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The edge tenacity of a split graph

Bahareh Bafandeh Mayvan*[1]

[1]Department of Computer Engineering, Ferdowsi University of Mashhad.

ABSTRACT

The edge tenacity $T_e(G)$ of a graph G is defined as:

$$T_e(G) = min\{\frac{|X|+\tau(G-X)}{\omega(G-X)-1}|X \subseteq E(G) \text{ and } \omega(G-X) > 1\}$$

where the minimum is taken over every edge-cutset X that separates G into $\omega(G-X)$ components, and by $\tau(G-X)$ we denote the order of a largest component of G. The objective of this paper is to determine this quantity for split graphs. Let $G = (Z, I, E)$ be a noncomplete connected split graph with minimum vertex degree $\delta(G)$ we prove that if $\delta(G) \geq \frac{|E(G)|}{|V(G)|-1}$ then its edge-tenacity is $\frac{|E(G)|}{|V(G)|-1}$.

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1 Introduction

We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. Our terminology will be standard except as indicated. We denote by $V(G)$, $E(G)$ and $| V(G) |$ the set of vertices, the set of edges and the order of G, respectively.
A graph $G = (V, E)$ is called a split graph if its vertex set V can be partitioned into a clique Z and an independent set I. Usually, the split graph G is denoted by $G = (Z, I, E)$.

*Email: bahareh.bafandeh@gmail.com

If $N(I) \neq Z$, where $N(I)$ denote a neighborhood of vertices in I, then by choosing a vertex $v \in Z \backslash N(I)$, and replacing Z by $Z \backslash \{v\}$ and I by $I \bigcup \{v\}$, G can be rewritten as $G = (Z \backslash \{v\}, I \bigcup \{v\}, E)$, in which $N(I \bigcup \{v\}) = Z \backslash \{v\}$. Hence, in the following we always assume that $N(I) = Z$ for any split graph $G = (Z, I, E)$.

Edge-tenacity of graphs was first studied by Moazzami and Salehian in [14] where they defined the edge-tenacity of a graph G as

$$T_e(G) = \min_{A \subset E(G)} \{\frac{\mid A \mid + \tau(G-A)}{\omega(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of edges) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$.
Any undefined terms can be found in the standard references on graph theory, including Bondy and Murty [1].

2 Edge-tenacity of split graphs

The concept of tenacity of a graph G was introduced in [2,3], as a useful measure of the "vulnerability" of G. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [4 -25], the authors studied more about this new invariant. In the following, subject to some conditions, we show that the edge-tenacity of split graphs can be obtained directly from a formula.

Theorem 1. Let G be a graph of order p and size q , Then $\mathrm{T}_e(G) \leq \frac{q}{p-1}$.
Proof: In the worst case of computing $\mathrm{T}_e(G)$ of a graph, we should select all of its edges to be in the cut .i.e $|X| = q$, in this case the number of components is p, and largest component is 0 therefore $\mathrm{T}_e(G)$ will be $\frac{q}{p-1}$. In any other case (i.e $|x| < q$) $\mathrm{T}_e(G)$ should be less than or equal to $\frac{q}{p-1}$. $\square$

Theorem 2. Let $G = (Z, I, E)$ be a noncomplete split graph with $\delta(G) \geq \frac{|E(G)|}{|V(G)|-1}$ then $\mathrm{T}_e(G) = \frac{|E(G)|}{|V(G)|-1}$.

Proof: Let u be a vertex of minimum degree. If $u \in Z$, then by our assumption $N(I) = Z$ and the definition of split graphs, we have $d(u) \geq |Z| \geq \delta(G)$. If $d(u) = \delta(G)$, then $\delta(G) = |Z|$, and u is adjacent to exactly one vertex v in I. Since G is noncomplete, there must be another vertex $w \in I$ such that $uw \notin E(G)$. This implies that $d(w) < \delta(G)$, a contradiction. So, if u is a vertex of minimum degree, then $u \in I$. Let X be an arbitrary edge cut of G. In the following, we will prove that $\frac{|X|+\tau(G-X)}{\omega(G-X)-1} \geq \frac{|E(G)|}{|V(G)|-1}$ always holds. We distinguish three cases.

Case 1. $X \subseteq [Z, I]$

It is clear that the components of $G - X$ can be divided into two classes. One class contains only one component, which includes all vertices of C, while in the other class, every component is a vertex of I. Suppose that there are f_2 components in the second class. Then $|X| \geq f_2\delta(G)$ and $\omega(G - X) = f_2 + 1$. Thus

$$\frac{|X|+\tau(G-X)}{\omega(G-X)-1} \geq \frac{f_2\delta(G)}{(f_2+1)-1} = \delta(G) \geq \frac{|E(G)|}{|V(G)|-1}$$

Case 2. $X \subseteq E(Z)$

Denote the components of $G - X$ by $G_1, G_2, ..., G_f$ and $Z_i = V(G_i) \bigcap Z$ for $i = 1, 2, ..., f$. Since $N(I) = Z$, each component Z_i must contain at least one vertex $v_i \in I$. Clearly $N(v_i) \subseteq Z_i$. So $\delta(G) \leq d(v_i) \leq |Z_i|$. Then we have $|X| \geq \frac{f(f-1)}{2}\delta(G)^2$. Thus,

$$\frac{|X|+\tau(G-X)}{\omega(G-X)-1} \geq \frac{\frac{f(f-1)}{2}\delta(G)^2}{f-1} = \frac{f}{2}\delta(G)^2 \geq \delta(G)^2 \geq \delta(G) \geq \frac{|E(G)|}{|V(G)|-1}$$

Case 3. $X \bigcap [Z, I] \neq \phi$ and $X \bigcap E(Z) \neq \phi$.

As in the proof of Case 2, we denote the components of $G - X$ by $G_1, G_2, ..., G_f$ and let $Z_i = V(G_i) \bigcap Z$ for $i = 1, 2, ..., f$.

Case 3.1. $|Z_i| \geq \delta(G)$ for some i with $1 \leq i \leq f$.

Without loss of generality, we assume $|Z_i| \geq \delta(G)$ for $i = 1, 2, ..., f_1$, $0 < |Z_i| < \delta(G)$ for $i = f_1+1, f_1+2, ..., f_1+f_2$ and $|Z_i| = 0$ for $i = f_1+f_2+1, f_1+f_2+2, ..., f_1+f_2+f_3 = f$. It is easy to see that

$$\begin{aligned} |X| &\geq \frac{f_1(f_1-1)}{2}\delta(G)^2 + f_1f_2\delta(G) + \frac{f_2(f_2-1)}{2} + f_3\delta(G) \\ &\geq \frac{f_1(f_1-1)}{2}\delta(G)^2 + f_1f_2\delta(G) + f_3\delta(G). \end{aligned}$$

Then we have

$$\begin{aligned} \frac{|X|+\tau(G-X)}{\omega(G-X)-1} &\geq \frac{\frac{f_1(f_1-1)}{2}\delta(G)^2+f_1f_2\delta(G)+f_3\delta(G)}{(f_1+f_2+f_3)-1} \\ &\frac{\frac{f_1(f_1-1)}{2}\delta(G)+f_1f_2+f_3}{(f_1+f_2+f_3)-1}\delta(G). \end{aligned}$$

It is not difficult to check that the inequality $\frac{f_1(f_1-1)}{2}\delta(G) + f_1f_2 + f_3 \geq (f_1 + f_2 + f_3) - 1$ holds for any positive integers f_1 and $\delta(G)$, and any nonnegative integers f_2 and f_3. So we have

$$\frac{|X|+\tau(G-X)}{\omega(G-X)-1} \geq \delta(G) \geq \frac{|E(G)|}{|V(G)|-1}.$$

Case 3.2. $|Z_i| < \delta(G)$ for $i = 1, 2, ..., f$.

Suppose that $|V(G_i)| \geq 2$ for $i = 1, 2, ..., f_1$, $|V(G_i)| = 1$ and $V(G_i) \subseteq C$ for $i = f_1+1, f_1+2, ..., f_1+f_2$, and $|V(G_i)| = 1$ and $V(G_i) \subseteq I$ for $i = f_1+f_2+1, f_1+f_2+2, ..., f_1+f_2+f_3 = f$. Then G_i must contain at least one vertex of Z when $i = 1, 2, ..., f_1$.

If $f_1 = 0$, then $X = E(G)$ and $\omega(G - X) = |V(G)|$. This implies that

$$\frac{|X|+\tau(G-X)}{\omega(G-X)-1} \geq \frac{|E(G)|}{|V(G)|-1}.$$

So we assume $f_1 \geq 1$.
Let $l = \min |C_i| : i = 1, 2, ..., f_1$. Without loss of generality, assume $|C_1| = l$ and let $|V(G_1)| = n_1$. Thus $0 < l < \delta(G)$. So we have

$$\begin{aligned} |X| &\geq \tfrac{f_1(f_1-1)}{2}l^2 + f_1f_2l + \tfrac{f_2(f_2-1)}{2} + f_3\delta(G) \\ &\geq \tfrac{f_1(f_1-1)}{2}l^2 + f_1f_2l + f_3l \end{aligned}$$

Set $X_1 = X \bigcup E(G_1)$. Then $|X_1| \leq |X| + l(n_1 - \frac{l+1}{2})$ and $\omega(G - X_1) = \omega(G - X) + n_1 - 1$ hold. Therefore,

$$\begin{aligned} &\frac{|X|}{\omega(G-X)-1} - \frac{|X_1|}{\omega(G-X_1)-1} \\ &\geq \frac{|X|}{(f_1+f_2+f_3)-1} - \frac{|X|+l(n_1-\frac{l+1}{2})}{(f_1+f_2+f_3)-1+n_1-1} \\ &= \frac{(n_1-1)|X|-(f_1+f_2+f_3-1)l(n_1-\frac{l+1}{2})}{(f_1+f_2+f_3-1)(f_1+f_2+f_3+n_1-2)} \\ &\geq \frac{(n_1-1)(\frac{f_1(f_1-1)}{2}l^2+f_1f_2l+f_3l)-(f_1+f_2+f_3-1)l(n_1-\frac{l+1}{2})}{(f_1+f_2+f_3-1)(f_1+f_2+f_3+n_1-2)} \\ &= \frac{(n_1-1)(\frac{f_1(f_1-1)}{2}l+f_1f_2+f_3)-(f_1+f_2+f_3-1)(n_1-\frac{l+1}{2})}{(f_1+f_2+f_3-1)(f_1+f_2+f_3+n_1-2)}l \end{aligned}$$

Since f_1 and l are positive integers, f_2 and f_3 are nonnegative integers, we have $(n_1 - 1) \geq (n_1 - \frac{l+1}{2})$. Therefore,

$$(n_1 - 1)(\tfrac{f_1(f_1-1)}{2}l + f_1f_2 + f_3) - (f_1 + f_2 + f_3 - 1)(n_1 - \tfrac{l+1}{2}) \geq 0.$$

Thus, we get

$$\frac{|X|}{\omega(G-X)-1} \geq \frac{|X_1|}{\omega(G-X_1)-1}.$$

If $f_1 = 1$, then $X_1 = E(G)$ and $\omega(G - X_1) = |V(G)|$. Then

$$\frac{|X|+\tau(G-X)}{\omega(G-X)-1} \geq \frac{|X_1|}{\omega(G-X_1)-1} \geq \frac{|E(G)|}{|V(G)|-1}.$$

If $f_1 > 1$, then $G - X_1$ has $f_1 - 1$ components with at least two vertices, and each component of $G - X_1$ has less than $\delta(G)$ vertices. Repeating the above process, we can get a sequence of edge cuts $X_1, X_2, ..., X_{k_1}$ such that

$$\frac{|X|}{\omega(G-X)-1} \geq \frac{|X_1|}{\omega(G-X_1)-1} \geq ... \geq \frac{|X_{k_1}|}{\omega(G-X_{K_1})-1},$$

$X_{k_1} = E(G)$ and $\omega(G - X_{k_1}) = |V(G)|$. So we have

$$\frac{|X|+\tau(G-X)}{\omega(G-X)-1} \geq \frac{|E(G)|}{|V(G)|-1}$$

This completes the proof.

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