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# On Generalized Weak Structures

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**ABSTRACT**

Avila and Molina [8] introduced the notion of generalized weak structures which naturally generalize minimal structures, generalized topologies and weak structures and the structures $\alpha(g)$,$\pi(g)$,$\sigma(g)$ and $\beta(g)$. This work is a further investigation of generalized weak structures due to Avila and Molina. Further we introduce the structures $ro(g)$ and $rc(g)$ and study the properties of the structures $ro(g)$, $rc(g)$, and also further properties of $\alpha(g)$,$\pi(g)$,$\sigma(g)$ and $\beta(g)$ due to [8]

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## 1 Introduction

The study of more general structure than that of a topological space has taken several directions over the last twenty years. In 1996, Maki[9] studied minimal structures, or

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shortly $m$-structures, on a set $X$ that is a collection of subsets of $X$ containing the empty set and $X$, with no other restriction. In 1997, Csaszar[3] introduced a generalized structure on a non-empty set $X$ called a generalized topology. Also, Csaszar[2, 4] introduced and studied generalized operators. As a natural generalization of the above mentioned structures, in 2011, Csaszar[5] introduced the notion of a weak structure which is a collection of subsets of $X$ containing the empty set. He defined the interior and the closure in the new context and showed the important properties of these operators. Let $X$ be a non empty set and $P$ be its power set. A structure on $X$ is a subset of $P$ and an operation on $X$ is a function from $P$ to $P$. A structure $w$ on $X$ is called a weak strucutre on $X$ if and only if $\phi \in w$. Weak structures are briefly noted as $WS$. If w is a $WS$ on $X$, then every member of $w$ is known as $w$-open and complement of a $w$-open set is known as $w$-closed. Let $w$ be a $WS$ on $X$ and $A \subset X$ then the union of all $w$-open subsets of $A$ is denoted as $i\ w\ A$ and the intersection of all $w$-closed sets containing $A$ is denoted as $c\ w\ A$. Further with the help of $i\ w$ and $c\ w$, several other structures such as $\alpha(w)$, $\beta(w)$, $\sigma(w)$, $\pi(w)$ and $\rho(w)$ have been introduced and studied. Ekici [7], studied the properties of the structures $\alpha(w)$, $\beta(w)$, $\sigma(w)$, $\pi(w)$ and $\rho(w)$ and introduced the structures $r(w)$ and $rc(w)$. Navaneethakrishnan and Tamaraiselvi [8] extended the study of weak structures and $m$ structures. Das [6] shown that under some conditions $r(w)$ is a topology on $X$ and studied the comparison of two weak structures. Avila and Molina [1] defined the notion of generalized weak structures (GWS) as an extention of weak structures[5]. For that they introduced the interior, the closure and other related notions and also proved that many properties of these familiar notions remain valid under the general assumptions.
The generalized weak structure (GWS) on a non-empty set $X$ is a non-empty class $g$ of subsets of $X$. If $g$ is a generalized weak structure on $X$ then each element of $g$ is said to be $g$-open and the complement (in $X$) of a $g$-open set is called a $g$-closed set. It is clear that each generalized topology [3], minimal structure [9] and weak structure [5] are GWS. Let $g$ be a GWS on $X$ and $A \subset X$. Then the $g-$closure of $A$ is defined by $c_g(A) = \cap\{F : A \subset F, F \text{ is } g-closed\}$ if there is any $F$ such that $A \subset F$ and $F$ is g-closed otherwise $c_g(A) = X$ and the $g-$interior of $A$ is defined by $i_g(A) = \cup\{G : G \subset A, G \text{ is } g-open\}$ if there is any $G$ such that $G \subset A$ and $G$ is g-open otherwise $i_g(A) = \emptyset$.

In this paper we introduce and study the structures $ro(g)$ and $rc(g)$ in GWS and the properties of $ro(g)$. In addition we study the properties of the structures $\alpha(g)$, $\pi(g)$,$\sigma(g)$ and $\beta(g)$. The following results were proved in [8].

**Theorem 1.1.** *[8] Let $g$ be a generalized weak structure on $X$ and $A, B \subset X$. Then the following propertes hold:*
*(i)*$i_g(i_g(A)) = i_g(A)$.
*(ii)*$c_g(c_g(A)) = c_g(A)$.
*(iii)*$A \subset B$ *implies that* $i_g(A) \subset i_g(B)$ *and* $c_g(A) \subset c_g(B)$.
*(iv)If* $A \in g$, *then* $A = i_g(A)$.

*(v)If $A$ is $g$-closed, then $A = c_g(A)$.*

**Theorem 1.2.** *[8] Let $g$ be a generalized weak structure on $X$ and $A \subset X$. Then*
*(i)$x \in c_g(A)$ if and only if $G \cap A \neq \emptyset$ whenever $x \in G \in g$.*
*(ii)$c_g(X - A) = X - i_g(A)$ and $i_g(X - A) = X - c_g(A)$.*

**Definition 1.3.** [8] Let $g$ be a GWS on $X$ and $A \subset X$. Then, we define the following:
(i)$A \in \alpha(g)$if $A \subset i_g c_g i_g(A)$.
(ii)$A \in \pi(g)$ if $A \subset i_g c_g(A)$.
(iii)$A \in \sigma(g)$ if $A \subset c_g i_g(A)$.
(iv)$A \in \beta(g)$ if $A \subset c_g i_g c_g(A)$.

**Lemma 1.4.** *[8] Let $g$ be a GWS on $X$. Then $i_g c_g i_g c_g = i_g c_g$ and $c_g i_g c_g i_g = c_g i_g$.*

# 2 Properties of the structure ro(g)

Let $g$ be a GWS on $X$ and $A \subset X$. Then, we define $ro(g)$ and $rc(g)$ as follows.
(1) $A \in ro(g)$ if $A = i_g(c_g(A))$.
(2)$A \in rc(g)$ if $A = c_g(i_g(A))$.

**Theorem 2.1.** *For a GWS $g$ on $X$ and $A \subset X$, $A \in ro(g)$ if and only if $A \in \alpha(g)$ and $X - A \in \beta(g)$.*

*Proof.* Let $A \in ro(g)$. We have $A = i_g(c_g(A))$. By Theorem 1.1,$i_g(A) = i_g(i_g(c_g(A)))$ $= i_g(c_g(A)) = A$. Then, we have $A = i_g(A) \subset c_g(i_g(A))$. It follows that $A = i_g(A) = i_g(i_g(A)) \subset i_g(c_g(i_g(A)))$. Thus $A \subset i_g(c_g(i_g(A)))$ and hence $A \in \alpha(g)$. On the other hand, since $A = i_g(c_g(A))$, then $X - A = X - i_g(c_g(A))$. By Theorem 1.2, we have $X - A = c_g(i_g(X - A))$. Moreover, $c_g(i_g(X - A)) \subset c_g(i_g(c_g(X - A)))$. This implies that $X - A = c_g(X - A) = c_g(i_g(X - A)) \subset c_g(i_g(c_g(X - A)))$. Thus, $X - A \subset c_g(i_g(c_g(X - A)))$ and hence $X - A \in \beta(g)$.
Conversely, let $A \in \alpha(g)$ and $X - A \in \beta(g)$. We have $A \subset i_g(c_g(i_g(A)))$ and $i_g(c_g(i_g(A))) \subset A$. Thus,$A = i_g(c_g(i_g(A)))$and by Lemma 1.4 $A \in ro(g)$. □

**Theorem 2.2.** *For a GWS $g$ on $X$ and $A \subset X$, $A \in ro(g)$ if and only if $A \in \pi(g)$ and $X - A \in \sigma(g)$.*

*Proof.* Let $A \in \pi(g)$ and $X - A \in \sigma(g)$. We have $A \subset i_g(c_g(A))$ and $i_g(c_g(A) \subset A$. Thus, $A = i_g(c_g(A))$ and hence $A \in ro(g)$. The converse follows from the fact that $A = i_g(c_g(A)$. □

**Theorem 2.3.** *For a GWS $g$ on $X$ and $A \subset X$. Then $A \in \pi(g)$ if and only if there exists $B \in ro(g)$ such that $A \subset B$ and $c_g(A) = c_g(B)$.*

*Proof.* Let $A \in \pi(g)$. We have $A \subset i_g(c_g(A))$. If we take $B = i_g(c_g(A))$, then $B \in ro(g)$ and also $A \subset B$ and $c_g(A) = c_g(B)$.
Converesely, suppose that $B \in ro(g)$ such that $A \subset B$ and $c_g(A) = c_g(B)$. Then $i_g c_g(A) = i_g c_g(B) = B$ and hence $A \subset i_g c_g(A)$ which implies that $A \in \pi(g)$. □

Let $g$ be a GWS on $X$ and $A \subset X$. Then $A$ is said to be $g$-dense if $c_g(A) = X$.

**Theorem 2.4.** *Let $g$ be a GWS on $X$ such that $g$ is closed under finite intersection and $A, B \subset X$. Then the following hold.*
*(a)$c_g(A) \cup c_g(B) = c_g(A \cup B)$.*
*(b)$i_g(A \cap B) = i_g(A) \cap i_g(B)$.*
*(c) $G \cap c_g(A) \subset c_g(G \cap A)$ for every $G \in g$ and $A \subset X$.*
*(d) $c_g(G \cap c_g(A)) = c_g(G \cap A)$ for every $G \in g$ and $A \subset X$.*
*(e) $c_g(G) = c_g(G \cap A)$ for every $G \in g$ and every $g$-dense set $A$.*

*Proof.* (a)Suppose $x \notin c_g(A) \cup c_g(B)$. Then $x \notin c_g(A)$ and $x \notin c_g(B)$. Then there exist $G, H \in g$ contining x such that $G \cap A = \emptyset$ and $H \cap B = \emptyset$. If $x \in G \cap H \in g$ such that $(G \cap H) \cap (A \cup B) = ((G \cap H) \cap A) \cup ((G \cap H) \cap B) \subset (G \cap A) \cup (H \cap B) = \emptyset$ and so $x \notin c_g(A \cup B)$. Hence $c_g(A \cup B) \subset c_g(A) \cup c_g(B)$ and, by using Theorem 1.1 we obtain $c_g(A) \cup c_g(B) = c_g(A \cup B)$.
(b)The proof follows from (a) and Theorem 1.2.
(c) Let $x \in G \cap c_g(A)$. Then $x \in G$ and $x \in c_g(A)$. If $x \in H \in g$, then $x \in H \cap G \in g$ and so $(H \cap G) \cap A \neq \emptyset$which implies that $H \cap (G \cap A) \neq \emptyset$. Hence $x \in c_g(G \cap A)$ which implies that $G \cap c_g(A) \subset c_g(G \cap A)$.
(d) By (c), $G \cap c_g(A) \subset c_g(G \cap A)$ and so $c_g(G \cap c_g(A)) \subset c_g(G \cap A)$. But $G \cap A \subset G \cap c_g(A) \subset c_g(G \cap c_g(A))$ and so $c_g(G \cap A) \subset c_g(G \cap c_g(A))$. Hence $c_g(G \cap c_g(A)) = c_g(G \cap A)$.
(e) The proof follows from (d). □

**Theorem 2.5.** *Let $g$ be a GWS on $X$ and $A \subset X$. If $A \in \pi(g)$, then $A$ is the intersection of $B \in ro(g)$ and a $g$-dense set $C$.*

*Proof.* Let $A \in \pi(g)$. By Theorem 2.3, there exists a $B \in ro(g)$ such that $A \subset B$ and $c_g(A) = c_g(B)$. If we take $C = A \cup (X - B)$, then we have $X = c_g(B) \cup c_g(X - B) = c_g(A) \cup c_g(X - B) \subset c_g(A \cup (X - B)) = c_g(C)$. Thus $C$ is $g$-dense and hence $A = B \cap C$. □

The following example shows that the converse of Theorem 2.5 is not true in general.

**Example 2.6.** *Let $X = \{a, b, c\}$ and $g = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}\}$. Then for the sets $A = \{b, c\}$ and $B = \{a, c\}$, $A \in ro(g)$ and $B$ is $g$-dense but $A \cap B = \{c\} \notin \pi(g)$.*

# 3 Properties of the structures $\alpha(g)$,$\pi(g)$,$\sigma(g)$,$\beta(g)$

**Theorem 3.1.** *Let $g$ be a GWS on $X$ and $A \subset X$. If $A$ is $g$-open and $g$-closed, then $A \in \alpha(g)$ and $X - A \in \pi(g)$.*

*Proof.* Let $A$ be $g$-open and $g$-closed. By Theorem 1.1, $A = i_g(A)$ and $A = c_g(A)$. We have $A = i_g(A) \subset c_g(i_g(A))$. By Theorem 1.1, $A = i_g(A) = i_g(i_g(A)) \subset i_g(c_g(i_g(A)))$. Thus, $A \subset i_g(c_g(i_g(A)))$and hence, $A \in \alpha(g)$. On the other hand, since $A = i_g(A)$ and $A = c_g(A)$, then $X - A = X - i_g(A) = c_g(X - A)$ and $X - A = X - c_g(A) = i_g(X - A)$. This implies $X - A = i_g(X - A) \subset i_g(c_g(X - A))$. Thus, $X - A \subset i_g(c_g(X - A))$ and hence $X - A \in \pi(g)$. □

The following example shows that the converse of Theorem 3.1 is not true in general.

**Example 3.2.** *Let $X = \{a, b, c, d\}$ and $g = \{\emptyset, \{d\}, \{a, b\}, \{b, c\}, \{a, b, d\}\}$. Then $A = \{a, b, c\} \in \alpha(g)$ and $X - A \in \pi(g)$ but $A$ is not $g$-open.*

**Theorem 3.3.** *Let $g$ be a GWS on $X$ and $A \subset X$. If there exists a $g$-open set $B$ such that $B \subset A \subset c_g(B)$, then $A \in \sigma(g)$.*

*Proof.* Let $B \subset A \subset c_g(B)$ for a $g$-open set $B$. Since $B \subset A$, then $B \subset i_g(A)$. This implies $c_g(B) \subset c_g(i_g(A))$ and then $A \subset c_g(i_g(A))$. Thus, $A \in \sigma(g)$. □

The converse of Theorem 3.3 is not true in general as shown in the following example.

**Example 3.4.** *Let $X = \{a, b, c, d\}$ and $g = \{\emptyset, \{b, d\}, \{a, d\}, \{a, c\}, \{b, c\}\}$. Then $A = \{a, b, d\} \in \sigma(g)$ but we do not have any $g$-open set $B$ such that $B \subset A \subset c_g(B)$.*

**Theorem 3.5.** *Let $g$ be a GWS on $X$ and $C \subset X$. If $C \in \beta(g)$, then $C = A \cap B$, where $A \in \sigma(g)$ and $B$ is $g$-dense.*

*Proof.* Let $C \in \beta(g)$. Then $C \subset c_g(i_g(c_g(C)))$. By Theorem 1.1, we have $c_g(C) \subset c_g(c_g(i_g(c_g(C)))) = c_g(i_g(c_g(C)))$. Also, $i_g(c_g(C)) \subset c_g(C)$ and then $c_g(i_g(c_g(C))) \subset c_g(c_g(C)) = c_g(C)$. We have $c_g(C) = c_g(i_g(c_g(C)))$. This implies that $A = c_g(C) \in \sigma(g)$. If we take $B = C \cup (X - c_g(C))$, then $B$ is $g$-dense and $C = A \cap B$. □

The converse of Theorem 3.5 is not true in general as shown in the following example.

**Example 3.6.** *Let $X = \{a, b, c, d\}$ and $g = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}\}$. If we take $A = \{b, c\}$ and $B = \{a, c, d\}$, then $A \in \sigma(g)$ and $B$ is $g$-dense but $A \cap B = \{c\} \notin \beta(g)$.*

**Theorem 3.7.** *For a GWS $g$ on $X$ and $A \subset X$, the following properties are equivalent.*
*(i) $A \in \beta(g)$.*
*(ii) there exists $B \in \pi(g)$ such that $B \subset c_g(A) \subset c_g(B)$.*
*(iii) $c_g(A) \in rc(g)$.*

*Proof.* (i)$\Longrightarrow$(ii). Let $A \in \beta(g)$. Then $A \subset c_g(i_g(c_g(A)))$. Put $B = i_g(c_g(A))$. This implies that $B \in \pi(g)$. Since $A \subset c_g(B)$, by Theorem 1.1 $c_g(A) \subset c_g(c_g(B)) = c_g(B)$. Hence, $B = i_g(c_g(A) \subset c_g(A) \subset c_g(B)$.
(ii)$\Longrightarrow$(iii). Let $B \in \pi(g)$ such that $B \subset c_g(A) \subset c_g(B)$. Then $B \subset i_g(c_g(B))$ and $c_g(B) \subset c_g(i_g(c_g(B)))$. We have $c_g(i_g(c_g(B))) \subset c_g(i_g(c_g(A)))$. Since $c_g(A) \subset c_g(B)$, then by Theorem 1.1, $i_g(c_g(A)) \subset c_g(B)$ and then $c_g(i_g(c_g(A))) \subset c_g(c_g(B) = c_g(B)$. Since $B \subset c_g(A)$, then $c_g(B) \subset c_g c_g(A)) = c_g(A)$. This implies that $c_g(B) \subset c_g(i_g(c_g(B))) \subset c_g(i_g(c_g(A))) \subset c_g(B) \subset c_g(A) \subset c_g(B)$ and $c_g(A) = c_g(i_g(c_g(A)))$. Hence $c_g(A) \in rc(g)$.
(iii)$\Longrightarrow$(i) Let $c_g(A) \in rc(g)$. We have $c_g(A) = c_g(i_g(c_g(A)))$. Since $A \subset c_g(A) = c_g(i_g(c_g(A)))$, then $A \in \beta(g)$. □

**Theorem 3.8.** *Let $g$ be a GWS on $X$. If $A \subset B \subset c_g(A)$ and $A \in \beta(g)$, then $B \in \beta(g)$.*

*Proof.* Let $A \subset B \subset c_g(A)$ and $A \in \beta(g)$. Then $A \subset c_g(i_g(c_g(A)))$. Now $B \subset c_g(A) \subset c_g(c_g(i_g(c_g(A)))) = c_g(i_g(c_g(A))) \subset c_g(i_g(c_g(B)))$. Thus, $B \subset c_g(i_g(c_g(B)))$ and hence $B \in \beta(g)$. □

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