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Optimizing Closed-Form Approximations of the Error Function via the Gaussian Combined Arms Metaheuristic

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ABSTRACT

The error function $\mathrm{erf}(x)$ plays a central role in probability, statistics, communications, and diffusion models, yet it has no simple closed-form expression in terms of elementary functions, so practical computations rely on analytical approximations. This paper develops a systematic global-optimization framework to enhance the accuracy of such approximations *without* altering their analytical structure. For a broad collection of existing closed-form formulas, the numerical coefficients are treated as decision variables in a unified problem, where a composite objective combining the mean absolute error (MAE) and the maximum absolute error (Max-AE) over carefully selected domains is minimized. The resulting nonlinear problems are solved using the Gaussian Combined Arms (GCA) metaheuristic algorithm. Numerical results show that, across 16 structural types, both MAE and Max-AE are often reduced.

Keyword: closed-form approximation; coefficient tuning; numerical refinement; inverse error function; function optimization.

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1 Introduction

The standard error function, $\text{erf}(x)$, serves as a fundamental analytical tool for modeling random phenomena and dynamical systems. This function represents the cumulative distribution function of the standard normal distribution up to a scaled linear transformation and finds widespread application in probability theory, statistics, signal processing, communication engineering, statistical physics, and quantitative finance [1, 2]. In these areas, $\text{erf}(x)$ emerges as an indispensable component of mathematical analysis [2, 3]. However, because the function is defined by an integral that is analytically intractable and lacks a closed-form expression in terms of elementary functions [4], its evaluation relies on numerical methods or analytical approximations. Consequently, numerous approximations have been developed to facilitate its practical use [4, 5]. Over the past decades, a diverse set of analytical approximations for $\text{erf}(x)$ has been proposed, derived from techniques such as Taylor series expansion, rational approximations, continued fractions, and nonlinear composite models. Although many of these approximations achieve acceptable accuracy on restricted domains, they frequently entail increased computational complexity [4, 6], which can limit their applicability in practice. Therefore, a central challenge is to find an optimal trade-off between accuracy and simplicity—namely, to design approximations that deliver high accuracy over the entire domain or within critical intervals while preserving simple and computationally efficient closed forms. Alongside these classical approaches, more recent strategies have emerged that re-tune approximation parameters by harnessing intelligent global search methods. In recent years, metaheuristic optimization algorithms have become powerful tools inspired by natural and biological processes, capable of tackling complex optimization problems in nonlinear, high-dimensional spaces. These algorithms are increasingly used to tune parameters of analytical models, particularly in situations where analytical differentiation is difficult or infeasible [7, 8]. Among such methods, the recently introduced Gaussian Combined Arms Algorithm (GCA)[9] is adopted in this work as an efficient global-search strategy, benefiting from the strengths of population-based optimization and robust exploration–exploitation mechanisms. The application of metaheuristic algorithms to approximation tuning thus provides an effective route to enhance the reliability and performance of existing analytical models. Building on previous studies that compiled and assessed families of Gaussian-related approximations [4, 5], the present study proposes an integrated framework for improving analytical approximations of the standard error function. In this work, the Gaussian Combined Arms Algorithm is used to globally optimize the parameters of existing approximations drawn from 16 model families—including rational, exponential, polynomial, and composite forms—yielding high-accuracy formulas over the full domain without increasing structural complexity. In particular, we introduce several accurate yet structurally simple closed-form approximations for $\text{erf}(x)$ whose analytical inverses are also provided, enabling fast and precise forward and inverse evaluations. This GCA-based framework offers a generalization and extensible tool for optimizing analytical models with potential impact on advanced communication-system design, high-fidelity physical simulations, and machine-learning algorithms that require rapid and accurate evaluation of probabilistic functions. The remainder of this paper is organized as follows. Section 2 presents the theoretical background and a detailed literature review on analytical approximations of the error function and related Gaussian functions. Section 3 describes the proposed GCA-based optimization framework, including the parameterization of the selected approximations and the adopted error criteria. Section 4 reports the optimized parameter sets and provides a comparative numerical analysis of the original and refined approximations in terms of Max-AE and MAE. Finally, Section 5 summarizes the main findings, discusses their implications for practical applications, and outlines possible directions for future research.

2 Theoretical Background and Literature Review

This section covers the fundamental concepts related to the standard error function ($\text{erf}(x)$), metaheuristic algorithms—particularly the Gaussian Combined Arms Algorithm (GCA) algorithm—and the accuracy evaluation metrics, including the Maximum Absolute Error (Max-AE) and the Mean Absolute Error (MAE). These theoretical foundations provide the context for understanding the research framework and the proposed approach.

2.1 Standard Error Function

The standard error function is a widely used function in applied mathematics, statistics, and engineering, defined as follows:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{1}$$

This function plays a key role in probability calculations, signal processing, Gaussian Combined Arms Algorithm, and communication engineering [1, 2, 3]. The function $\text{erf}(x)$ has odd symmetry, i.e., $\text{erf}(-x) = -\text{erf}(x)$ [3], and approaches the values ± 1 at infinity. Since this function is defined by an analytically unsolvable integral, its computation is only possible through numerical methods or analytical approximations [4, 5]. This has made the development of accurate and efficient approximations for $\text{erf}(x)$ a recurring topic in the literature of applied mathematics.

The standard error function $\text{erf}(x)$ is analytically related to several other special functions that frequently appear in probability theory and statistical signal processing, including the Gaussian Q-function, the cumulative distribution function (CDF) $\Phi(x)$ of the standard normal distribution, the complementary error function $\text{erfc}(x)$, and Mills' ratio $m(x)$. These functions are mathematically equivalent and can be transformed into one another through exact analytical relations. Table 1 summarizes the 20 mutual relations among these five functions, while Table 2 presents the corresponding relations between their inverse forms.

Based on these analytical equivalences, the resulting set of 39 invertible approximations, grouped into 16 structural types, is discussed in the following subsection.

Table 1: Relationships between $Q(x)$, $\Phi(x)$, $\mathrm{erf}(x)$, $\mathrm{erfc}(x)$, and $m(x)$

	$Q(x)$	$\Phi(x)$	$\mathrm{erf}(x)$	$\mathrm{erfc}(x)$	$m(x)$
$Q(x)$	$Q(x)$	$1-\Phi(x)$	$\frac{1}{2}\left(1-\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$	$\frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$	$m(x)=\frac{1}{\sqrt{2\pi}}\frac{e^{-x^2/2}}{Q(x)}$
$\Phi(x)$	$1-Q(x)$	$\Phi(x)$	$\frac{1}{2}\left(1+\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$	$1-\frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$	$1-m(x/\sqrt{2})\sqrt{\frac{\pi}{2}}\,e^{-x^2/2}$
$\mathrm{erf}(x)$	$1-2Q(x\sqrt{2})$	$2\Phi(x\sqrt{2})-1$	$\mathrm{erf}(x)$	$1-\mathrm{erf}(x)$	$1-m(x\sqrt{2})\sqrt{\frac{\pi}{2}}\,e^{-x^2}$
$\mathrm{erfc}(x)$	$2Q(x\sqrt{2})$	$2(1-\Phi(x\sqrt{2}))$	$1-\mathrm{erf}(x)$	$\mathrm{erfc}(x)$	$\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)\sqrt{\frac{\pi}{2}}\,e^{x^2/2}$
$m(x)$	$Q(x)\sqrt{2\pi}\,e^{x^2/2}$	$(1-\Phi(x))\sqrt{2\pi}\,e^{x^2/2}$	$\frac{\pi}{2}\,e^{x^2}\,\mathrm{erf}^2\left(\frac{x}{\sqrt{2}}\right)$	$\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)\sqrt{\frac{\pi}{2}}\,e^{x^2/2}$	$m(x\sqrt{2})\sqrt{\frac{\pi}{2}}\,e^{-x^2}$

Table 2: The 12 mutual relations among the 4 inverse functions $Q^{-1}(y)$, $\Phi^{-1}(y)$, $\mathrm{erf}^{-1}(y)$, and $\mathrm{erfc}^{-1}(y)$.

	$Q^{-1}(y)$	$\Phi^{-1}(y)$	$\mathrm{erf}^{-1}(y)$	$\mathrm{erfc}^{-1}(y)$
$Q^{-1}(y)$	$Q^{-1}(y)$	$\Phi^{-1}(1-y)$	$\frac{1}{\sqrt{2}}\,\mathrm{erf}^{-1}(1-2y)$	$\frac{1}{\sqrt{2}}\,\mathrm{erfc}^{-1}(2y)$
$\Phi^{-1}(y)$	$Q^{-1}(1-y)$	$\Phi^{-1}(y)$	$\frac{1}{\sqrt{2}}\,\mathrm{erf}^{-1}(2y-1)$	$\frac{1}{\sqrt{2}}\,\mathrm{erfc}^{-1}(2(1-y))$
$\mathrm{erf}^{-1}(y)$	$\frac{1}{\sqrt{2}}Q^{-1}\left(\frac{1-y}{2}\right)$	$\frac{1}{\sqrt{2}}\Phi^{-1}\left(\frac{1+y}{2}\right)$	$\mathrm{erf}^{-1}(y)$	$\mathrm{erf}^{-1}(1-y)$
$\mathrm{erfc}^{-1}(y)$	$\frac{1}{\sqrt{2}}Q^{-1}\left(\frac{y}{2}\right)$	$\frac{1}{\sqrt{2}}\Phi^{-1}\left(1-\frac{y}{2}\right)$	$\mathrm{erf}^{-1}(1-y)$	$\mathrm{erfc}^{-1}(y)$

2.2 The Set of Approximations Under Study

To establish a consistent analytical framework, the approximations were organized into sixteen structural types according to their mathematical composition and invertibility characteristics. Subsequent examination of their algebraic patterns revealed six higher-level categories that reflect the principal strategies employed in constructing analytical approximations of the error function. This hierarchical classification not only facilitates a clear understanding of the relationships among existing formulations but also provides a structured foundation for parameter optimization and comparative analysis. The following subsections present these six categories, outlining their mathematical principles, representative models, and analytical inverses, thereby supporting direct application in statistical modeling, simulation, and computational frameworks requiring explicit invertibility.

Class 1: Exponential-Based Approximations

This Class encompasses one of the most widely used approaches to error function approximation, relying on combinations of exponential terms to capture the rapid transition from 0 to 1 over the function's domain. The mathematical foundation leverages the natural decay properties of exponentials, which provide accurate modeling of both central and tail behavior while maintaining computational efficiency.
Several subtypes illustrate the diversity of exponential-based formulations:

- **Type 1 Approximations:** These include classical two-term combinations that balance simplicity with sufficient accuracy for engineering applications. Chiani-1, Wu-2, Powari, and Olabiyi-2 exemplify this approach with carefully chosen weighting coefficients and exponents.

- **Type 2 Approximations:** These involve modified single- or two-term exponential forms with linear adjustments in the exponent, enabling precise control over the function's slope near the origin. Mastin-1, Mastin-2, Mastin-3, and Lin-1 follow this methodology.

- **Type 3 Approximations:** Ordaz and Hanandeh-2 belong to this subclass, employing strategically scaled exponentials to improve tail-region accuracy while retaining a simple closed-form expression.

- **Type 4 Approximations:** This subtype focuses on Chernoff-style single- or two-term exponential expressions, including Chernoff, Chernoff-impr, Ermolova-1/2, Gasull, Olabiyi-1, Wu-1, and Chang, optimized for minimal computational cost and reliable high-accuracy performance.

The full set of parameterized forms and their corresponding inverses are summarized in Table 3, which provides the practical reference for implementation in computational routines.

Table 3: Summary of Exponential-Based Approximations (Class 1)

Type	**Name (Source)**	**Approximation** $\mathrm{erf}(x)$	**Functional Form**	**Inverse** $\mathrm{erf}^{-1}(\mathbf{y})$
Type 1	$\mathrm{erf}_{\text{Chiani-1}}$ [14]	$1-\left(\frac{1}{2}e^{-2x^2}+\frac{1}{2}e^{-x^2}\right)$	$1-2\left(ae^{-b(\sqrt{2}x)^2}+ce^{-2b(\sqrt{2}x)^2}\right)$	$\frac{1}{\sqrt{2}}\sqrt{\frac{1}{b}\ln\left(\frac{a+\sqrt{a^2+2c(1-y)}}{1-y}\right)}$
	$\mathrm{erf}_{\text{Wu-2}}$ [15]	$1-\left(\frac{1}{3}e^{-2x^2}+\frac{1}{3}e^{-x^2}\right)$		
	$\mathrm{erf}_{\text{Powari}}$ [16]	$1-\left(\frac{2}{3}e^{-2x^2}+\frac{1}{6}e^{-x^2}\right)$		
	$\mathrm{erf}_{\text{Olabiyi-2}}$ [17]	$1-2\left(0.15085e^{-0.5255(\sqrt{2}x)^2}+0.21945e^{-1.051(\sqrt{2}x)^2}\right)$		
Type 2	$\mathrm{erf}_{\text{Mastin-1}}$ [18]	$1-e^{-x^2-2\sqrt{\frac{1}{\pi}}x}$	$1-e^{-2ax^2-b\sqrt{2}x}$	$\frac{-b+\sqrt{b^2-4a\ln(1-y)}}{2a\sqrt{2}}$
	$\mathrm{erf}_{\text{Mastin-2}}$ [18]	$1-e^{-\frac{2}{\pi}x^2-2\sqrt{\frac{1}{\pi}}x}$		
	$\mathrm{erf}_{\text{Mastin-3}}$ [18]	$1-e^{-0.748x^2-0.777\sqrt{2}x}$		
	$\mathrm{erf}_{\text{Lin-1}}$ [19]	$1-e^{-0.832x^2-0.717\sqrt{2}x}$		
Type 3	$\mathrm{erf}_{\text{Ordaz}}$ [20]	$1-2\left(0.6931e^{-\left(\frac{9\sqrt{2}x+8}{14}\right)^2}\right)$	$1-2ae^{-\left(\frac{9\sqrt{2}x+8}{14}\right)^2}$	$\frac{2}{9\sqrt{2}}\left(7\sqrt{\ln\left(\frac{2a}{1-y}\right)}-4\right)$
	$\mathrm{erf}_{\text{Hanandeh-2}}$ [21]	$1-2\left(0.688182e^{-\left(\frac{9\sqrt{2}x+8}{14}\right)^2}\right)$		
Type 4	$\mathrm{erf}_{\text{Chernoff}}$ [22]	$1-2e^{-x^2}$	$1-2ae^{-2bx^2}$	$\sqrt{\frac{1}{2b}\ln\left(\frac{2a}{1-y}\right)}$
	$\mathrm{erf}_{\text{Chernoff-impr}}$ [22]	$1-e^{-x^2}$		
	$\mathrm{erf}_{\text{Ermolova-2}}$ [23]	$1-2\left(0.3e^{-1.01x^2}\right)$		
	$\mathrm{erf}_{\text{Gasull}}$ [24]	$1-2\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{11}{5}x^2}\right)$		
	$\mathrm{erf}_{\text{Olabiyi-1}}$ [17]	$1-2\left(0.24015e^{-1.1232x^2}\right)$		
	$\mathrm{erf}_{\text{Wu-1}}$ [15]	$1-2\left(\frac{1}{4}e^{-\frac{4}{\pi}x^2}\right)$		
	$\mathrm{erf}_{\text{Ermolova-1}}$ [23]	$1-2\left(0.28e^{-1.275x^2}\right)$		
	$\mathrm{erf}_{\text{Chang}}$ [25]	$1-2\left(\sqrt{\frac{e}{2\pi}}\frac{\sqrt{1.080-1}}{1.080}e^{-1.080x^2}\right)$		

Class 2: Radical–Exponential Approximations

This Class includes approximations defined by a characteristic square-root–exponential composition that inherently preserves the boundedness and symmetry of the error function. The underlying structure exploits the mathematical connection between Gaussian integrals and radical transformations, providing natural domain constraints while maintaining smooth differentiability over the entire real line. The specific approximations for each type are summarized in Table 4.
Three principal structural patterns can be identified according to the complexity of the inner transformation:

- **Polynomial Inner Function (Type 5):** Incorporating linear or quadratic polynomials within the exponential argument, this subgroup provides balanced accuracy across the domain. The Hamaker approximation exemplifies this design, with polynomial coefficients tuned to control both central behavior and asymptotic decay.
- **Rational Inner Function (Type 6):** Employing rational expressions inside the exponential term, this formulation enhances tail control through improved asymptotic modeling. The Lin-2 approximation demonstrates how rational inner functions can extend accuracy in extreme regions while preserving analytical simplicity.
- **Linear Inner Function (Type 7):** Representing the most streamlined variant, these approximations use simple linear transformations to achieve maximum computational efficiency. The classical Pólya formulation and its modern adaptations illustrate how minimal parameterization can still deliver satisfactory accuracy for broad practical applications.

The radical–exponential framework ensures intrinsic numerical stability and automatic range enforcement, making it particularly suitable for iterative or feedback-driven algorithms where robustness is essential. The progression from linear to rational inner functions clearly demonstrates the trade-off between parametric complexity and refinement of approximation accuracy.

Table 4: Summary of Radical-Exponential Approximations (Class 2)

Type	**Name (Source)**	**Approximation** $\mathrm{erf}(x)$	**Functional Form**	**Inverse** $\mathrm{erf}^{-1}(\mathbf{y})$
Type 5	$\mathrm{erf}_{\text{Hamaker}}$ [26]	$\sqrt{1-e^{-[0.806(\sqrt{2}x)(1-0.018(\sqrt{2}x))]^2}}$	$\sqrt{1-e^{-[a(\sqrt{2}x)(1-b(\sqrt{2}x))]^2}}$	$\frac{a}{\sqrt{2}}\sqrt{-\ln(1-y^2)}\left(1+b\sqrt{-\ln(1-y^2)}\right)$
Type 6	$\mathrm{erf}_{\text{Lin-2}}$ [27]	$\sqrt{1-e^{-\left[\frac{\sqrt{2}x}{1.237+0.0249(\sqrt{2}x)}\right]^2}}$	$\sqrt{1-e^{-\left[\frac{\sqrt{2}x}{a+b(\sqrt{2}x)}\right]^2}}$	$\frac{a}{\sqrt{2}}\frac{\sqrt{-\ln(1-y^2)}}{1-b\sqrt{-\ln(1-y^2)}}$
Type 7	$\mathrm{erf}_{\text{Pólya}}$ [29]	$\sqrt{1-e^{-\frac{4}{\pi}x^2}}$	$\sqrt{1-e^{-2ax^2}}$	$\sqrt{-\frac{1}{2a}\ln(1-y^2)}$
	$\mathrm{erf}_{\text{Boiroju-2}}$ [30]	$\sqrt{1-e^{-1.27x^2}}$		
	$\mathrm{erf}_{\text{Aludaat}}$ [1]	$\sqrt{1-e^{-\sqrt{\frac{\pi}{2}}x^2}}$		
	$\mathrm{erf}_{\text{Eidous}}$ [31]	$\sqrt{1-e^{-\frac{5}{4}x^2}}$		
	$\mathrm{erf}_{\text{Abderrahmane-2}}$ [32] (2016)	$\sqrt{1-e^{-1.24612358x^2}}$		
	$\mathrm{erf}_{\text{Hanandeh-4}}$ [33]	$\sqrt{1-e^{-\frac{81}{65}x^2}}$		

Class 3: Logistic-Based Approximations

This Class exploits the intrinsic similarity between the error function and logistic-type sigmoidal curves, both of which exhibit smooth S-shaped transitions, bounded asymptotes, and symmetry about the origin. The mathematical foundation draws upon the close connection between cumulative distribution functions and logistic transformations, enabling intuitive parameterization under natural range constraints.
Two main analytical approaches can be distinguished within this Class (see Table 5):

- **Standard Logistic Forms (Type 8):** These approximations employ the classical logistic function with variable scaling parameters. The Tocher approximation represents the earliest formulation in this direction, forming the basis for later refinements by Johnson, Bowling, and Hanandeh, each optimizing the steepness parameter for different application contexts.
- **Generalized Fractional Logistic Forms (Type 9):** This extended formulation introduces separate control over the numerator and denominator, creating a three-parameter family that allows fine-tuned modulation of amplitude, shift, and transition behavior. The Abderrahmane-3 approximation exemplifies this structure, achieving improved geometric fidelity while preserving analytical simplicity.

Logistic-based approximations are particularly advantageous in scenarios requiring both computational efficiency and closed-form invertibility, since their inverse functions involve only elementary logarithmic operations. This property makes them well-suited for embedded systems, real-time computations, and pedagogical applications where speed and clarity are equally important. The transition from standard to fractional formulations illustrates the continuous refinement of sigmoidal modeling for accurate and analytically tractable error function approximation.

Table 5: Summary of Logistic-Based Approximations (Class 3)

Type	**Name (Source)**	**Approximation** $\operatorname{erf}(x)$	**Functional Form**	**Inverse** $\operatorname{erf}^{-1}(y)$
Type 8	$\operatorname{erf}_{\text{Tocher}}$ [34]	$1-\dfrac{2}{1+e^{\sqrt{\frac{1}{\pi}}4x}}$	$1-\dfrac{2}{1+e^{a\sqrt{2}x}}$	$\dfrac{1}{a\sqrt{2}}\ln\left(\dfrac{1+y}{1-y}\right)$
	$\operatorname{erf}_{\text{Hanandeh-1}}$ [21]	$1-\dfrac{2}{1+e^{1.7017\sqrt{2}x}}$		
	$\operatorname{erf}_{\text{Bowling-1}}$ [35]	$1-\dfrac{2}{1+e^{1.702\sqrt{2}x}}$		
	$\operatorname{erf}_{\text{Johnson-1}}$ [36]	$1-\dfrac{2}{1+e^{\frac{\pi\sqrt{2}x}{\sqrt{3}}}}$		
Type 9	$\operatorname{erf}_{\text{Abderrahmane-3}}$ [37]	$-1+\dfrac{2\times 0.97186}{0.96628+e^{-1.69075\sqrt{2}x}}$	$-1+\dfrac{2a}{b+e^{-c\sqrt{2}x}}$	$-\dfrac{1}{c\sqrt{2}}\ln\left(\dfrac{2a}{1+y}-b\right)$

Class 4: Power–Rational Approximations

This Class represents an advanced analytical approach to error function approximation, combining power transformations with rational expressions to achieve flexible control over

functional behavior. The mathematical foundation exploits the adaptability of power laws together with the asymptotic precision of rational forms, allowing accurate tuning of both central and tail regions.
Two principal methodological perspectives illustrate the power–rational paradigm (see Table 6):

- **Burr-Type Formulations (Type 10):** Inspired by statistical distribution theory, these approximations embed nested power transformations within rational frameworks. The Burr approximation exemplifies this design through its four-parameter system, enabling independent regulation of scaling, shaping, and tail decay via coordinated parameter interactions.
- **Power–Exponential Transformations (Type 11):** These formulations generalize the classical exponential form by replacing the fixed quadratic exponent with a tunable power parameter. The Hanandeh-3 and Zogheib-2 approximations demonstrate how power-law modulation can yield a continuum of decay behaviors ranging from sub-exponential to super-exponential characteristics.

Power–rational approximations are particularly effective in applications requiring domain-specific control across multiple functional regions, such as extreme-value analysis and risk modeling, where conventional methods may exhibit systematic bias. Their parametric flexibility allows targeted optimization while preserving the analytical transparency essential for statistical inference and computational implementation.

Table 6: Summary of Power-Rational Approximations (Class 4)

Type	Name (Source)	Approximation $\mathrm{erf}(x)$	Functional Form	Inverse $\mathrm{erf}^{-1}(y)$
Type 10	$\mathrm{erf}_{\text{Burr}}$ [28]	$1-2\left[1+(0.6447+0.1620\sqrt{2}x)^{4.874}\right]^{-6.158}$	$1-2\left[1+(a+b\sqrt{2}x)^{c}\right]^{-d}$	$\frac{1}{b\sqrt{2}}\left[\left(\left(\frac{1-y}{2}\right)^{-1/d}-1\right)^{1/c}-a\right]$
Type 11	$\mathrm{erf}_{\text{Hanandeh-3}}$ [21]	$1-e^{-1.2(\sqrt{2}x)^{1.275}}$	$1-e^{-a(\sqrt{2}x)^{b}}$	$\frac{1}{\sqrt{2}}\left(-\frac{1}{a}\ln(1-y)\right)^{1/b}$
	$\mathrm{erf}_{\text{Zogheib-2}}$ [38]	$1-e^{-1.2(\sqrt{2}x)^{1.3}}$		

Class 5: Nested Exponential Approximations

This Class introduces a paradigm shift in error function approximation by employing deeply nested exponential hierarchies, generating complex functional behavior through iterative composition. The mathematical foundation leverages the rapid growth of exponential functions to construct sophisticated decay profiles while maintaining analytical transparency through carefully structured nesting. The specific approximations for each type are summarized in Table 7.
Two primary nesting strategies illustrate this Class:

- **Power–Exponential Nesting (Type 12):** This strategy employs a three-layer hierarchy with power-transformed intermediate levels. The Soranzo-3 approximation utilizes bases 54 and 1.3671 in a carefully calibrated arrangement, where the outer layer establishes the primary decay scale and inner layers fine-tune transition behavior and asymptotic characteristics.

- **Constant–Base Nesting (Type 13):** This approach uses fixed numerical bases in a strategic composition that optimizes computational efficiency. The Soranzo-4 approximation employs bases 22 and 41 with a scaling factor of 10, creating a robust architecture that avoids parameter optimization overhead while delivering high accuracy through numerical synergy.

Nested exponential approximations offer several distinct advantages: hierarchical error distribution across layers, inherent numerical stability due to the well-conditioned exponential operations, and preservation of closed-form invertibility despite functional complexity. These properties make them particularly valuable for high-precision applications in scientific computing, financial modeling, and engineering simulations, where conventional approximations may produce unacceptable tail-region errors.

Table 7: Summary of Nested Exponential Approximations (Class 5)

Type	**Name (Source)**	**Approximation** $\mathrm{erf}(x)$	**Functional Form**	**Inverse** $\mathrm{erf}^{-1}(\mathbf{y})$
Type 12	$\mathrm{erf}_{\text{Soranzo-3}}$ [39]	$1-2\left(1-2^{-54^{1.3671(\sqrt{2}x)}}\right)$	$1-2\left(1-2^{-\alpha^{\beta(\sqrt{2}x)}}\right)$	$\frac{1}{\sqrt{2}\ln\beta}\ln\left(1-\frac{\ln\left(-\ln\left(1-\frac{1-y}{2}\right)/\ln 2\right)}{\ln\alpha}\right)$
Type 13	$\mathrm{erf}_{\text{Soranzo-4}}$ [39]	$1-2\left(1-2^{-22^{1-41^{(\sqrt{2}x)/10}}}\right)$	$1-2\left(1-2^{-\gamma^{1-\delta^{(\sqrt{2}x)/\epsilon}}}\right)$	$\frac{\epsilon}{\sqrt{2}\ln\delta}\ln\left(1-\frac{\ln\left(-\ln\left(1-\frac{1-y}{2}\right)/\ln 2\right)}{\ln\gamma}\right)$

Class 6: Advanced Composite Approximations

This Class encompasses the most sophisticated error function approximations, transcending conventional paradigms through innovative synthesis of diverse functional elements. These hybrid architectures represent the forefront of analytical approximation methodology, combining multiple mathematical concepts into structures with exceptional properties unattainable by single-mechanism approaches. The specific approximations for each type are summarized in Table 8.
The unifying characteristic of this Class lies in the strategic integration of heterogeneous functional operations, where each approximation employs a distinct combination of affine transformations, radical terms, logarithmic scaling, and exponential decay arranged in a coherent analytical framework.
Three primary architectural philosophies illustrate this advanced Class:

- **Additive–Radical Synthesis (Type 14):** Combines affine transformations with radical-exponential operations in an additive framework. The Abderrahmane-1 approximation demonstrates how this combination achieves complementary error cancellation, with each element dominating different regions of the function domain.

- **Logarithmic–Exponential Cascading (Type 15):** Implements a multi-stage pipeline employing softplus activation, logarithmic transformation, power scaling, and Pareto-type tail shaping. The Lipoth approximation showcases how layered functional composition creates smooth transitions between behavioral regimes while maintaining full analytical tractability.

- **Extreme Value Theoretic Framework (Type 16):** Draws inspiration from extreme value distribution theory, particularly the Gumbel distribution, using double-exponential composition with external power modulation. The Kudu approximation naturally embeds extreme value characteristics, making it suitable for tail modeling and rare-event analysis.

Advanced composite approximations represent the culmination of error function approximation theory, illustrating how strategic functional synthesis can overcome fundamental trade-offs among accuracy, complexity, and analytical tractability. Their sophisticated architectures make them especially valuable in quantitative finance, risk analysis, scientific computing, and engineering simulations, where conventional approximations may fall short.

Table 8: Summary of Advanced Composite Approximations (Class 6)

Type	**Name (Source)**	**Approximation** $\mathrm{erf}(x)$	**Functional Form**	**Inverse** $\mathrm{erf}^{-1}(\mathbf{y})$
Type 14	$\mathrm{erf}_{\text{Abderrahmane-1}}$ [37]	$0.49897 - 0.49794\sqrt{1 - e^{-1.25264x^2}}$	$a - b\sqrt{1 - e^{-2cx^2}}$	$\sqrt{-\frac{1}{2c}\ln\left(1 - \left(\frac{a+y-1}{b}\right)^2\right)}$
Type 15	$\mathrm{erf}_{\text{Lipoth}}$ [4]	$1 - 2\left(1 - \left(1 + 0.00162\left[\ln\left(1 + e^{-1.217x+3.26863}\right)\right]^{3.38692}\right)^{-7.80501}\right)$	$1 - 2\left(1 - \left(1 + a\left[\ln\left(1 + e^{-(\sqrt{2}x)/h+c}\right)\right]^{b}\right)^{-d}\right)$	$\frac{h}{\sqrt{2}}\left(c - \ln\left(e^{\left(\frac{\left(\frac{2}{1+y}\right)^{1/d}-1}{a}\right)^{1/b}} - 1\right)\right)$
Type 16	$\mathrm{erf}_{\text{Kundu}}$ [40]	$1 - 2\left(1 - \left(1 - e^{-e^{0.540x+1.07925}}\right)^{12.8}\right)$	$1 - 2\left(1 - \left(1 - e^{-e^{a\sqrt{2}x+b}}\right)^{c}\right)$	$\frac{1}{a\sqrt{2}}\left(\ln\left(-\ln\left(1 - \left(\frac{1+y}{2}\right)^{1/c}\right)\right) - b\right)$

2.3 Metaheuristic Algorithms

In complex optimization problems, especially those with nonlinear objective functions, classical optimization methods often face challenges such as getting trapped in local optima, requiring differentiability, and slow convergence [10, 11]. Under these conditions, metaheuristic algorithms emerge as effective and flexible solutions. Metaheuristic algorithms are a class of intelligent search methods inspired by natural, biological, or collective behaviors, aimed at finding near-optimal solutions in large and complex search spaces [13]. By combining random search and structured rules, these methods can effectively explore the search space and escape local optima. Although there is no guarantee of reaching the global optimum [12], these algorithms tend to find high-quality solutions within a limited number of iterations and demonstrate better performance compared to traditional methods for complex problems. In this context, the Gaussian Combined Arms (GCA) algorithm is a recently developed and effective metaheuristic for high-dimensional, nonlinear, and complex optimization problems [9]. Proposed in 2025 by Etesami et al., GCA is inspired by combined-arms military strategies and uses two coordinated groups of search agents, mimicking ground and air forces, to jointly perform intensive local refinement and wide-range global exploration. By employing Gaussian-based update rules, the algorithm adaptively balances exploration and exploitation, which makes it a powerful tool for challenging engineering optimization tasks. The main steps of the GCA algorithm are summarized below.

2.4 GCA Algorithm Steps

Step 1: Population Initialization

The GCA algorithm operates in a d-dimensional search space using two groups of agents: ground forces (G_F) and air forces (A_F). To ensure sufficient diversity and exploration capability, the total number of agents N is chosen such that $N > 2d$.
More precisely:

- The number of air-force agents is set to the problem dimension, i.e., $|A_F| = d$.
- The remaining $n = N - d$ agents are assigned to the ground forces, i.e., $|G_F| = n$.

The initial population is represented by two matrices:

$$A_F^{(1)} = \begin{bmatrix} x_{11}^{(1)} & x_{12}^{(1)} & \dots & x_{1d}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} & \dots & x_{2d}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{d1}^{(1)} & x_{d2}^{(1)} & \dots & x_{dd}^{(1)} \end{bmatrix}_{d\times d}, \quad G_F^{(1)} = \begin{bmatrix} x_{(d+1),1}^{(1)} & \dots & x_{(d+1),d}^{(1)} \\ \vdots & \ddots & \vdots \\ x_{N1}^{(1)} & \dots & x_{Nd}^{(1)} \end{bmatrix}_{n\times d}.$$

The initial positions are randomly generated within the predefined search bounds:

$$x_{ij}^{(1)} = lb_j + \text{rand} \times (ub_j - lb_j),$$

where lb_j and ub_j are the lower and upper bounds of the j-th dimension, respectively, and rand is a uniformly distributed random number in $[0, 1]$.

Step 2: Objective Function Evaluation

At iteration t, the complete population matrix is given by

$$X^{(t)} = \begin{bmatrix} G_F^{(t)} \\ A_F^{(t)} \end{bmatrix}_{N\times d}, \quad t = 1, 2, \dots, T,$$

where T denotes the total number of iterations.
Each individual $X_i^{(t)}$ represents a candidate solution, whose fitness is evaluated by the objective function $f(X_i^{(t)})$. For a minimization problem, the global best position at iteration t, denoted by $G_{\text{best}}^{(t)}$, is determined as

$$G_{\text{best}}^{(t)} = \arg \min_{1\le i\le N} f(X_i^{(t)}).$$

The corresponding position guides the search process in subsequent iterations.

Step 3: Position Update

The positions of both air and ground forces are updated using normal (Gaussian) distributions, where $X \sim \mathcal{N}(\mu, \sigma)$ denotes a normal distribution with mean μ and standard deviation σ.

Air Forces Update. For the i-th air-force agent and the j-th dimension, the position at iteration $t+1$ is sampled as

$$x_{ij}^{(t+1)} \sim \mathcal{N}\big(\mu_{ij}^{(t+1)}, \sigma_{ij}^{(t+1)}\big), \quad i = 1, \dots, d,\ j = 1, \dots, d,$$

with

$$\mu_{ij}^{(t+1)} = \lambda_1 x_{ij}^{(t)} + (1 - \lambda_1) g_j^{(t)}, \qquad \sigma_{ij}^{(t+1)} = \lambda_2 \big| x_{ij}^{(t)} - g_j^{(t)} \big|,$$

where $0 \leq \lambda_1 \leq 1$ and $\lambda_2 > 0$. The parameters λ_1 and λ_2 control the balance between local exploitation around the current position and global exploration around the best-known solution.

Ground Forces Update. For the i-th ground-force agent and the j-th dimension, the position at iteration $t+1$ is updated as

$$x_{ij}^{(t+1)} \sim \mathcal{N}\big(\mu_{ij}^{(t+1)}, \sigma_{ij}^{(t+1)}\big), \quad i = d+1, \dots, N,\ j = 1, \dots, d,$$

with

$$\mu_{ij}^{(t+1)} = \omega_1 x_{ij}^{(t)} + (1 - \omega_1) g_j^{(t)}, \qquad \sigma_{ij}^{(t+1)} = \omega_2 \big| x_{ij}^{(t)} - g_j^{(t)} \big|,$$

where $0 \leq \omega_1 \leq 1$ and $\omega_2 > 0$. Here, ω_1 shapes the convergence behavior of the ground forces toward the global best, while ω_2 regulates the spread of the search and thus the degree of exploration.

3 Research Method

This research develops a systematic framework to enhance the accuracy of existing analytical approximations for the standard error function, $\mathrm{erf}(x)$, while preserving their original mathematical structure. The proposed approach treats the numerical constants in each approximation as optimizable parameters and formulates the refinement as a global optimization problem. By applying a Gaussian Combined Arms Algorithm, the method identifies parameter values that minimize both the mean and maximum absolute errors, ensuring reliable performance across critical regions of the function domain. The overall procedure is organized into four steps: selection of approximations, parameterization, formulation as an optimization problem, and extraction of optimized parameters.

Step 1: Selection and Parameterization of Approximations

Following the introduction of the approximations, their different structures, and corresponding formulas in the previous section, for the initiation of the optimization process, the Functional Formulas of each approximation type were selected and treated parametrically. In this step, the numerical constants in the original formulas were converted into variable parameters $\theta_1, \theta_2, \ldots, \theta_n$, transforming each model into a parametric function that allows the determination of optimal values without altering the overall structure of the approximation. This approach preserves both the simplicity and clarity of the models while providing the necessary flexibility to enhance the accuracy of the approximations. The parametric formulas of each approximation are presented in the table below.

Table 9: Parameterized Analytical Approximations of erf(x)

Types of erf(x)	Parametric Functional Forms	Parameters
Type 1	$\theta_1 e^{-\theta_2 x^2} + \theta_3 e^{-2\theta_2 x^2}$	$\theta_1, \theta_2, \theta_3$
Type 2	$\frac{1}{2} e^{-\theta_1 x^2 - \theta_2 x}$	θ_1, θ_2
Type 3	$\theta_1 e^{-((9x+8)/14)^2}$	θ_1
Type 4	$\theta_1 e^{-\theta_2 x^2}$	θ_1, θ_2
Type 5	$\frac{1}{2} - \frac{1}{2}\sqrt{1 - e^{-(\theta_1 x(1-\theta_2 x))^2}}$	θ_1, θ_2
Type 6	$\frac{1}{2} - \frac{1}{2}\sqrt{1 - e^{-(x/(\theta_1+\theta_2 x))^2}}$	θ_1, θ_2
Type 7	$\frac{1}{2} - \frac{1}{2}\sqrt{1 - e^{-\theta_1 x^2}}$	θ_1
Type 8	$\frac{1}{1+e^{-\theta_1 x}}$	θ_1
Type 9	$1 - \frac{\theta_1}{\theta_2 + e^{-\theta_3 x}}$	$\theta_1, \theta_2, \theta_3$
Type 10	$(1 + (\theta_1 + \theta_2 x)^{\theta_3})^{-\theta_4}$	$\theta_1, \theta_2, \theta_3, \theta_4$
Type 11	$\frac{1}{2} e^{-\theta_1 x^{\theta_2}}$	θ_1, θ_2
Type 12	$1 - 2^{-\theta_1^{1-\theta_2 x}}$	θ_1, θ_2
Type 13	$1 - 2^{-\theta_1^{1-41^{x/10}}}$	θ_1, θ_2
Type 14	$\theta_1 - \theta_2\sqrt{1 - e^{-\theta_3 x^2}}$	$\theta_1, \theta_2, \theta_3$
Type 15	$1 - (1 + \theta_1(\ln(1 + e^{-x/\theta_5 + \theta_3}))^{\theta_2})^{-\theta_4}$	$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$
Type 16	$1 - (1 - e^{-e^{\theta_1 x + \theta_2}})^{\theta_3}$	$\theta_1, \theta_2, \theta_3$

Step 2: Formulation of the Optimization Problem

The task of improving analytical approximations of the standard error function erf(x) was formulated as a global optimization problem. The objective is to find the optimal parameter vector

$$\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_n]^T$$

that globally minimizes the error between the parameterized approximation $\mathrm{erf}_{\mathrm{approx}}(x;\boldsymbol{\theta})$ and the exact function $\mathrm{erf}(x)$ over a specified domain $D \subset \mathbb{R}$. The optimization problem is defined using two primary accuracy criteria: Maximum Absolute Error (Max-AE) and Mean Absolute Error (MAE), as follows:

$$\min_{\boldsymbol{\theta}} \max_{x\in D} \left|\mathrm{erf}(x) - \mathrm{erf}_{\mathrm{approx}}(x;\boldsymbol{\theta})\right|, \tag{2}$$

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{i=1}^{N} \left|\mathrm{erf}(x_i) - \mathrm{erf}_{\mathrm{approx}}(x_i;\boldsymbol{\theta})\right|, \tag{3}$$

where the first objective corresponds to Max-AE and the second to MAE. To ensure balanced performance, a combined objective function was employed:

$$\hat{\boldsymbol{\theta}}_i = \arg\min_{\boldsymbol{\theta}} \left(\frac{1}{n}\sum_{j=1}^{n} \left|\Phi(z_j) - \hat{\Phi}_i(z_j;\boldsymbol{\theta})\right| + \max_{j} \left|\Phi(z_j) - \hat{\Phi}_i(z_j;\boldsymbol{\theta})\right| \right), \tag{4}$$

where $\hat{\boldsymbol{\theta}}_i$ is the optimized parameter vector for the i-th approximation, $\Phi(z_j)$ is the exact cumulative distribution function value at z_j, and $\hat{\Phi}_i(z_j;\boldsymbol{\theta})$ is the parameterized approximation. The first term corresponds to MAE and the second term to Max-AE. This combination prevents excessive errors at critical points while preserving average accuracy.

Evaluation Metrics

For the quantitative evaluation of the performance of optimized approximations, standard absolute error metrics are used. The two main metrics employed are the Maximum Absolute Error (Max-AE) and the Mean Absolute Error (MAE), which are defined as follows [6]:

$$\text{Max-AE} = \max_{x\in D} \left|\mathrm{erf}_{\mathrm{exact}}(x) - \mathrm{erf}_{\mathrm{approx}}(x)\right|, \tag{5}$$

$$\text{MAE} = \frac{1}{N}\sum_{i=1}^{N} \left|\mathrm{erf}_{\mathrm{exact}}(x_i) - \mathrm{erf}_{\mathrm{approx}}(x_i)\right|, \tag{6}$$

where D is the desired domain for evaluation (for N=5001 points in the interval $D =$ 0 to $D = 5$ with a step of 0.001) and $\mathrm{erf}_{\mathrm{exact}}(x)$ and $\mathrm{erf}_{\mathrm{approx}}(x)$ denote the exact and approximate values of the error function at point x, respectively. The Max-AE metric represents the maximum deviation between the approximation and the true value and is typically used to guarantee a global error upper bound, whereas the MAE evaluates the mean absolute error over the entire domain and represents the overall performance of the approximation [41, 42].

4 Results

Following the optimization procedure described previously, the optimal parameter sets were obtained for sixteen analytical approximations of the error function. Table 10 lists the corresponding optimized coefficients, and Table 11 summarizes the numerical performance of both the original and optimized forms in terms of Maximum Absolute Error (Max-AE) and Mean Absolute Error (MAE). As observed in Table 11, the optimized approximations exhibit distinct performance improvements across the different function types. For the majority of the models, both Max-AE and MAE values were notably reduced after optimization, confirming enhanced numerical accuracy. The improvement is particularly significant for several families such as Type 2 (Mastin-family), Type 4 (Chernoff-based), Type 6 (Lin-2), Type 7 (Polya-family), and Type 10 (Burr), where optimization led to considerable reductions in both maximum and mean absolute errors. In contrast, for certain cases such as Type 9 (Abderrahmane-3), Type 11 (Hanandeh-3/Zogheib-2), and Type 13 (Soranzo-4), the optimization yielded only marginal or negligible improvements, indicating that the original formulations were already highly accurate. Similarly, for Type 14 and Type 15, the optimized results differ only slightly from the original, reflecting near-optimal initial parameterization. Overall, the optimized functions preserve the analytical simplicity of the original expressions while delivering improved accuracy profiles in most cases, as reflected by the reduced Max-AE and MAE metrics presented in Table 11.

5 Conclusions and Future Directions

In this paper, we presented a systematic framework for improving analytical approximations of the standard error function $\operatorname{erf}(x)$ by globally optimizing their parameters using the Gaussian Combined Arms Algorithm (GCA). Considering a broad set of existing closed-form formulas from 16 model families, we treated their numerical coefficients as decision variables and formulated a unified optimization problem combining the mean absolute error (MAE) and the maximum absolute error (Max-AE) over selected domains. This design allowed us to enhance both average and worst-case accuracy while preserving the original analytical structure of each approximation. The numerical results show that the GCA-based optimization yields substantial accuracy improvements for most families, with MAE and Max-AE often reduced by up to an order of magnitude, without increasing structural complexity or computational cost. Notably, several exponential and rational–exponential types benefit from significantly tighter fits to the true error function, whereas already highly accurate formulas exhibit only marginal gains, confirming their near-optimality under the chosen criteria. A further contribution is the identification of structurally simple, high-accuracy approximations for $\operatorname{erf}(x)$ whose analytical inverses are also available, enabling efficient forward and inverse evaluations in practical applications. The proposed approach is general and can be extended to other Gaussian-related and special functions, as well as to multi-objective formulations that jointly balance accuracy and computational efficiency. These findings highlight GCA-based global optimization as

Table 10: Optimized parameters for the 16 approximations of $\mathrm{erf}(x)$

Type of $Q(x)$	**Optimized Parameters**
Type 1	$\theta_1 = 0.1796414$, $\theta_2 = 0.5963829$, $\theta_3 = 0.2223313$
Type 2	$\theta_1 = 0.3818009$, $\theta_2 = 0.7701928$
Type 3	$\theta_1 = 0.6878569$
Type 4	$\theta_1 = 0.3969323$, $\theta_2 = 0.8547055$
Type 5	$\theta_1 = 0.3750000$, $\theta_2 = 0.7768232$
Type 6	$\theta_1 = 1.24228965$, $\theta_2 = 0.02100437$
Type 7	$\theta_1 = 0.623077$
Type 8	$\theta_1 = 1.701698$
Type 9	$\theta_1 = 0.97185$, $\theta_2 = 0.96628$, $\theta_3 = 1.69074$
Type 10	$\theta_1 = 0.6538930$, $\theta_2 = 0.1528689$, $\theta_3 = 5.0718128$, $\theta_4 = 6.3500000$
Type 11	$\theta_1 = 1.1900$, $\theta_2 = 1.2755$
Type 12	$\theta_1 = 10.000000$, $\theta_2 = 2.603895$
Type 13	$\theta_1 = 21.97274$, $\theta_2 = 41.02409$
Type 14	$\theta_1 = 0.4979669$, $\theta_2 = 0.4974308$, $\theta_3 = 0.6226144$
Type 15	$\theta_1 = 0.00161983$, $\theta_2 = 3.38628287$, $\theta_3 = 3.26842910$, $\theta_4 = 7.80450553$, $\theta_5 = 0.82112000$
Type 16	$\theta_1 = 0.3865424$, $\theta_2 = 1.0647756$, $\theta_3 = 12.2569109$

an effective and versatile tool for refining closed-form approximations in applied mathematics, statistics, and engineering.

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Table 11: Performance metrics of original and optimized $\mathrm{erf}(x)$ approximations

Type	Function	erf_i Max. AE	erf_i MAE	$(\hat{\mathrm{erf}}_i)$ Max. AE	$(\hat{\mathrm{erf}}_i)$ MAE
Type 1	$\mathrm{erf}_{\text{Chiani-1}}$	1.089857×10^{-1}	3.39817×10^{-2}	9.80274×10^{-2}	7.2769×10^{-3}
	$\mathrm{erf}_{\text{Wu-2}}$	1.089857×10^{-1}	3.39817×10^{-2}		
	$\mathrm{erf}_{\text{Powari}}$	8.33333×10^{-2}	8.3793×10^{-3}		
	$\mathrm{erf}_{\text{Olabiyi-2}}$	1.297000×10^{-1}	8.3847×10^{-3}		
Type 2	$\mathrm{erf}_{\text{Mastin-1}}$	2.22023×10^{-2}	8.2099×10^{-3}	9.391×10^{-4}	4.418×10^{-4}
	$\mathrm{erf}_{\text{Mastin-2}}$	7.0100×10^{-3}	2.9787×10^{-3}		
	$\mathrm{erf}_{\text{Mastin-3}}$	9.419×10^{-4}	4.530×10^{-4}		
	$\mathrm{erf}_{\text{Lin-1}}$	6.5853×10^{-3}	1.3422×10^{-3}		
Type 3	$\mathrm{erf}_{\text{Ordaz}}$	4.3991×10^{-3}	9.645×10^{-4}	3.7647×10^{-3}	7.206×10^{-4}
	$\mathrm{erf}_{\text{Hanandeh-2}}$	3.5302×10^{-3}	7.218×10^{-4}		
Type 4	$\mathrm{erf}_{\text{Chernoff}}$	5.785386×10^{-1}	2.135758×10^{-1}	1.030677×10^{-1}	9.5939×10^{-3}
	$\mathrm{erf}_{\text{Chernoff-impr}}$	1.512199×10^{-1}	5.69191×10^{-2}		
	$\mathrm{erf}_{\text{Ermolova-2}}$	2.000000×10^{-1}	2.49864×10^{-2}		
	$\mathrm{erf}_{\text{Gasull}}$	1.010577×10^{-1}	1.54597×10^{-2}		
	$\mathrm{erf}_{\text{Olabiyi-1}}$	2.598500×10^{-1}	2.99921×10^{-2}		
	$\mathrm{erf}_{\text{Wu-1}}$	2.500000×10^{-1}	3.03165×10^{-2}		
	$\mathrm{erf}_{\text{Ermolova-1}}$	2.200000×10^{-1}	2.20397×10^{-2}		
	$\mathrm{erf}_{\text{Chang}}$	1.733898×10^{-1}	6.34209×10^{-2}		
Type 5	$\mathrm{erf}_{\text{Hamaker}}$	1.911988×10^{-1}	5.19135×10^{-2}	1.807550×10^{-1}	4.79819×10^{-2}
Type 6	$\mathrm{erf}_{\text{Lin-2}}$	9.302×10^{-4}	1.983×10^{-4}	5.119×10^{-4}	1.480×10^{-4}
Type 7	$\mathrm{erf}_{\text{Polya}}$	3.1458×10^{-3}	1.2641×10^{-3}	1.627×10^{-3}	8.325×10^{-4}
	$\mathrm{erf}_{\text{Boiroju-2}}$	2.9355×10^{-3}	1.1491×10^{-3}		
	$\mathrm{erf}_{\text{Aludaat}}$	1.9732×10^{-3}	8.425×10^{-4}		
	$\mathrm{erf}_{\text{Eidous}}$	1.8079×10^{-3}	8.300×10^{-4}		
	$\mathrm{erf}_{\text{Abderrahmane-2}}$	1.6256×10^{-3}	8.326×10^{-4}		
	$\mathrm{erf}_{\text{Hanandeh-4}}$	1.6270×10^{-3}	8.325×10^{-4}		
Type 8	$\mathrm{erf}_{\text{Tocher}}$ [32]	1.76712×10^{-2}	8.5929×10^{-3}	9.4601×10^{-3}	5.5415×10^{-3}
	$\mathrm{erf}_{\text{Hanandeh-1}}$ [33]	9.4600×10^{-3}	5.5415×10^{-3}		
	$\mathrm{erf}_{\text{Bowling-1}}$ [95]	9.4863×10^{-3}	5.5414×10^{-3}		
	$\mathrm{erf}_{\text{Johnson-1}}$ [96]	2.26628×10^{-2}	7.3114×10^{-3}		
Type 9	$\mathrm{erf}_{\text{Abderrahmane-3}}$	5.7393×10^{-3}	3.3280×10^{-3}	5.7499×10^{-3}	3.3276×10^{-3}
Type 10	$\mathrm{erf}_{\text{Burr}}$	3.9724×10^{-3}	1.0246×10^{-3}	1.7505×10^{-3}	3.367×10^{-4}
Type 11	$\mathrm{erf}_{\text{Hanandeh-3}}$ [33]	9.1654×10^{-3}	3.8181×10^{-3}	9.514×10^{-3}	3.7356×10^{-3}
	$\mathrm{erf}_{\text{Zogheib-2}}$ [82]	1.12029×10^{-2}	3.5195×10^{-3}		
Type 12	$\mathrm{erf}_{\text{Soranzo-3}}$ [22]	6.408835×10^{-1}	1.203785×10^{-1}	5.183275×10^{-1}	7.74961×10^{-2}
Type 13	$\mathrm{erf}_{\text{Soranzo-4}}$ [22]	1.274×10^{-4}	6.62×10^{-5}	1.249×10^{-4}	6.42×10^{-5}
Type 14	$\mathrm{erf}_{\text{Abderrahmane-1}}$	1.0300×10^{-3}	6.641×10^{-4}	1.0923×10^{-3}	6.544×10^{-4}
Type 15	$\mathrm{erf}_{\text{Lipoth}}$ [30]	2.72×10^{-5}	1.43×10^{-5}	2.43×10^{-5}	1.38×10^{-5}
Type 16	$\mathrm{erf}_{\text{Kundu}}$ [99]	3.152×10^{-4}	1.311×10^{-4}	2.309×10^{-4}	1.033×10^{-4}

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