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Minimum Tenacity of Toroidal graphs

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ABSTRACT

The tenacity of a graph G, $T(G)$, is defined by $T(G) = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph $G-S$, and $\omega(G-S)$ be the number of components of $G-S$.In this paper a lower bound for the tenacity $T(G)$ of a graph with genus γ (G) is obtained using the graph's connectivity $\kappa(G)$. Then we show that such a bound for almost all toroidal graphs is best possible.

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1 Introduction

The concept of tenacity of a graph G was introduced in [4,5], as a useful measure of the "vulnerability" of G. In [5] Cozzens et al. calculated tenacity of the first and second case of the Harary Graphs but they didn't show the complete proof of the third case. In [17] they showed a new and complete proof for case three of the Harary Graphs. In [11], they compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [7 - 27], the authors studied more about this new invariant. We consider only graphs without loops or multiple edges. We use $V(G)$, and $\omega(G)$ to denote the vertex set and number of components in

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a graph G, respectively. We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. We denote by $V(G)$, $E(G)$ and $\mid V(G) \mid$ the set of vertices, the set of edges and the order of G, respectively.
The tenacity of a graph G, $T(G)$, is defined by $T(G) = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph $G-S$, and $\omega(G-S)$ be the number of components of $G-S$. A connected graph G is called T-tenacious if $\mid S \mid +\tau(G-S) \geq T\omega(G-S)$ holds for any subset S of vertices of G with $\omega(G-S) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $S \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|S|+\tau(G-S)}{\omega(G-S)}$.
The Mix-tenacity $T_m(G)$ of a graph G is defined as

$$T_m(G) = \min_{A\subset E(G)}\{\frac{\mid A \mid +\tau(G-A)}{\omega(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of vertices) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$. Cozzens et al. in [4], called this parameter Edge-tenacity, but Moazzami changed the name of this parameter to Mix-tenacity in [15]. It seems Mix-tenacity is a better name for this parameter. $T(G)$ and $T_m(G)$ turn out to have interesting properties.
After the pioneering work of Cozzens, Moazzami, and Stueckle in [4,5], several groups of researchers have investigated tenacity, and its related problems. In [19] and [20] Piazza et al. used the $T_m(G)$ as Edge-tenacity. But this parameter is a combination of cutset $A \subset E(G)$ and the number of vertices of a largest component, $\tau(G-A)$. It may be observed that in the definition of $T_m(G)$, the number of edges removed is added to the number of vertices in a largest component of the remaining graph. Also this parameter didn't seem very satisfactory for Edge-tenacity. Thus Moazzami and Salehian introduced a new measure of vulnerability, the Edge-tenacity, $T_e(G)$, in [15]. The Edge-tenacity $T_e(G)$ of a graph G is defined as

$$T_e = \min_{A\subset E(G)}\{\frac{\mid A \mid +\tau(G-A)}{\omega(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of edges) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$. This new measure of vulnerability involves edges only and thus is called the Edge-tenacity. Since 1992 there were several interesting questions. But the question " How difficult is it to recognize T-tenacious graphs? " has remained an interesting open problem for some time. The question was first raised by Moazzami in [10]. Our purpose in [18] was to show that for any fixed positive rational number T, it is NP-hard to recognize T-tenacious graphs. To prove this we showed that it is NP-hard to recognize T-tenacious graphs by reducing a well-known NP-complete variant of INDEPENDENT SET.

The toughness of a graph G was defined in [1] as $t(G) = min\{\frac{|A|}{\omega(G-A)}\}$, where the minimum is taken over all cut-sets A of G. A subset A of $V(G)$ is said to be a t-set of G if $t(G) = \frac{|A|}{\omega(G-A)}$. Note that if G is disconnected then the set A may be empty.

2 Preliminary results

Let G be a simple non-complete graph, we know that

$$\kappa = \left\{ \frac{2g(1+2\gamma/\nu-2/\nu)}{g-2} \right\} \tag{1}$$

$$\varepsilon(G) \leq \frac{g(\nu+2\gamma-2)}{g-2} \tag{2}$$

W. Goddard et al. [6] proved that

$$t\left(G\right) > \frac{\kappa}{2} - 1 \qquad if \gamma = 0 \tag{3}$$

$$t\left(G\right) \geq \frac{\kappa(\kappa-2)}{2(\kappa-2+2\gamma)} \qquad if \gamma = 1 \tag{4}$$

3 Lower bounds

Theorem 1. For all graphs we have the following:

$$T\left(G\right) \geq \frac{1}{2\left(\kappa-2+2\gamma\right)} \left(\kappa\left(\kappa-2\right) + \frac{\kappa^2+4\left(\gamma-1\right)}{\nu}\right) \qquad if \gamma \geq 1 \tag{5}$$

$$T\left(G\right) \geq \frac{\kappa\left(\kappa-2\right)}{2\left(\kappa-2+2\gamma\right)} + \frac{\kappa}{2\nu} \qquad if \gamma \geq 0 \tag{6}$$

Proof. If $\gamma = 1$, from the definition of toughness

$$t\left(G\right) = min\left\{ \frac{|S|}{\omega(G-S)} \mid S \subset V\left(G\right),\ \omega\left(G-S\right) \geq 2 \right\} \tag{7}$$

Now we know that

$$\begin{aligned}
\forall S \subset V(G): \quad & \frac{|S|}{\omega(G-S)} \geq t(G) \geq \frac{\kappa(\kappa-2)}{2(\kappa-2+2\gamma)} && (8)\\
& \nu(G) \geq |S| + \omega(G-S) \\
& \geq \omega(G-S)\left(\frac{\kappa(\kappa-2)}{2(\kappa-2+2\gamma)}+1\right) \\
& = \omega(G-S)\left(\frac{\kappa^2+4(\gamma-1)}{2(\kappa-2+2\gamma)}\right) && (9)\\
& \text{using [5] and knowing } \tau(G-S) \geq 1 \\
\Rightarrow & \frac{|S|+\tau(G-S)}{\omega(G-S)} \geq \frac{|S|}{\omega(G-S)} + \frac{1}{\omega(G-S)} \\
& \geq \frac{\kappa(\kappa-2)}{2(\kappa-2+2\gamma)} + \frac{\kappa^2+4(\gamma-1)}{2\nu(G)\times(\kappa-2+2\gamma)} && (10)\\
\text{If } \gamma = 0 \quad & : \\
\forall S \subset V(G): \quad & \frac{|S|}{\omega(G-S)} \geq t(G) \geq \frac{\kappa}{2} - 1 \\
\Rightarrow \quad & \nu(G) \geq |S| + \omega(G-S) \geq \omega(G-S)\frac{\kappa}{2} \\
\Rightarrow \quad & T(G) \geq \frac{\kappa(\kappa-2)}{2(\kappa-2+2\gamma)} + \frac{\kappa}{2\nu} && (11)
\end{aligned}$$

□

Considering toroidal graphs, it is known that for such a graph,

$$T(G) \leq \frac{(\kappa-2)}{2} + \frac{\kappa}{2\nu(G)} \text{ and } \kappa \leq \frac{2g}{g-2} \Leftrightarrow g \leq \frac{2\kappa}{\kappa-2}$$

Thus a toroidal graph (G) has connectivity no more than 6 and girth of anything from 3 to $\frac{2\kappa}{\kappa-2}$. Interestingly, all example graphs given in [6] for $\kappa \geq 4$ for toughness equal to the lower bound, leave all remaining components after the cut with a lone vertex. I use these examples with others for $\kappa = 3$ and study graphs by their connectivity with the formula mentioned previously.

3.1 Connectivity 6

$\kappa = 6 \Rightarrow g = 3$ By Euler's formula,

$$\begin{aligned}
& \delta \geq \kappa \Rightarrow \nu\kappa \leq 2\varepsilon, \; Fg \leq 2\varepsilon \\
& \Rightarrow 0 = \nu(G) - \varepsilon(G) + F \leq \frac{2\varepsilon(G)}{6} - \varepsilon(G) + \frac{2\varepsilon(G)}{3} = 0 && (12)\\
& \Rightarrow \nu(G) = \frac{2\varepsilon(G)}{\kappa} \text{ and } F = \frac{2\varepsilon(G)}{g} && (13)
\end{aligned}$$

(where $\nu(G)$ is the number of vertices of the graph, $\varepsilon(G)$ the number of edges and F the number of faces when embedded on the torus). Thus the graph is 6-regular with every face a triangle.
Imagine the bipartite 'honeycomb' graph (G) embedded on the torus. By Euler's formula this graph is 3-connected, 3-regular with every face a hexagon. By inserting a vertex in each face and joining to all surrounding vertices to this vertex, we have a toroidal graph (H) with connectivity 6 and girth 3.
By deleting all vertices of the original honeycomb $|S| = \nu(G)$ we leave $\omega\left(H-G\right) = \nu\left(H-G\right) = \frac{1}{2}\nu(G)$ components each containing a single vertex. Thus, $T\left(H\right) = \frac{\nu(G)+1}{1/2\nu(G)} = 2 + \frac{2}{\nu(G)} = 2 + \frac{3}{\nu(H)}$ which is equal to the lower bound thus this lower bound is best possible.

3.2 Connectivity 5

$\kappa = 5 \Rightarrow g = 3$ Let us consider graph G where

$$V\left(G\right) = \{\ a_i,\ b_i,\ c_i,\ d_i,\ e_i,\ f_i\ \left| i = 0, 1, 2, \ldots, n-1\right.\}\ \text{and} \tag{14}$$

$$E\left(G\right) = \{a_ib_i,\ a_ic_i,\ b_id_i,\ c_ie_i,\ e_if_i,\ d_if_i,\ a_ia_{i+1},\ d_ib_{i+1},\ e_ic_{i+1},\ f_if_{i+1}\left| i = 0, 1, \ldots, n-1\right.\} \tag{15}$$

It is clear that this graph embedded on the torus has every face a pentagon, by inserting a vertex into each face and joining it to all vertices surrounding the face, we have a 5-connected toroidal graph (H) with girth 3.

$$\begin{aligned} S &= \{u\left| u \in \nu(G)\right.\} \\ &\Rightarrow \omega\left(H-G\right) = \nu\left(H-G\right) = \frac{2}{3}\nu\left(G\right) = \frac{2}{5}\nu\left(H\right), \qquad \tau\left(H-G\right) = 1 \\ &\Rightarrow T\left(G\right) = \frac{3}{2} + \frac{5/2}{\nu(H)} \end{aligned} \tag{16}$$

3.3 Connectivity 4

$\kappa = 4 \Rightarrow g = 3\ or\ 4.$ If g=4, by Euler's formula we know this graph is 4-regular with every face a quadrilateral. Consider $H_n = C_4 \times C_n$, as it is also mentioned in [6], this graph is bipartite and so has toughness 1. It is clear that both parts must have an equal number of vertices so by deleting all of one part we have:

$$\omega\left(H_n\right) = \frac{\nu(H_n)}{2},\ |S| = \frac{\nu(H_n)}{2}, \tau\left(H_n\right) = 1\ \Rightarrow\ T\left(G\right) = 1 + \frac{2}{\nu(H_n)} \tag{17}$$

By adding one edge to two opposite corners of one of the faces, (two corners which are in the part which is our cutset), we have a graph with the same tenacity with girth 3.

3.4 Connectivity 3

$\kappa = 3 \Rightarrow g \leq 6$. If g=6, W. Cao and M. J. Pelsmajer [1] proved that all such graphs are bipartite and as they are also 6-regular have toughness exactly 1. We know this graph is bipartite with both parts of equal order, so by deleting all vertices in one we again have equation {18}, a higher number than the lower bound suggested by the theorem.
If $g = 4$, we know that $k_{3,6}$ has girth 4, connectivity 3 and genus 1, [21], and also by deleting the 3 vertices with degree 6, we have 6 lone vertices in 6 components so

$$T(G) = \frac{1}{2} + \frac{1}{6} = \frac{\kappa - 2}{2} + \frac{\kappa}{2n} \tag{18}$$

If g=3, consider the graph $k_{3,6}$ to which one edge has been added joining 2 of the vertices with degree 6. The graph still is toroidal and the tenacity is equal to the above (if g=4) and so for g=3, the lower bound is also best possible.
If g=5, the example given in [2] results in components, the number of which are double the number of vertices cut, but which are each dodecahedrons and have 20 vertices. So

$$\begin{aligned} & 20\omega(G-S) + \frac{1}{2}\omega(G-S) = \nu(G) \\ \Rightarrow & \frac{20}{\omega(G-S)} = \frac{410}{\nu(G)} \\ \Rightarrow & T(G) = \frac{1}{2} + \frac{20}{\omega(G-S)} = \frac{1}{2} + \frac{410}{\nu(G)} \end{aligned} \tag{19}$$

However, the lower bound is $T(G) = \frac{1}{2} + \frac{3}{2\nu(G)}$ which is only possible if all components after the cut contain 1 vertex only. I claim this is not possible

Theorem 2. The lower bound for the tenacity cannot be reached if g=5 and $T(G) \geq \frac{1}{2} + \frac{21}{2(\nu-6)}$

Proof. Suppose we have achieved the lower bound, then we have cut S vertices and there are now $2|S|$ components remaining each containing a single vertex thus $3|S| = \nu(G)$. Each remaining component has at least 3 adjacent vertices in S and so $3 \times 2|S| \leq \varepsilon(G) \leq \frac{g(\nu+2\gamma-2)}{g-2} = \frac{5(3|S|+2-2)}{3} = 5|S|$ which is a contradiction.
Generally I can say $\tau(G-S) \geq 7$. Let τ be the order of the largest component. Let for every $1 = i = \tau$, e_i be the number of edge with both ends in a component with i vertices, E_i be the number of edges with at least one end in this component and x_i be the number of components with i vertices.
We know that as in this component the girth is at least 5. So for every 4 vertices, there are a maximum of 3 edges, if there were more, there would definitely be a cycle of length 4 or less. So if we count the number of "groups of 4 vertices" multiplied by 3, we have definitely counted each edge at least the number of times it can appear in a "group of 4" so

$$3 \times \binom{i}{4} \geq e_i \times \binom{i-2}{2} \tag{20}$$
$$\Rightarrow \frac{3 \times i!}{4! \times (i-4)!} \geq \frac{e_i \times (i-2)(i-3)}{2}$$
$$\Rightarrow \frac{i(i-1)}{4} \geq e_i$$

$$\delta \geq 3 \Rightarrow (E_i - e_i) + 2e_i \geq 3i \Rightarrow E_i \geq 3i - e_i \geq 3i - \frac{i(i-1)}{4} = \frac{13}{4}i - \frac{i^2}{4} \tag{21}$$

$$\varepsilon(G) \leq \frac{g(\nu + 2\gamma - 2)}{g-2} = \frac{5\nu}{3} \qquad \text{and} \qquad |S| = \frac{1}{2}\omega(G-S) = \frac{1}{2}\sum_{i=1}^{\tau} x_i$$
$$\Rightarrow \sum_{1}^{\tau}\left(\frac{13}{4}i - \frac{i^2}{4}\right)x_i \leq \sum_{1}^{\tau} E_i x_i \leq \varepsilon(G) \leq \frac{5\nu}{3} = \frac{5}{3}\left(\sum_{i=1}^{\tau} i x_i + \frac{1}{2}\sum_{i=1}^{\tau} x_i\right)$$
$$\Rightarrow \exists i: \ \frac{13}{4}i - \frac{i^2}{4} \leq \frac{5i}{3} + \frac{1}{2} \Rightarrow \frac{3i^2 - 19i + 6}{12} \geq 0 \Rightarrow i \geq 6 \tag{22}$$

But for $i = 6$, e_i is at most 6 (because if more, there will be a cycle shorter than 5) and so E_i is at least 12 and $\frac{5}{3}\left(i + \frac{1}{2}\right) = 10 + \frac{5}{6} < 12$, so there has to exist one component with $i \geq 6$ but $i \neq 6$ which means $i \geq 7$, meaning that $\tau \geq 7$.
Therefore, for a toroidal graph with connectivity 3 and girth 5, $T(G) \geq \frac{1}{2} + \frac{21}{2(\nu - 6)}$ □

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