \pm(\frac{p+q}{2})\}$  when  $p+q$  is even such that  $f(uv) = f(u) + f(v)$ . A graph that admits a super pair sum labeling is called a *super pair sum graph*. Here we study about the super pair sum labeling of some standard graphs.

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## 1 Introduction

Throughout this paper, by a graph we mean a finite, undirected simple graph. Let  $G(V, E)$  be a graph with  $p$  vertices and  $q$  edges. For notations and terminology, we follow [2].

Path on  $n$  vertices is denoted by  $P_n$  and a cycle on  $n$  vertices is denoted by  $C_n$ .  $K_{1,m}$  is called a *star* and it is denoted by  $S_m$ . The bistar  $B_{m,n}$  is the graph obtained

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from  $K_2$  by identifying the central vertices of  $K_{1,m}$  and  $K_{1,n}$  at the end vertices of  $K_2$  respectively.  $B_{m,m}$  is often denoted by  $B(m)$ . The corona of a graph  $G$  on  $p$  vertices  $v_1, v_2, \dots, v_p$  is the graph obtained from  $G$  by adding  $p$  new vertices  $u_1, u_2, \dots, u_p$  and the new edges  $u_i v_i$  for  $1 \leq i \leq p$ . The corona of  $G$  is denoted by  $G \odot K_1$ . The graph  $P_n \odot K_1$  is called a *comb*. Let  $S_m$  be a star with central vertex  $v_0$  and pendant vertices  $v_1, v_2, \dots, v_m$  and let  $[P_{2n}; S_m]$  be the graph obtained from  $2n$  copies of  $S_m$  with vertices  $v_{0_j}, v_{1_j}, \dots, v_{m_j}$  ( $1 \leq j \leq 2n$ ) and joining  $v_{0_j}$  and  $v_{0_{j+1}}$  by means of an edge,  $1 \leq j \leq 2n - 1$ . A caterpillar  $T$  is a tree with a path  $P_n = u_1, u_2, \dots, u_n$  called *spine* with leaves (pendant vertices) known as feet attached to the vertices of the spine by edges known as legs. It is noted that every spine vertex  $u_i$  is attached to  $X_i$  (possibly zero) number of leaves  $b_{i_j}$  ( $1 \leq j \leq X_i, 1 \leq i \leq n$ ). The caterpillar is denoted as  $T = S(X_1, X_2, \dots, X_n)$ . The graceful labelings of graphs was first introduced by Rosa in 1961 [1] and super vertex graceful labeling of some standard graphs was discussed in [5]. The concept of pair sum labeling was introduced and studied by R. Ponraj et al. [3, 4]. It motivates us to define a new concept called *super pair sum labeling of graphs*. Let  $G$  be a  $(p, q)$  graph. A one-one map  $f : V(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm p\}$  is said to be a pair sum labeling if the induced edge function  $f_e : E(G) \rightarrow Z - \{0\}$  defined by  $f_e(uv) = f(u) + f(v)$  is one-one and  $f_e(E(G))$  is either of the form  $\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{q}{2}}\}$  or  $\{\pm k_1, \pm k_2, \dots, \pm k_{\frac{q-1}{2}}\}$  according as  $q$  is even or odd. A graph with a pair sum labeling defined on it is called a *pair sum graph*.

A graph  $G$  with  $p$  vertices and  $q$  edges is said to have a super pair sum labeling if there exists a bijection  $f$  from  $V(G) \cup E(G)$  to  $\{0, \pm 1, \pm 2, \dots, \pm(\frac{p+q-1}{2})\}$  when  $p + q$  is odd and from  $V(G) \cup E(G)$  to  $\{\pm 1, \pm 2, \dots, \pm(\frac{p+q}{2})\}$  when  $p + q$  is even such that  $f(uv) = f(u) + f(v)$ . A graph that admits a super pair sum labeling is called a *super pair sum graph*.

A super pair sum labeling of a graph  $B(2)$  is given in Figure 1

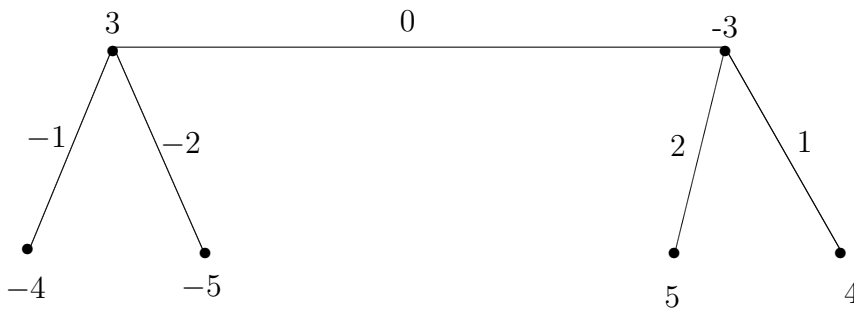


Figure 1: A super pair sum labeling of  $B(2)$

In this paper, we prove that the graphs  $P_n, K_{1,m}$ , bistar  $B_{m,n}$  for  $m \geq 1, n \geq 1, [P_{2n}; S_m]$  for  $n \geq 1, m \geq 1$ , comb,  $C_{2n}, K_{1,m} \cup K_{1,n}$  and banana tree  $S(m, 0, 0, \dots, 0), m \geq 1$  are super pair sum graphs.

## 2 Super Pair Sum Graphs

**Theorem 2.1.** Any path is a super pair sum graph.

*Proof.* Let  $u_1, u_2, \dots, u_n$  be the vertices of the path  $P_n$ .

**Case i.**  $n$  is odd.

Define  $f : V(G) \cup E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm(n - 1)\}$  as follows:

$$f(u_i) = \begin{cases} \frac{i+1-2n}{2} & \text{if } i \text{ is odd} \\ \frac{n+i-1}{2} & \text{if } i \text{ is even} \end{cases}$$

and  $f(u_i u_{i+1}) = \frac{2i - n + 1}{2}, 1 \leq i \leq n - 1.$

Thus,  $f$  is a super pair sum labeling.

**Case ii.**  $n$  is even.

Define  $f : V(G) \cup E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm(n - 1)\}$  as follows:

$$f(u_i) = \begin{cases} \frac{3n+2i-4}{4} & 1 \leq i \leq \frac{n}{2}, i \text{ is odd and } n \equiv 2(mod 4) \\ & 2 \leq i \leq \frac{n}{2}, i \text{ is even and } n \equiv 0(mod 4) \\ \frac{n+2i-4}{4} & \frac{n+4}{2} \leq i \leq n - 1, i \text{ is odd and } n \equiv 2(mod 4) \\ & \frac{n+4}{2} \leq i \leq n, i \text{ is even and } n \equiv 0(mod 4) \\ \frac{2-3n+2i}{4} & 2 \leq i \leq \frac{n-2}{2}, i \text{ is even and } n \equiv 2(mod 4) \\ & 1 \leq i \leq \frac{n-2}{2}, i \text{ is odd and } n \equiv 0(mod 4) \\ \frac{2-5n+2i}{4} & \frac{n+2}{2} \leq i \leq n, i \text{ is even and } n \equiv 2(mod 4) \\ & \frac{n+2}{2} \leq i \leq n - 1, i \text{ is odd and } n \equiv 0(mod 4), \end{cases}$$

$$f(u_i u_{i+1}) = i, \text{ for } 1 \leq i \leq \frac{n-2}{2},$$

$$f(u_{\frac{n}{2}} u_{\frac{n+2}{2}}) = 0 \text{ and}$$

$$f(u_i u_{i+1}) = i - n, \text{ for } \frac{n+2}{2} \leq i \leq n - 1.$$

Thus,  $f$  is a super pair sum labeling and hence  $P_n$  is a super pair sum graph.  $\square$

For example, super pair sum labelings of  $P_{11}, P_{12}$  and  $P_{10}$  are shown in Figure 2.

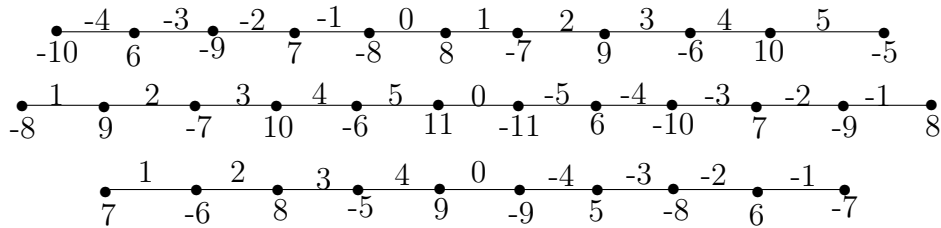


Figure 2: Super pair sum labelings of  $P_{11}, P_{12}$  and  $P_{10}$

**Theorem 2.2.** *Every star graph  $S_m$  is a super pair sum graph for  $m \geq 1$ .*

*Proof.* Let  $v_0, v_1, v_2, \dots, v_m$  be the vertices of the star  $K_{1,m}$  with  $v_0$  as the central vertex.

Define  $f : V(G) \cup E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm m\}$  as follows:

$$\begin{aligned} f(v_0) &= m, \\ f(v_i) &= i - 1 - m, 1 \leq i \leq m \text{ and} \\ f(v_0v_i) &= i - 1, 1 \leq i \leq m. \end{aligned}$$

Thus,  $f$  is a super pair sum labeling and hence  $S_m$  is a super pair sum graph for  $m \geq 1$ .  $\square$

A super pair sum labeling of  $S_8$  is shown in Figure 3.

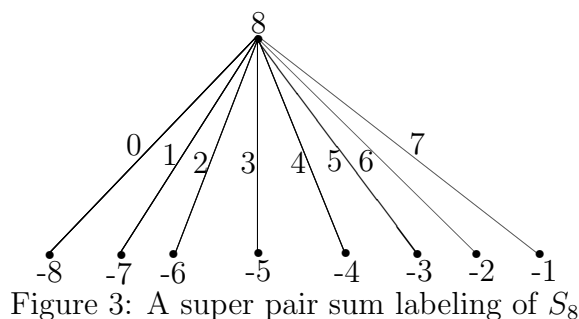


Figure 3: A super pair sum labeling of  $S_8$

**Theorem 2.3.** *Bistar  $B_{m,n}$  is a super pair sum graph for  $m \geq 1, n \geq 1$ .*

*Proof.* Let  $V(K_2) = \{u, v\}$  and  $u_i (1 \leq i \leq m), v_j (1 \leq j \leq n)$  be the vertices adjacent to  $u$  and  $v$  respectively.  $B_{m,n}$  has  $m + n + 2$  vertices and  $m + n + 1$  edges.

Define  $f : V(G) \cup E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm(m + n + 1)\}$  as follows:

$$\begin{aligned} f(u) &= -(m + n + 1), \\ f(v) &= m + n + 1, \\ f(u_i) &= m + n + 1 - i, 1 \leq i \leq m, \\ f(v_j) &= -(m + n + 1 - j), 1 \leq j \leq n, \\ f(uu_i) &= -i, 1 \leq i \leq m, \\ f(vv_j) &= i, 1 \leq j \leq n \text{ and} \\ f(uv) &= 0. \end{aligned}$$

Thus,  $f$  is a super pair sum labeling and hence  $B_{m,n}$  is a super pair sum graph for  $m \geq 1, n \geq 1$ .  $\square$

A super pair sum labeling of  $B_{7,8}$  is shown in Figure 4.

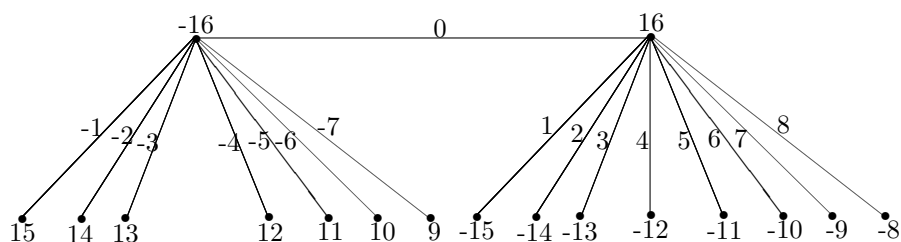


Figure 4: A super pair sum labeling of  $B_{7,8}$

**Theorem 2.4.**  $[P_{2n}; S_m]$  is a super pair sum graph for  $n \geq 1, m \geq 1$ .

*Proof.* Let  $v_{0_j}, v_{1_j}, v_{2_j}, \dots, v_{m_j}$  be the vertices in the  $j^{th}$  copy of  $S_m, 1 \leq j \leq 2n$ . The number of vertices and edges of  $[P_{2n}; S_m]$  are  $2n(m + 1)$  and  $2n(m + 1) - 1$  respectively.

**Case i.**  $n \equiv 0(mod 2)$ .

Define  $f : V(G) \cup E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm(2n(m + 1) - 1)\}$  as follows:

$$f(v_{0_j}) = \begin{cases} \frac{(m+1)(3n-j+1)}{2} - 1 & \text{if } j \text{ is odd, } 1 \leq j \leq n - 1 \\ 1 - \frac{(m+1)(3n+j)}{2} & \text{if } j \text{ is even, } 2 \leq j \leq n, \end{cases}$$

$$f(v_{0_j}) = -f(v_{0_{2n+1-j}}), n + 1 \leq j \leq 2n,$$

$$f(v_{i_j}) = \begin{cases} 1 - i - \frac{(m+1)(3n+j-1)}{2} & \text{if } j \text{ is odd, } 1 \leq j \leq n - 1 \\ \frac{(m+1)(3n-j+2)}{2} - i - 1 & \text{if } j \text{ is even, } 2 \leq j \leq n, \end{cases}$$

$$f(v_{i_j}) = -f(v_{i_{2n+1-j}}), n + 1 \leq j \leq 2n, 1 \leq i \leq m,$$

$$f(v_{0_j}v_{0_{j+1}}) = -j(m + 1), 1 \leq j \leq n - 1,$$

$$f(v_{0_n}v_{0_{n+1}}) = 0,$$

$$f(v_{0_j}v_{0_{j+1}}) = (2n - j)(m + 1), n + 1 \leq j \leq 2n - 1, \text{ and}$$

$$f(v_{0_j}v_{i_j}) = \begin{cases} (1 - j)(m + 1) - i & 1 \leq j \leq n, 1 \leq i \leq m \\ (2n - j)(m + 1) + i, & n + 1 \leq j \leq 2n, 1 \leq i \leq m. \end{cases}$$

Thus,  $f$  is a super pair sum labeling.

**Case ii.**  $n \equiv 1(mod 2)$ .

Define  $f : V(G) \cup E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm(2n(m + 1) - 1)\}$  as follows:

$$\begin{aligned}
 f(v_{0_j}) &= \begin{cases} \frac{1-(m+1)(3n+j)}{2} & \text{if } j \text{ is odd, } 1 \leq j \leq n \\ \frac{(m+1)(3n-j+1)}{2} - 1 & \text{if } j \text{ is even, } 2 \leq j \leq n - 1, \end{cases} \\
 f(v_{0_j}) &= -f(v_{0_{2n+1-j}}), n + 1 \leq j \leq 2n, \\
 f(v_{i_j}) &= \begin{cases} \frac{(m+1)(3n-j+2)}{2} - i - 1 & \text{if } j \text{ is odd, } 1 \leq j \leq n \\ 1 - i - \frac{(m+1)(3n+j-1)}{2} & \text{if } j \text{ is even, } 2 \leq j \leq n - 1, \end{cases} \\
 f(v_{i_j}) &= -f(v_{i_{2n+1-j}}), n + 1 \leq j \leq 2n, 1 \leq i \leq m, \\
 f(v_{0_j}v_{0_{j+1}}) &= -j(m + 1), 1 \leq j \leq n - 1, \\
 f(v_{0_n}v_{0_{n+1}}) &= 0, \\
 f(v_{0_j}v_{0_{j+1}}) &= (2n - j)(m + 1), n + 1 \leq j \leq 2n - 1 \text{ and} \\
 f(v_{0_j}v_{i_j}) &= \begin{cases} (1 - j)(m + 1) - i, & 1 \leq j \leq n, 1 \leq i \leq m, \\ (2n - j)(m + 1) + i, & n + 1 \leq j \leq 2n, 1 \leq i \leq m. \end{cases}
 \end{aligned}$$

Thus,  $f$  is a super pair sum labeling. Hence,  $[P_{2n}; S_m]$  is a super pair sum graph.  $\square$

For example, super pair sum labelings of  $[P_8; S_3]$  and  $[P_6; S_4]$  are shown in Figure 5.

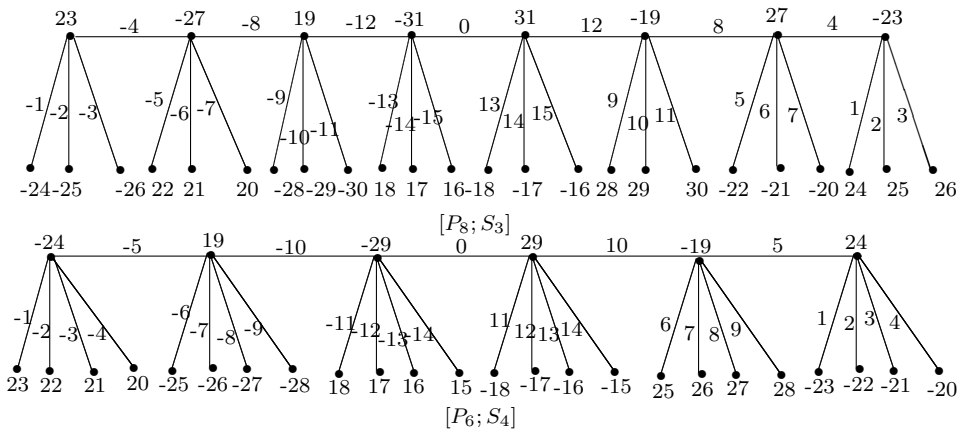


Figure 5: Super pair sum labelings of  $[P_8; S_3]$  and  $[P_6; S_4]$

**Theorem 2.5.** Any comb is a super pair sum graph

*Proof.* Let  $G$  be the comb obtained from a path  $P_n : v_1, v_2, \dots, v_n$  by joining a vertex  $u_i$  to  $v_i(1 \leq i \leq n)$ .

**Case i.**  $n \equiv 1(mod 4)$ .

Define  $f : V(G) \cup E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm(2n - 1)\}$  as follows:

$$f(v_i) = \begin{cases} 1 - 2i & \text{if } i \text{ is odd, } 1 \leq i \leq n \\ 2(n - i) + 1 & \text{if } i \text{ is even, } 1 \leq i \leq n, \end{cases}$$

$$f(u_i) = \begin{cases} f(v_{i+1}) + 2, & 1 \leq i \leq n - 1 \\ 1, & i = n, \end{cases}$$

$$f(v_i v_{i+1}) = 2n - 4i, 1 \leq i \leq n - 1 \text{ and}$$

$$f(u_i v_i) = 2n - 4i + 2, 1 \leq i \leq n.$$

Then,  $f$  is a super pair sum labeling.

**Case ii.**  $n \equiv 3 \pmod{4}$ .

Define  $f : V(G) \cup E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm(2n - 1)\}$  as follows:

$$f(v_i) = \begin{cases} 2(n - i) + 1 & \text{if } i \text{ is odd, } 1 \leq i \leq n \\ 1 - 2i & \text{if } i \text{ is even, } 1 \leq i \leq n, \end{cases}$$

$$f(u_i) = \begin{cases} f(v_{i+1}) + 2, & 1 \leq i \leq n - 1 \\ 1 - 2n, & i = n, \end{cases}$$

$$f(v_i u_{i+1}) = 2n - 4i, 1 \leq i \leq n - 1 \text{ and}$$

$$f(u_i v_i) = 2n - 4i + 2, 1 \leq i \leq n.$$

Thus,  $f$  is a super pair sum labeling. When  $n$  is even and  $m = 1$ , the result follows from Theorem 2.4. Hence, any comb is a super pair sum graph.  $\square$

For example, super pair sum labelings of  $P_9 \odot K_1$  and  $P_{11} \odot K_1$  are shown in Figure 6.

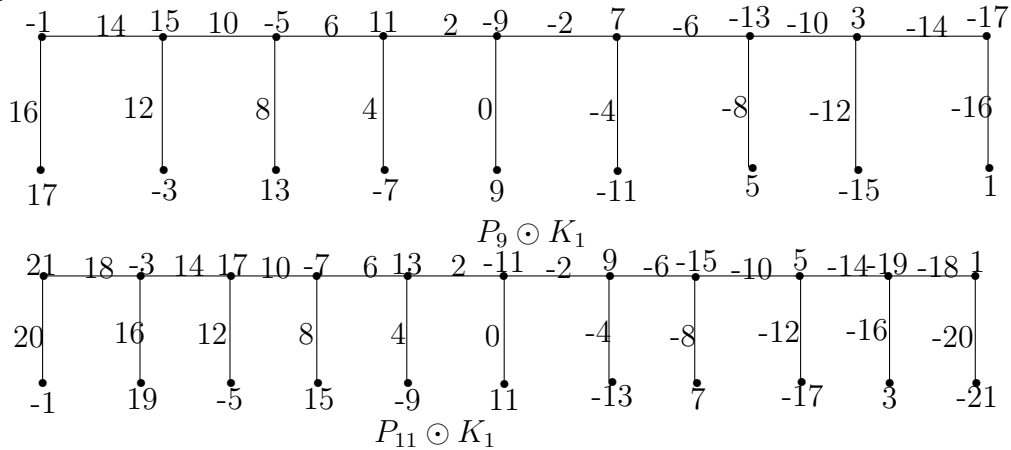


Figure 6: Super pair sum labelings of  $P_9 \odot K_1$  and  $P_{11} \odot K_1$

**Theorem 2.6.**  $C_{2n}$  is a super pair sum graph for  $n \geq 1$ .

*Proof.* Let  $v_1, v_2, \dots, v_{2n}$  be the vertices of the cycle  $C_{2n}$ . Define  $f : V(G) \cup E(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm 2n\}$  as follows:

$$f(v_1) = 1,$$

$$f(v_{2i+1}) = 1 - n - i, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor,$$

$$f(v_{2i}) = 2n - i, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor,$$

$$f(v_i) = -f(v_{i-n}), n + 1 \leq i \leq 2n,$$

$$\begin{aligned}
 f(v_1v_2) &= 2n, \\
 f(v_iv_{i+1}) &= n - i + 1, 2 \leq i \leq n - 1, \\
 f(v_nv_{n+1}) &= \frac{3n - 2}{2}, \\
 f(v_iv_{i+1}) &= -f(v_{i-n}v_{i+1-n}), n + 1 \leq i \leq 2n - 1 \text{ and} \\
 f(v_{2n}v_1) &= -f(v_nv_{n+1}).
 \end{aligned}$$

Thus,  $f$  is a super pair sum labeling and hence  $C_{2n}$  is a super pair sum graph.  $\square$

For example, a super pair sum labeling of  $C_{12}$  is shown in Figure 7.

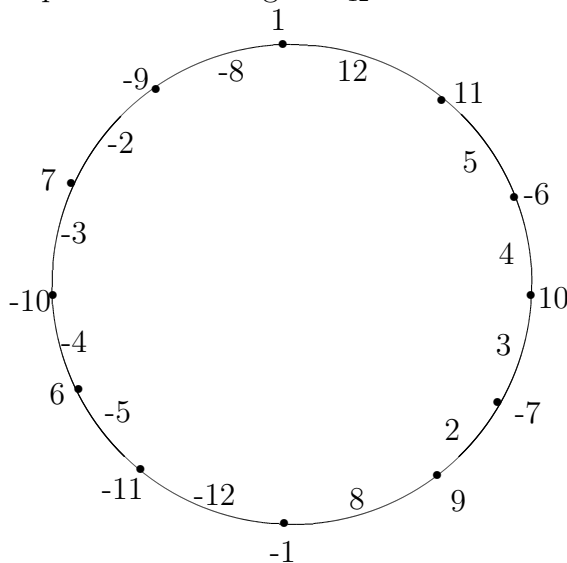


Figure 7: A super pair sum labeling of  $C_{12}$

**Theorem 2.7.**  $K_{1,m} \cup K_{1,n}$  is a super pair sum graph.

*Proof.* Let  $u_0, u_1, u_2, \dots, u_m$  be the vertices of  $K_{1,m}$  and  $E(K_{1,m}) = \{u_0u_i : 1 \leq i \leq m\}$ . Let  $v_0, v_1, v_2, \dots, v_n$  be the vertices of  $K_{1,n}$  and  $E(K_{1,n}) = \{v_0v_i : 1 \leq i \leq n\}$ . Without loss of generality assume that  $m < n$ .

Define  $f : V(G) \cup E(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm(m + n + 1)\}$  as follows:

$$\begin{aligned}
 f(u_0) &= -(m + n + 1), \\
 f(u_i) &= m + n + 1 - 2i, 1 \leq i \leq m, \\
 f(v_0) &= m + n + 1, \\
 f(v_i) &= \begin{cases} 1 - 2i, & 1 \leq i \leq m \\ -m - i, & m + 1 \leq i \leq n, \end{cases} \\
 f(u_0u_i) &= -2i, 1 \leq i \leq m, \\
 f(v_0v_i) &= \begin{cases} m + n - 2(i - 1), & 1 \leq i \leq m \\ n + 1 - i, & m + 1 \leq i \leq n. \end{cases}
 \end{aligned}$$

Thus,  $f$  is a super pair sum labeling and hence  $K_{1,m} \cup K_{1,n}$  is a super pair sum graph.  $\square$



For example, a super pair sum labeling of  $K_{1,4} \cup K_{1,7}$  is shown in Figure 8.

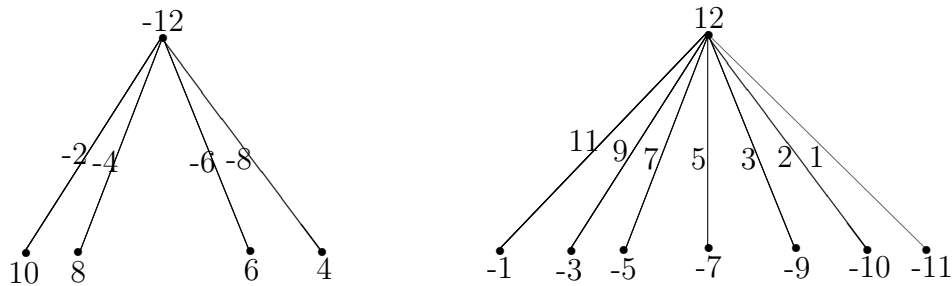


Figure 8: A super pair sum labeling of  $K_{1,4} \cup K_{1,7}$

**Theorem 2.8.** *The caterpillar  $S(X_1, X_2, \dots, X_n)$  where  $X_1 = m, X_2 = X_3 = \dots = X_n = 0$  is a super pair sum graph.*

*Proof.* Let  $u_1, u_2, \dots, u_n$  be the vertices of the path  $P_n$ . The vertex  $u_1$  is attached to  $X_1 = m$  number of leaves  $b_{1_j} (1 \leq j \leq m)$ .

Define  $f : V(G) \cup E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm(m+n)\}$  as follows:

$$f(u_i) = \begin{cases} -\lfloor \frac{2n-i}{2} \rfloor - m & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ \lfloor \frac{n+i-1}{2} \rfloor + m & \text{if } i \text{ is even and } 2 \leq i \leq n, \end{cases}$$

$$f(b_{1_j}) = \lfloor \frac{n-1}{2} \rfloor + j, 1 \leq j \leq m,$$

$$f(u_1 b_{1_j}) = -m - \lfloor \frac{n}{2} \rfloor + j, 1 \leq j \leq m \text{ and}$$

$$f(u_i u_{i+1}) = i - \lfloor \frac{n}{2} \rfloor, 1 \leq i \leq n - 1.$$

Then,  $f$  is a super pair sum labeling and hence  $S(m, 0, 0, \dots, 0)$  is a super pair sum graph. □

For example, a super pair sum labeling of  $S(6, 0, 0, 0, 0, 0, 0, 0)$  is shown in Figure 9.

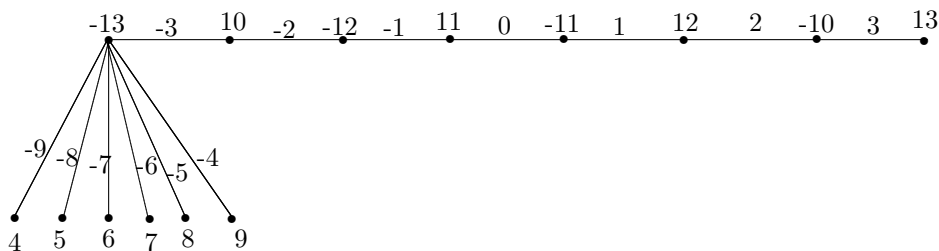


Figure 9: A super pair sum labeling of  $S(6, 0, 0, 0, 0, 0, 0, 0)$

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