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Constructing Graceful Graphs with Caterpillars

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ABSTRACT

A graceful labeling of a graph G of size n is an injective assignment of integers from $\{0, 1, \dots, n\}$ to the vertices of G, such that when each edge of G has assigned a weight, given by the absolute difference of the labels of its end vertices, the set of weights is $\{1, 2, \dots, n\}$. If a graceful labeling f of a bipartite graph G assigns the smaller labels to one of the two stable sets of G, then f is called an α-labeling and G is said to be an α-graph. A tree is a caterpillar if the deletion of all its leaves results in a path. In this work we study graceful labelings of the disjoint union of a cycle and a caterpillar. We present necessary conditions for this union to be graceful and, in the case where the cycle has even size, to be an α-graph. In addition, we present a new family of graceful trees constructed using α-labeled caterpillars.

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1 Introduction

Let G be a graph of order m and size n, where $n+1 \geq m$. A function f is a *graceful labeling* of G if f is an injection from $V(G)$ to the set $\{0, 1, \dots, n\}$ such that when each edge uv of G has assigned the *weight* $|f(u) - f(v)|$, the resulting weights form the set $\{1, 2, \dots, n\}$.

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A graph that admits a graceful labeling is said to be *graceful*. Let f be a graceful labeling of a bipartite graph G; if there exists an integer λ such that $f(u) \leq \lambda < f(v)$ for every $uv \in E(G)$ with $f(u) < f(v)$, then f is called an *α-labeling* and G is an *α-graph*. The number λ is called the *boundary value* of f.
It is well-known that not all graphs are graceful. This fact motivated Golomb [10] to introduce the concept of gracefulness. The *gracefulness* of a graph G of size n, denoted by $grac(G)$, is the smallest positive integer k for which it is possible to label the vertices of G with distinct elements from the set $\{0, 1, \ldots, k\}$ in such a way that distinct edges have distinct weights. A graph G is graceful is $grac(G) = n$. Golomb suggested that the main questions in this area are to determine the relationship between $grac(G)$ and n, identifying families of graphs for which $grac(G) = n$, $grac(G) > n$, and to find better bounds for $grac(G) - n$.
The smallest graph, in order and size that is not graceful, is the disjoint union of the cycle C_3 and the path P_2. In this case, $grac(C_3 \cup P_2) = 5$, this value is obtained with the labeling $(0, 2, 5)(3, 4)$. Frucht and Salinas [6] studied the gracefulness of graphs of the form $C_m \cup P_n$, they proved that $C_4 \cup P_n$ is graceful for every $n \geq 3$, in addition, they conjectured that $C_m \cup P_n$ is graceful if $m + n \geq 7$. Choudum and Kishore [4] proved this conjecture for every $m \geq 5$ and $n \geq (m + 5)/2$. In [7], Frucht proved that $C_3 \cup P_n$ is graceful whenever $n \geq 4$. The conjecture was completely proved by Traetta [16]. Note that the graph $C_3 \cup P_2$ can also be seen as $C_3 \cup K_{1,1}$; Choudum and Kishore [5] proved that $C_m \cup K_{1,n}$ is graceful for every $m \geq 7$ and $n \geq 1$. Barrientos [2] proved that $C_6 \cup K_{1,n}$ is graceful if and only if n is odd or $n = 2, 4$, and that $C_5 \cup K_{1,n}$ is not graceful for every $n \geq 2$. Seoud and Youssef [14] proved that neither $C_3 \cup K_{1,n}$ nor $C_4 \cup K_{1,n}$ (in this case $n \neq 2$) are graceful. Thus, the gracefulness of $C_m \cup G$ has been completely determined when G is a path or a star of size n.
In this paper we analyze this problem from a more general perspective, here we study graceful labelings of graphs of the form $C_r \cup G_n$ where G_n is a caterpillar of size n. We provide sufficient conditions on the structure of G_n to obtain a graceful labeling of $C_r \cup G_n$ when r is odd and an α-labeling when r is even.
In Section 2, we present all the existing results required to prove our conclusions. In Section 3 we show how to constuct the desired labelings of $C_r \cup G_n$. In Section 4 we use α-labeled caterpillars of even size to construct new α-trees.
In this work we follow the notation and terminology used in [3] and [9].

2 Preliminary Results

Recall that every α-graph is bipartite. Let G be a graph of size n and let f be an α-labeling of G with boundary value λ. Let A and B be the stable sets of $V(G)$, without loss of generality, we assume that $A = \{v \in V(G) : f(v) \leq \lambda\}$ and $B = \{v \in V(G) : f(v) > \lambda\}$. When a positive constant d is added to every vertex in B, the set of induced weights is $\{d+1, d+2, \ldots, d+n\}$, and the resulting labeling is a *d-graceful labeling*. This definition, as well as the process to transform α-labelings into d-graceful labelings, was introduced

independently by Maheo and Thuiller [11] and Slater [15]. Note that when an α-labeling of a tree of size n is transformed into a d-graceful labeling, the set formed by the labels assigned is $\{0, 1, \ldots, \lambda\} \cup \{d + \lambda + 1, d + \lambda + 2, \ldots, d + n\}$.
Let G be a graph of order m and size n, and let f be a graceful labeling of G. When a constant c is added to every vertex label in G, the set of weights induced by this new labeling is the same, that is, $\{1, 2, \ldots, n\}$; the set of labels used are $\{c+l_1, c+l_2, \ldots, c+l_m\}$ where the l_i's are the labels originally assigned by f.
Modifying the permissable vertex labels and/or the weights of a graceful labeling, Rosa [13] introduced the concept of $\hat{\rho}$-labelings. An injective function $f : V(G) \to \{0, 1, \ldots, n+1\}$ is called a *$\hat{\rho}$-labeling* if the set of induced weights is either $\{1, 2, \ldots, n-1, n+1\}$ or $\{1, 2, \ldots, n\}$. Frucht [8] used the term *nearly graceful* to refer to the $\hat{\rho}$-labeling of Rosa.
The following two theorems are due to Rosa [12], we use them together with Theorem 3 to prove the results in the coming sections. We do not prove them but we provide the corresponding labelings.

Theorem 2.1. *If $m \equiv 0, 3 (mod\ 4)$, then the cycle C_m is graceful.*

Let $V(C_m) = \{v_1, v_2, \ldots, v_m\}$ and $E(C_m) = \{v_i v_{i+1} : 1 \le i \le m\}$, the addition is taken modulo m. The graceful labeling of C_m is given by

$$f(v_i) = \begin{cases} (i-1)/2 & \text{if } i \text{ is odd,} \\ m - (i-2)/2 & \text{if } i \text{ is even, } 2 \le i \le \lceil m/2 \rceil, \\ m - (i/2) & \text{if } i \text{ is even, } \lceil m/2 \rceil < i \le m. \end{cases}$$

Note that when $m \equiv 0 (\text{mod } 4)$, f is an α-labeling with boundary value $\lambda = m/2 - 1$ and that the number $3m/4$ is not a label of C_m. When $m \equiv 3 (\text{mod } 4)$, the number $(3m-1)/4$ is not a label of C_m.

Theorem 2.2. *If a tree T of size n is a path or a caterpillar, then there exists an α-labeling of T.*

For the sake of completeness we present in Figure 1, the labeling scheme given by Rosa. We must observe that the label 0 can be assigned to any of the four vertices highlighted in the figure.

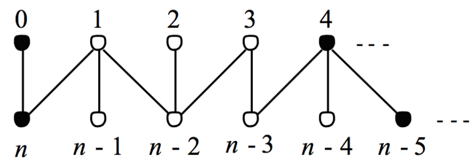

Figure 1: Rosa's α-labeling scheme for a caterpillar

When $m \equiv 1, 2 (\text{mod } 4)$, the cycle C_m is not graceful [12]. In the following theorem, proved by Barrientos [1], we present a nearly graceful labeling of these cycles where the set of induced weights is $\{1, 2, \ldots, m-1, m+1\}$.

Theorem 2.3. *The cycle C_m is nearly graceful, with weights $1, 2, \ldots, m-1, m+1$ if and only if $m \equiv 1, 2 (mod\ 4)$.*

The labeling given below is the complementary labeling of the one provided in [1].

$$f(v_i) = \begin{cases} m+1 & \text{if } i = 1, \\ m-(i-1)/2 & \text{if } i \text{ is odd, } 3 \le i \le \lfloor (m-3)/2 \rfloor, \\ m-(i+1)/2 & \text{if } i \text{ is odd, } \lceil m/2 \rceil \le i \le m, \\ (i-2)/2 & \text{if } i \text{ is even.} \end{cases}$$

Note that when $m \equiv 2(\text{mod } 4)$, f is a bipartite labeling with boundary value $\lambda = (m-2)/2$, that is, the vertices in A receive the labels $0, 1, \ldots, \lambda$, while the vertices in B get the labels $\lambda+1, \lambda+2, \ldots, (m+2)/4, (m+10)/4, (m+14)/4, \ldots, m-1$, and $m+1$.
We must also observe that when $m = 6$, the numbers 5 and 6 are not labels of C_6; for $m \ge 10$, the numbers $(m+6)/4$ and m are not labels of C_m. Similarly, when $m \equiv 1(\text{mod } 4)$ the numbers $(3m+1)/4$ and m are not labels of C_m.
Therefore, when $m \equiv 0, 3(\text{mod } 4)$, $grac(C_m) = m$, and $grac(C_m) = m+1$ when $m \equiv 1, 2(\text{mod } 4)$.

3 Graceful Labeling of $C_r \cup G_n$

In this section we construct graceful and α-labelings of graphs of the form $C_r \cup G_n$, using the labelings introduced in Section 2. Since the labeling of C_r depends on the congruence modulo 4 of r, we analyze two cases based on the gracefulness of C_r.
Let G_n be a caterpillar of size n with stable sets A and B, where $|A| = a$ and $|B| = b$. From this point, we assume that all caterpillars have been labeled using Rosa's α-labeling in such a way that the vertex labeled 0 is in A.
Let m be a positive integer. Suppose $r = 4m+i$, where $i \in \{-1, 0\}$. We denote by $\mathcal{G}_i$ the family of all caterpillars of size $n \ge m+2$ such that the vertex u of A labeled $a-m-1$, is adjacent to a leaf v. In the next theorem, we prove that when $G_n \in \mathcal{G}_i$, $C_{4m+i} \cup G_n$ is a graceful graph when $i = -1$ and is an α-graph when $i = 0$.

Theorem 3.1. *If $G_n \in \mathcal{G}_i$, then $C_{4m+i} \cup G_n$ is a graceful graph when $i = -1$ and it is an α-graph when $i = 0$.*

Proof. Suppose that C_{4m+i} has been labeled using the labeling in Theorem 1. Add to every vertex label of C_{4m+i} the constant a. The new labeling of C_{4m+i} uses all the numbers in $\{a, a+1, \ldots, a+4m+i\}$ except $a+3m+i$, and induces the weights $1, 2, \ldots, 4m+i$.
Let u be the vertex of G_n adjacent to the leaf v. The α-labeling of $G'_n = G_n - uv$ is transformed into a $(4m+1+i)$-graceful labeling by adding the constant $4m+1+i$ to every vertex label in $B' = B - \{v\}$. The labels in G'_n form the set $\{0, 1, \ldots, a-1\} \cup \{a+4m+1+i, a+4m+2+i, \ldots, 4m+n+i\}$, and the weights form the set $\{4m+2+i, 4m+3+i, \ldots, 4m+n+i\}$. By assigning the label $a+3m+i$ to the vertex v, the edge uv has now weight $4m+1+i$. Therefore the final labeling of $C_{4m+i} \cup G_n$

is graceful. Moreover, the labeling of $C_{4m} \cup G_n$ is an α-labeling with boundary value $\lambda = 2m + a - 2$. □

In Figure 2 we show an example for a graph $C_8 \cup G_{13}$, exhibiting the inital labelings of each component and the final labeling; the vertices u and v are highlighted.

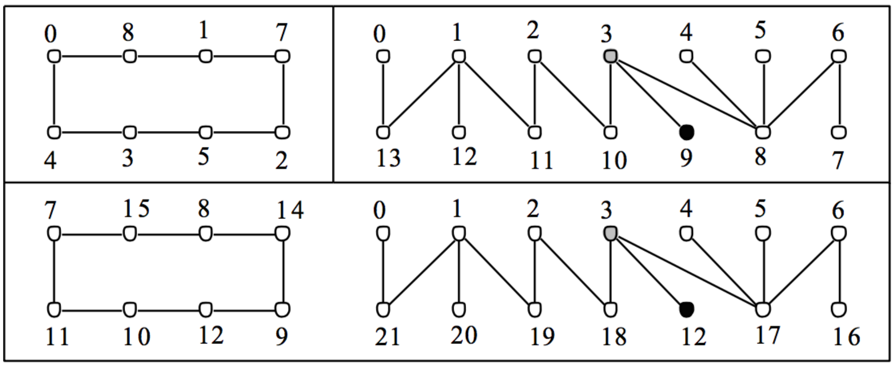

Figure 2: α-labeling of $C_8 \cup G_{13}$.

Now we analyze the case where $r = 4m + i$, $i \in \{1, 2\}$. Let $\mathcal{G}_i$ be the family of all caterpillars of size $n \geq m + 2$ such that the vertices u_1 and u_2 of A, labeled $a - m$ and $a - 2$, respectively, are adjacent to the leaves v_1 and v_2. Note that when $m = 2$, $a - m = a - 2$, that is, the vertex u in A, labeled $a - 2$, is adjacent to the leaves v_1 and v_2. In the next theorem, we show that when $G_n \in \mathcal{G}_i$, $C_{4m+i} \cup G_n$ is a graceful graph when $i = 1$ and it is an α-graph when $i = 2$.

Theorem 3.2. *If $G_n \in \mathcal{G}_i$, then $C_{4m+i} \cup G_n$ is a graceful graph when $i = 1$ and it is an α-graph when $i = 2$.*

Proof. Suppose that C_{4m+i} has been labeled using the labeling in Theorem 3. Add to every vertex label of C_{4m+i} the constant a. The new labeling of C_{4m+i} uses the labels in the set $\{a, a+1, \ldots, a+4m+1+i\} - \{a+3m+i, a+4m+i\}$, and induces the weights $1, 2, \ldots, 4m-1+i$, and $4m+1+i$.
Let u_1 and u_2 be the vertices of G_n adjacent to the leaves v_1 and v_2, respectively. Recall that when $m = 2$, $u_1 = u_2$. The α-labeling of $G'_n = G_n - u_1v_1 - u_2v_2$ is transformed into a $(4m + 2 + i)$-graceful labeling by adding the constant $4m + 2 + i$ to each vertex label in $B' = B - \{v_1, v_2\}$. The labels used on G'_n form the set $\{0, 1, \ldots, a - 1\} \cup \{a + 4m + 2 + i, a + 4m + 3 + i, \ldots, 4m + n + i\}$ and the set of induced weights is $\{4m+3+i, 4m+4+i, \ldots, 4m+n+i\}$. The vertices v_1 and v_2 are labeled $a+3m+i$ and $a + 4m + i$, respectively. This implies that the edges u_1v_1 and u_2v_2 have weights $4m + i$ and $4m + 2 + i$. Therefore, $C_{4m+i} \cup G_n$ is graceful. Moreover, the labeling of $C_{4m+2} \cup G_n$ is, indeed, an α-labeling with boundary value $\lambda = 2m + a$. □

In Figure 3 we show an example of a graph of the form $C_{13} \cup G_{12}$.
Thus, we have proved the following theorem.

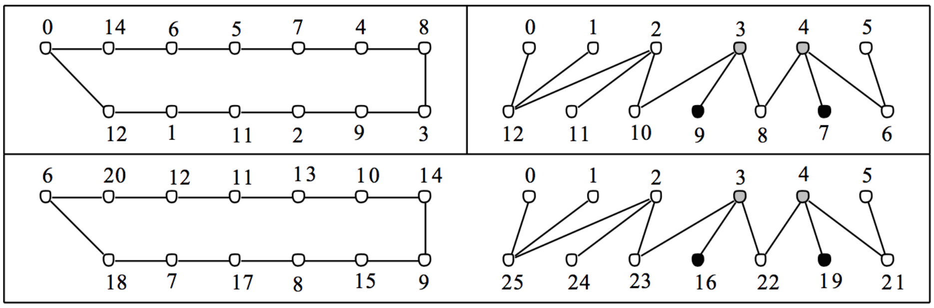

Figure 3: Graceful labeling of $C_{13} \cup G_{12}$.

Theorem 3.3. *If $i \in \{-1, 0, 1, 2\}$ and $G_n \in \mathcal{G}_i$, then $C_{4m+i} \cup G_n$ is graceful. Furthermore, if i is even, $C_{4m+i} \cup G_n$ is an α-graph.*

The families of caterpillars used in these results are quite robust, however not all caterpillars are included in these families. Therefore, it is an open problem to determine the gracefulness of $C_r \cup G_n$ for all the caterpillars that are not in these families.

4 A New Contruction of α-Trees

Let $\mathcal{G}$ be the family of all caterpillars G such that $|A| = |B| + 1$, where A and B are the stable sets of $V(G)$ and A contains the vertices of maximum eccentricity.
Let $G_1, G_2, \ldots, G_{2t} \in \mathcal{G}$ be caterpillars of size n. For each $i \in \{1, 2, \ldots, 2t\}$, let $u_i, v_i \in V(G_i)$ such that $dist(u_i, v_i) = diam(G_i)$. For every $i \in \{1, 2, \ldots, t\}$, the caterpillar Γ_i, of size $2n$ when $i = 1$ and size $2n + 1$ when $i > 1$ is constructed, via vertex amalgamation, identifying the vertices v_i and u_{t+i}; and for $i > 1$, attaching to x_i a pendant vertex y_i, where x_i is the vertex of Γ_i that results of the vertex amalgamation. For all $i \in \{1, 2, \ldots, t-1\}$, the tree H, of size $2t(n+1) - 2$, is obtained by connecting with an edge, the vertex x_i of Γ_i to the vertex y_{i+1} of Γ_{i+1}. Note that $\Gamma_i \in \mathcal{G}$ and $x_i \in A$. Let $\mathcal{H}$ be the family of all trees constructed in this manner, way claim that if $H \in \mathcal{H}$, then H is an α-tree.

Theorem 4.1. *If $H \in \mathcal{H}$, then H is an α-tree.*

Proof. For each $1 \leq i \leq t$, let f_i be the α-labeling of Γ_i, obtained using Theorem 2, that assign the label 0 to the vertex u_i. Thus, the vertex x_i has label $|A| - 1$, and the vertex y_i has label $n + |A|$ when $i \geq 2$. All these labelings have boundary value $\lambda = n$. Let g_i be the labeling of Γ_i defined by

$$g_i(v) = \begin{cases} f_i(v) + (i-1)(n+1) & \text{if } f_i(v) \leq n, \\ f_i(v) + (i-1)(n+1) + (t-i)(2n+2) & \text{if } f_i(v) > n. \end{cases}$$

Note that g_i is in fact a $((t-i)(2n+2))$-graceful labeling of Γ_i shifted $(i-1)(n+1)$ units. Therefore, the set of weights induced by g_i is $W_i = \{(t-i)(2n+2) + 1, (t-i)(2n+2) +$

$2, \ldots, (t-i)(2n+2)+\tau_i\}$ where $\tau_i = 2n$ when $i = 1$ and $\tau_i = 2n+1$ otherwise. Since $\min W_{i-1} - \max W_i = 2$ when $i \geq 2$, we have that

$$\bigcup_{i=1}^{t} W_i = \{1, 2, \ldots, 2t(n+1)-2\} - \{(t+1-i)(2n+2) : 2 \leq i \leq t\}.$$

The labels used on Γ_i form the set $P_i \cup Q_i$ where

$$P_i = \{(i-1)(n+1), (i-1)(n+1)+1, \ldots, (i-1)(n+1)+n\}$$

and

$$Q_i = \{(i-1)(n+1)+(t-i)(2n+2)+n+1, (i-1)(n+1)+ \\ (t-i)(2n+2)+n+2, \ldots, (i-1)(n+1)+(t-i)(2n+2)+\tau_i\}.$$

Since $\min P_i - \max P_{i+1} = 1$ and $\min Q_i - \max Q_{i+1} = 1$, we get that

$$P = \bigcup_{i=1}^{t} P_i = \{0, 1, \ldots, t(n+1)-1\}$$

and

$$Q = \bigcup_{i=1}^{t} Q_i = \{t(n+1), t(n+1)+1, \ldots, 2t(n+1)-2\}.$$

Hence, we have assigned injectively the labels $0, 1, \ldots, 2t(n+1)-2$ to the vertices of $\bigcup_{i=1}^{t} \Gamma_i$.

Now we turn our attention to the vertices x_i and y_i. For every $1 \leq i \leq t$, the vertex x_i in Γ_i has label $|A|-1+(i-1)(n+1)$ and for every $2 \leq i \leq t$, the vertex y_i in Γ_i has label $n+|A|+(i-1)(n+1)+(t-i)(2n+2)$. Thus, the edge $x_{i-1}y_i$ in H, for $2 \leq i \leq t$, has weight $(t+1-i)(2n+2)$. Hence, the weights of these edges form the set $\{(t+1-i)(2n+2) : 2 \leq i \leq t\}$.

Then, the weights induced on the edges of H are $1, 2, \ldots, 2t(n+1)-2$. Since $V(H) = \bigcup_{i=1}^{t} V(\Gamma_i)$ and $E(H) = \bigcup_{i=1}^{t} E(\Gamma_i) \cup \{x_{i-1}y_i : 2 \leq i \leq t\}$, we have that H is an α-graph; the boundary value of the labeling of H is $\lambda = t(n+1)-1$. □

In Figure 4 we show an example of this construction for a tree of size 64 formed by six caterpillars of size 10 and diameter 6. The vertices x_i are highlighted and the vertices y_i are represented by a small square.

We can see the potential of this construction, for example, by considering the fact that there are ten caterpillars of size 10 in $\mathcal{G}$, two of them are symmetric. Each of the non-symmetric caterpillars have two α-labelings (using Theorem 2), therefore there are 18 α-labeled caterpillars that can be used in the construction of a tree H. So we need to select $2t$ of them where repetition is allowed. When the symmetry of H is disregarded,

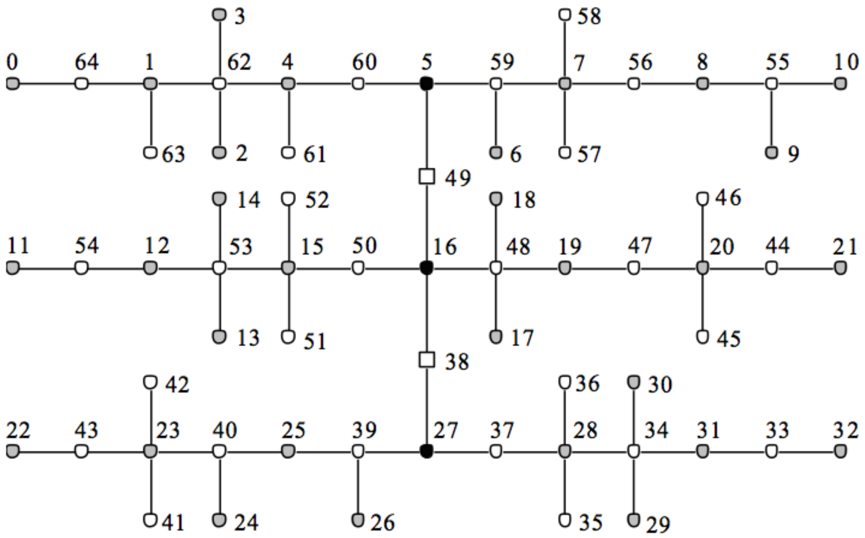

Figure 4: α-labeling of a tree H.

we have 18^t possibilities.

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