Group \(\{1, -1, i, -i\}\) Cordial Labeling of sum of \(C_n\) and \(K_m\) for some \(m\)

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**ABSTRACT**

Let \(G\) be a \((p,q)\) graph and \(A\) be a group. We denote the order of an element \(a \in A\) by \(o(a)\). Let \(f : V(G) \rightarrow A\) be a function. For each edge \(uv\) assign the label 1 if \((o(f(u)), o(f(v))) = 1\) or 0 otherwise. \(f\) is called a group \(A\) Cordial labeling if \(|v_f(a) - v_f(b)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\), where \(v_f(x)\) and \(e_f(n)\) respectively denote the number of vertices labelled with an element \(x\) and number of edges labelled with \(n (n = 0, 1)\). A graph which admits a group \(A\) Cordial labeling is called a group \(A\) Cordial graph. In this paper we define group \(\{1, -1, i, -i\}\) Cordial graphs and characterize the graphs \(C_n + K_m (2 \leq m \leq 5)\) that are group \(\{1, -1, i, -i\}\) Cordial.

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1 Introduction

Graphs considered here are finite, undirected and simple. Let $A$ be a group. The order of $a \in A$ is the least positive integer $n$ such that $a^n = e$. We denote the order of $a$ by $o(a)$. Cahit [3] introduced the concept of Cordial labeling. Motivated by this, we defined group $A$ cordial labeling and investigated some of its properties. We also defined group $\{1, -1, i, -i\}$ cordial labeling and discussed that labeling for some standard graphs [1]. In this paper we characterize $C_n + K_2, C_n + K_3, C_n + K_4$ and $C_n + K_5$ that are group $\{1, -1, i, -i\}$ Cordial. Terms not defined here are used in the sense of Harary[5] and Gallian [4].

The greatest common divisor of two integers $m$ and $n$ is denoted by $(m, n)$ and $m$ and $n$ are said to be relatively prime if $(m, n) = 1$. For any real number $x$, we denote by $\lfloor x \rfloor$, the greatest integer smaller than or equal to $x$ and by $\lceil x \rceil$, we mean the smallest integer greater than or equal to $x$.

Given two graphs $G$ and $H$, $G + H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv / u \in V(G), v \in V(H)\}$. We need the following theorem.

Theorem 1.1 [1]
The Complete graph $K_n$ is group $\{1, -1, i, -i\}$ Cordial iff $n \in \{1, 2, 3, 4, 7, 14, 21\}$.

Theorem 1.2 [2]
The Wheel $W_n$ is group $\{1, -1, i, -i\}$ Cordial iff $3 \leq n \leq 6$.

2 Group $\{1, -1, i, -i\}$ Cordial labeling of sum of $C_n$ and $K_m$

Definition 1. Let $G$ be a (p,q)graph and consider the group $A = \{1, -1, i, -i\}$ with multiplication. Let $f : V(G) \rightarrow A$ be a function. For each edge $uv$ assign the label 1 if $(o(f(u)), o(f(v))) = 1$ or 0 otherwise. $f$ is called a group $\{1, -1, i, -i\}$ Cordial labeling if $|v_f(a) - v_f(b)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(x)$ and $e_f(n)$ respectively denote the number of vertices labelled with an element $x$ and number of edges labelled with $n(n = 0, 1)$. A graph which admits a group $\{1, -1, i, -i\}$ Cordial labeling is called a group $\{1, -1, i, -i\}$ Cordial graph.

Example 2. A simple example of a group $\{1, -1, i, -i\}$ Cordial graph is given in Fig. 2.1.
We now investigate the group \( \{1, -1, i, -i\} \) Cordial labeling of \( C_n + K_m \) for \( 1 \leq m \leq 5 \). \( C_n + K_1 \) is the Wheel and theorem 1.2 characterizes the Wheels that are group \( \{1, -1, i, -i\} \) cordial.

**Theorem 3.** \( C_n + K_2 \) is group \( \{1, -1, i, -i\} \) cordial iff \( n \neq 3, 9 \).

**Proof.** Let the vertices of \( C_n \) be labelled as \( u_1, u_2, ... , u_n \) and let the vertices of \( K_2 \) be labelled as \( v_1, v_2 \). Number of vertices of \( C_n + K_2 \) is \( n + 2 \) and number of edges is \( 3n + 1 \). If \( n=3 \) , \( C_3 + K_2 \approx K_5 \) and by Theorem 1.1, \( K_5 \) is not group \( \{1, -1, i, -i\} \) Cordial. If \( n=9 \) , \( C_9 + K_2 \) has 11 vertices and 28 edges. There is no choice of 2 or 3 vertices so that 14 edges get label 1. So, \( C_9 + K_2 \) is not group \( \{1, -1, i, -i\} \) Cordial. Thus, \( n \neq 3, 9 \).

**Case (1):** \( n + 2 \equiv 0(\text{mod} \ 4) \).
Let \( n + 2 = 4k (k \in \mathbb{Z}, k \geq 2) \). Now each vertex label should appear \( k \) times. As number of edges is \( 12k - 5 \), one edge label appears \( 6k - 3 \) times and another \( 6k - 2 \) times. Label the vertices \( v_1, u_1, u_2, ..., u_{k-1} \) by 1. Label the remaining vertices arbitrarily so that \( k \) of them get label \(-1\), \( k \) of them get label \( i \) and \( k \) of them get label \(-i\).

**Case (2):** \( n + 2 \equiv 1(\text{mod} \ 4) \).
Let \( n + 2 = 4k + 1 (k \in \mathbb{Z}, k \geq 2) \). Now one vertex label should appear \( k + 1 \) times and each of the other three labels should appear \( k \) times. Number of edges = \( 3n + 1 = 3(4k - 1) + 1 = 12k - 2 \) and so each edge label appears \( 6k - 1 \) times . Label the vertices \( v_1, u_1, u_2, ..., u_{k-1} \) by 1. Label the remaining vertices arbitrarily so that \( k \) of them get label \(-1\), \( k \) of them get label \( i \) and \( k + 1 \) of them get label \(-i\).

**Case (3):** \( n + 2 \equiv 2(\text{mod} \ 4) \).
Let \( n + 2 = 4k + 2 (k \in \mathbb{Z}, k \geq 1) \). When \( k=1 \), a group \( \{1, -1, i, -i\} \) Cordial labeling of \( C_4 + K_2 \) is given in Table 1. Suppose \( k \geq 2 \). Now 2 vertex labels appear \( k + 1 \) times and 2 vertex labels appear \( k \) times. Number of edges = \( 12k + 1 \). So one edge label appears \( 6k \) times and another \( 6k + 1 \) times \( (k \geq 2) \). Label the vertices \( v_1, u_1, u_2, ..., u_{k-1} \) by 1. Label the remaining vertices arbitrarily so that \( k \) vertices get label 1, \( k + 1 \) vertices get label \( i \) and \( k + 1 \) vertices get label \(-i\).

**Case (4):** \( n + 2 \equiv 3(\text{mod} \ 4) \).
Let $n + 2 = 4k + 3$. If $k = 1$, a group $\{1, -1, i, -i\}$ Cordial labeling of $C_5 + K_2$ is given in Table 1. $k = 2$, is impossible by assumption. Suppose $k \geq 3$. In this case, 3 vertex labels appear $k + 1$ times and 1 vertex label appears $k$ times. Number of edges $= 3(4k + 1) + 1 = 12k + 4$ and so each edge label should appear $6k + 2$ times. Label the vertices $v_1, u_1, u_3, u_4, \ldots, u_k (k \geq 3)$ with label 1. Label the other vertices arbitrarily so that $k + 1$ vertices get label $-1$, $k + 1$ vertices get label $i$ and $k + 1$ vertices get label $-i$. That $C_n + K_2$ is group $\{1, -1, i, -i\}$ Cordial iff $n \neq 3$.

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<tr>
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<th>$v_f(-1)$</th>
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<th>$v_f(-i)$</th>
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<td>$k$</td>
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<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$6k - 1$</td>
<td>$6k - 1$</td>
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<td>$n + 2 \equiv 3 \pmod{4}$</td>
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<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$6k + 2$</td>
<td>$6k + 2$</td>
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</table>

Table 2

**Theorem 4.** $C_n + K_3$ is group $\{1, -1, i, -i\}$ Cordial iff $n \neq 3$.

**Proof.** Let the vertices of $C_n$ be labelled as $u_1, u_2, \ldots, u_n$ and let the vertices of $K_3$ be labelled as $v_1, v_2, v_3$. Number of vertices of $C_n + K_3$ is $n + 3$ and number of edges is $4n + 3$.

If $n = 3$, $C_3 + K_3 \simeq K_6$ which is not group $\{1, -1, i, -i\}$ Cordial by Theorem 1.1. Conversely, assume $n \neq 3$. We need to prove that $C_n + K_3$ is group $\{1, -1, i, -i\}$ Cordial.

**Case(1):** $n + 3 \equiv 0 \pmod{4}$.
Let $n + 3 = 4k (k \in \mathbb{Z}, k \geq 2)$. Now each vertex label should appear $k$ times. Number of edges $= 4(4k - 3) + 3 = 16k - 9$ and so one edge label appears $8k - 4$ times and another $8k - 5$ times. Label the vertices $v_1, u_1, u_3, u_5, \ldots, u_{2k-3}$ by 1. Label the remaining vertices arbitrarily so that $k$ of them get label $-1$, $k$ of them get label $i$ and $k$ of them get label $-i$. Number of edges with label 1 $= n + 2 + (k - 1)4 = 8k - 5$.

**Case(2):** $n + 3 \equiv 1 \pmod{4}$.
Let $n + 3 = 4k + 1 (k \in \mathbb{Z}, k \geq 2)$. If $k = 2$, a group $\{1, -1, i, -i\}$ Cordial labeling of $C_6 + K_3$ is given in Table 3. Suppose $k \geq 3$. Now one vertex label should appear $k + 1$ times and each of the other three labels should appear $k$ times. Number of edges is $16k - 5$.

Label the vertices $v_1, u_1, u_3, \ldots, u_{2k-5}, u_{2k-4}, u_{2k-3} (k \geq 3)$ by 1. Label the remaining vertices arbitrarily so that $k$ of them get label $-1$, $k$ of them get label $i$ and $k$ of them
get label $-i$. Number of edges with label 1 = $n + 2 + (k - 2)4 + 2 \times 3 = 8k - 2$.

Case(3): $n + 3 \equiv 2(\mod 4)$. Let $n + 3 = 4k + 2(k \in \mathbb{Z}, k \geq 2)$. Now 2 vertex labels appear $k + 1$ times and 2 vertex labels appear $k$ times. Number of edges $= 16k - 1$. So one edge label appears $8k$ times and another $8k - 1$ times. Label the vertices $v_1, u_1, u_3, \ldots, u_{2k-5}, u_{2k-3}, u_{2k-2}$ by 1. Label the remaining vertices arbitrarily so that $k + 1$ vertices get label $-1$, $k$ vertices get label $i$ and $k$ vertices get label $-i$. Number of edges with label 1 = $n + 2 + (2k - 2)2 + 3 = 8k$.

Case (4): $n + 3 \equiv 3(\mod 4)$. Let $n + 3 = 4k + 3(k \geq 1)$. If $k = 1$, $n = 4$. A group $\{1, -1, i, -i\}$ Cordial labeling of $C_4 + K_3$ is given in Table 3. Suppose $k \geq 2$. In this case, 3 vertex labels appear $k + 1$ times and 1 vertex label appears $k$ times. Label the vertices $v_1, u_1, u_3, u_5, \ldots, u_{2k-1}$ with label 1. Label the other vertices arbitrarily so that $k + 1$ vertices get label $-1$, $k + 1$ vertices get label $i$ and $k$ vertices get label $-i$. Number of edges with label 1 = $8k + 2$. That $C_n + K_2$ is group $\{1, -1, i, -i\}$ Cordial for $n \neq 3$ follows from Table 4.

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<th>$n$</th>
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<th>$v_3$</th>
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<th>$u_2$</th>
<th>$u_3$</th>
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</tr>
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Table 3

<table>
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<th>$v_f(1)$</th>
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<th>$v_f(i)$</th>
<th>$v_f(-i)$</th>
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</thead>
<tbody>
<tr>
<td>$n + 3 \equiv 0(\mod 4)$</td>
<td>$k$</td>
<td>$k$</td>
<td>$k$</td>
<td>$k$</td>
<td>$8k - 4$</td>
<td>$8k - 5$</td>
</tr>
<tr>
<td>$n + 3 \equiv 1(\mod 4)$</td>
<td>$k + 1$</td>
<td>$k$</td>
<td>$k$</td>
<td>$k$</td>
<td>$8k - 3$</td>
<td>$8k - 2$</td>
</tr>
<tr>
<td>$n + 3 \equiv 2(\mod 4)$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k$</td>
<td>$k$</td>
<td>$8k - 1$</td>
<td>$8k$</td>
</tr>
<tr>
<td>$n + 3 \equiv 3(\mod 4)$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k$</td>
<td>$8k + 1$</td>
<td>$8k + 2$</td>
</tr>
</tbody>
</table>

Table 4

**Theorem 5.** $C_n + K_4$ is group $\{1, -1, i, -i\}$ Cordial iff $n \in \{3, 4, 5, 6, 7, 9, 10, 11, 13, 17\}$.

**Proof.** Let the vertices of $C_n$ be labelled as $u_1, u_2, \ldots, u_n$ and let the vertices of $K_4$ be labelled as $v_1, v_2, v_3, v_4$. Number of vertices of $C_n + K_4$ is $n + 4$. Number of edges is $5n + 6$.

**Case(1):** $n + 4 \equiv 0(\mod 4)$. Let $n + 4 = 4k (k \geq 2)$. If 3 $v_i$’s are given label 1, we get $(n + 3) + (n + 2) + (n + 1) = 3n + 6 = 12k - 6$ edges with label 1. But we need only $10k - 7$ edges with label 1. So at most 2 $v_i$’s are given label 1.

**Subcase(1):** 2 $v_i$’s are given label 1.

If $k=2$, a group $\{1, -1, i, -i\}$ Cordial labeling of $C_4 + K_4$ is given in Table 5. Suppose $k \geq 3$. Minimum number of edges that can get label 1 using $k$ vertices is $(8k -
3) + 4 + (k - 3)3 = 11k - 8. So, the necessary condition is, 10k - 7 ≥ 11k - 8 and so
k ≤ 1 which is a contradiction.

Subcase(ii): One \( v_i \) is given label 1.
Maximum number of edges that can get label 1 now is \((4k - 1) + (k - 1)5 = 9k - 6\). To
get a group \{1, -1, i, -i\} Cordial labeling we need to have \(9k - 6 \geq 10k - 7\) i.e. \(k ≤ 1\),
which is impossible.

Subcase(iii): No \( v_i \) is given label 1.
We need to have \(k.6 \geq 10k - 7 \Rightarrow 4k \leq 7\) which is a contradiction. Thus in case 1, we
get \(n = 4\).

Case(2): \(n + 4 \equiv 1(\text{mod} \ 4)\).
Let \(n + 4 = 4k + 1(k \geq 2)\). If 3 \( v_i \)’s are given label 1, we get \((n + 3) + (n + 2) + (n + 1) =
3n + 6 = 12k - 3\) edges with label 1. But we need only at most \(10k - 4\) edges with label
1. So at most 2 \( v_i \)’s are given label 1.

Subcase(i): 2 \( v_i \)’s are given label 1.
If \(k = 2\), a group \{1, -1, i, -i\} Cordial labeling of \(C_5 + K_5\) is given in Table 5.
Suppose \(k ≥ 3\). Minimum number of edges that can get label 1 using \(k + 1\) vertices is
\((8k - 1) + 4 + (k - 2)3 = 11k - 3\). So the necessary condition is, \(10k - 4 \geq 11k - 3\) or
\(10k - 5 \geq 11k - 3\) i.e. \(k ≤ -1\) or \(k ≤ -2\), both not possible. Minimum number of edges
that can get label 1 using \(k\) vertices is, \((8k - 1) + 4 + (k - 3)3 = 11k - 1 + 4 - 9 = 11k - 6\).
So the necessary condition is, \(10k - 4 \geq 11k - 6\) or \(10k - 5 \geq 11k - 6\) i.e. \(k ≤ 2\) or \(k ≤ 1\).

Subcase(ii): One \( v_i \) is given label 1.
Now, maximum number of edges that can get label 1 using \(k+1\) vertices is, \((n+3) + k.5 =
4k + 5k = 9k\). So the necessary condition to get a group \{1, -1, i, -i\} Cordial labeling is
\(9k ≥ 10k - 4\) or \(9k ≥ 10k - 5\) i.e. \(k ≤ 5\). If \(3 ≤ k ≤ 5\), a group \{1, -1, i, -i\} Cordial
labeling of \(C_k + K_5\) is given in Tables 5 and 6.

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Table 5

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Table 6
Subcase(iii): No $v_i$ is given label 1.
We need to have $6(k+1) \geq 10k-5 \Rightarrow 4k \leq 11 \Rightarrow k \leq \frac{11}{4}$. Thus in Case 2, we have $n \in \{5, 9, 13, 17\}.$

Case(3): $n + 4 \equiv 2 \pmod{4}$.
Let $n + 4 = 4k + 2(k \geq 2)$. If $3 v_i$’s are given label 1, we get $3n + 6 = 12k$ edges with label 1. But we need only at most $10k - 2$ edges with label 1. Thus at least $2n + 5 = 8k + 1$ edges will have label 1. So $k \leq \frac{3}{2}$ which is impossible.

Subcase(ii): One $v_i$ is given label 1.
Maximum number of edges that can get label 1 using $k + 1$ vertices is $(n + 3) + k.5 = (4k - 2 + 3) + 5k = 9k + 1$. So the necessary condition to get a group $\{1, -1, i, -i\}$ Cordial labeling is $9k + 1 \geq 10k - 2 \Rightarrow k \leq 3$. If $k = 2, n = 6$ and if $k = 3, n = 10$. A group $\{1, -1, i, -i\}$ Cordial labeling of $C_6 + K_4$ and $C_{10} + K_4$ are given in Table 7.

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Table 7

Subcase(iii): No $v_i$ is given label 1.
We need to have $6(k+1) \geq 10k - 2 \Rightarrow 4k \leq 8 \Rightarrow k \leq 2$. Thus in Case 3, we have $n \in \{6, 10\}$.

Case(4): $n + 4 \equiv 3 \pmod{4}$.
Let $n + 4 = 4k + 3(k \geq 1)$. Number of edges = $5(4k - 1) + 6 = 20k + 1$. If $3 v_i$’s are given label 1, we get $3n + 6 = 12k + 3$ edges with label 1. But we need only at most $10k + 1$ edges with label 1. So at most 2 $v_i$’s are given label 1. Subcase(i): 2 $v_i$’s are given label 1.
Minimum number of edges that can get label 1 using $k$ vertices is $(n + 3) + (n + 2) + 4 + (k - 3)3 = 11k - 2$ and minimum number of edges that can get label 1 using $k + 1$ vertices is $11k + 1$. Thus $10k + 1 \geq 11k - 2 \Rightarrow k \leq 3$ or $10k + 1 \geq 11k + 1 \Rightarrow k \leq 0$, both impossible. If $k = 1$, a group $\{1, -1, i, -i\}$ Cordial labeling of $C_3 + K_4$ is given in Table 8.

If $k = 2$, and if $v_1$ and $v_2$ are labelled with 1 then 19 edges get label 1. Thus, there is no choice of 2 or 3 vertices so that 20 or 21 edges get label 1.

If $k = 3$, a group $\{1, -1, i, -i\}$ Cordial labeling of $C_{11} + K_4$ is given in Table 8.

Subcase(ii): One $v_i$ is given label 1.
Maximum number of edges that can get label 1 using $k$ vertices is $(n + 3) + (k - 1)5 = (4k - 1 + 3) + 5k - 5 = 9k - 3$. Maximum number of edges that can get label 1 using $k + 1$ vertices is $(n + 3) + k.5 = (4k - 1) + 3 + 5k = 9k + 2$. Thus $9k + 2 \geq 10k + 1$ or $9k + 2 \geq 10k$ so that $k \leq 1$ or $k \leq 2$. Also $9k - 3 \geq 10k + 1 \Rightarrow k \leq -4$ which is a contradiction. When $k = 2$, a group $\{1, -1, i, -i\}$ Cordial labeling of $C_7 + K_4$ is given in Table 8.
Subcase(iii): No \( v_i \) is given label 1.

\[
\begin{array}{cccccccccccc}
  n & v_1 & v_2 & v_3 & v_4 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} \\
 3 & 1 & 1 & -1 & -1 & i & i & -i & -i & & & & & & & \\
 7 & 1 & -1 & -1 & -1 & 1 & i & 1 & i & i & -i & -i & & & & \\
11 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & i & i & i & -i & -i & -i & -i & \\
\end{array}
\]

Table 8

We need to have \( 6(k+1) \geq 10k+1 \) or \( 6(k+1) \geq 10k \) so that \( 4k \leq 5 \) or \( 4k \leq 6 \). Otherwise we have, \( 6k \geq 10k+1 \) or \( 6k \geq 10k \), both impossible. Thus in this Case, \( n \in \{3,7,11\} \).

The labelings given for \( n \in \{3,4,5,6,7,9,10,11,13,17\} \) are group \( \{1,-1,i,-i\} \) Cordial is clear from Table 9.

\[
\begin{array}{cccccccc}
  n & v_f(1) & v_f(-1) & v_f(i) & v_f(-i) & e_f(0) & e_f(1) \\
 3 & 2 & 2 & 2 & 1 & 10 & 11 \\
 4 & 2 & 2 & 2 & 2 & 13 & 13 \\
 5 & 2 & 2 & 3 & 2 & 16 & 15 \\
 6 & 3 & 3 & 2 & 2 & 18 & 18 \\
 7 & 3 & 3 & 3 & 2 & 21 & 20 \\
 9 & 4 & 3 & 3 & 3 & 25 & 26 \\
10 & 4 & 4 & 3 & 3 & 28 & 28 \\
11 & 3 & 4 & 4 & 4 & 30 & 31 \\
13 & 5 & 4 & 4 & 4 & 35 & 36 \\
17 & 6 & 5 & 5 & 5 & 46 & 45 \\
\end{array}
\]

Table 9

Theorem 6. \( C_n + K_5 \) is group \( \{1,-1,i,-i\} \) Cordial iff \( n \) satisfies one of the following:

(i) \( n + 5 \equiv 0(\text{mod} \ 4) \) where \( n \geq 7 \).

(ii) \( n + 5 \equiv 1(\text{mod} \ 4) \) where \( n \geq 8 \).

(iii) \( n + 5 \equiv 2(\text{mod} \ 4) \) where \( n \geq 17 \).

(iv) \( n + 5 \equiv 3(\text{mod} \ 4) \) where \( n \geq 22 \).

Proof. Let the vertices of \( C_n \) be labelled as \( u_1, u_2, \ldots, u_n \) and let the vertices of \( K_5 \) be labelled as \( v_1, v_2, v_3, v_4, v_5 \). Number of vertices of \( C_n + K_5 \) is \( n + 5 \) and number of edges is \( 6n + 10 \).

Case(1): \( n + 5 \equiv 0(\text{mod} \ 4) \).

Let \( n + 5 = 4k (k \in \mathbb{Z}, k \geq 2) \). Now each vertex label should appear \( k \) times. Number of edges = \( 6n + 10 = 6(4k - 5) + 10 = 24k - 20 \) and so each edge label appears \( 12k - 10 \) times.

If \( k = 2 \), there is no choice of 2 vertices so that 14 edges get label 1. Suppose \( k \geq 3 \). Label the \( k \) vertices \( v_1, v_2, u_1, u_2, u_3, \ldots, u_{k-2} \) by 1. Label the remaining vertices arbitrarily.
so that $k$ of them get label $-1$, $k$ of them get label $i$ and $k$ of them get label $-i$. Number of edges with label $1 = (n + 4) + (n + 3) + 5 + (k - 3)4 = 12k - 10$. That this labeling is a group $\{1, -1, i, -i\}$ Cordial labeling of $C_n + K_5 (n \geq 7)$ is evident from Table 10.

**Case(2):** $n + 5 \equiv 1 \pmod{4}$

Let $n + 5 = 4k + 1 (k \in \mathbb{Z}, k \geq 2)$. Number of edges = $6n + 10 = 6(4k - 4) + 10 = 24k - 14$ and so each edge label appears $12k - 7$ times.

If $k = 2$, there is no choice of 2 or 3 vertices so that 17 edges get label 1. For $k = 3$, a group $\{1, -1, i, -i\}$ Cordial labeling of $C_8 + K_5$ is given in Table 10. Suppose $k \geq 4$. Label the $k$ vertices $v_1, v_2, u_1, u_3, u_4, u_5, ..., u_{k-1}$ by 1. Label the remaining vertices arbitrarily so that $k + 1$ of them get label $-1$, $k$ of them get label $i$ and $k$ of them get label $-i$. Number of edges with label $1 = (n + 4) + (n + 3) + 2.5 + (k - 4)4 = 12k - 7$. That this labeling is a group $\{1, -1, i, -i\}$ Cordial labeling of $C_n + K_5 (n \geq 8)$ is evident from Table 10.

<table>
<thead>
<tr>
<th>$n + 5$</th>
<th>$v_f(1)$</th>
<th>$v_f(-1)$</th>
<th>$v_f(i)$</th>
<th>$v_f(-i)$</th>
<th>$e_f(1)$</th>
<th>$e_f(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4k (k \geq 3)$</td>
<td>$k$</td>
<td>$k$</td>
<td>$k$</td>
<td>$k$</td>
<td>$12k - 10$</td>
<td>$12k - 10$</td>
</tr>
<tr>
<td>$4k + 1 (k \geq 3)$</td>
<td>$k$</td>
<td>$k + 1$</td>
<td>$k$</td>
<td>$k$</td>
<td>$12k - 7$</td>
<td>$12k - 7$</td>
</tr>
<tr>
<td>$4k + 2 (k \geq 5)$</td>
<td>$k$</td>
<td>$k$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$12k - 4$</td>
<td>$12k - 4$</td>
</tr>
<tr>
<td>$4k + 3 (k \geq 6)$</td>
<td>$k$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$k + 1$</td>
<td>$12k - 1$</td>
<td>$12k - 1$</td>
</tr>
</tbody>
</table>

**Case(3):** $n + 5 \equiv 2 \pmod{4}$

Let $n + 5 = 4k + 2 (k \in \mathbb{Z}, k \geq 2)$. Number of edges is $24k - 8$. So each edge label appears $12k - 4$ times. If 3 $v_i$’s are given label 1, we get $(n + 4) + (n + 3) + (n + 2) = 3n + 9 = 12k$ edges with label 1. But we need only 12k - 4 edges with label 1. So at most 2 $v_i$’s are given label 1.

**Subcase(i):** 2 $v_i$’s are given label 1.

Maximum number of edges that can get label 1 using $k$ vertices is $(n + 4) + (n + 3) + (k - 2)5 = 2(4k - 3) + 7 + 5k - 10 = 13k - 9$. So a necessary condition to get a group $\{1, -1, i, -i\}$ Cordial labeling is $13k - 9 \geq 12k - 4 \Rightarrow k \geq 5$. Maximum number of edges that can get label 1 using $k + 1$ vertices is $2(n + 7) + (k - 1)5 = 2(4k - 3) + 7 + 5k - 5 = 13k - 4$. So a necessary condition to get a group $\{1, -1, i, -i\}$ Cordial labeling is $13k - 4 \geq 12k - 4 \Rightarrow k \geq 0$. For $k \leq 4$, it is easy to observe that there is no group $\{1, -1, x, -i\}$ Cordial labeling. For $k \geq 5$, label the $k$ vertices $v_1, v_2, u_1, u_3, u_5, u_6, u_7, ..., u_k$ by 1. Label the remaining vertices arbitrarily so that $k$ of them get label $-1$, $k + 1$ of them get label $i$ and $k + 1$ of them get label $-i$. Number of edges with label 1 = $(n + 4) + (n + 3) + 15 + (k - 5)5 = 12k - 4$. That this labeling is a group $\{1, -1, i, -i\}$ Cordial labeling of $C_n + K_5 (n \geq 8)$ is evident from Table 10.

**Subcase(ii):** One $v_i$ is given label 1.

Maximum number of edges that can get label 1 using $k$ vertices is $(n + 3) + (k - 1)6 = 10k - 6$. So a necessary condition to get a group $\{1, -1, i, -i\}$ Cordial labeling is $10k - 6 \geq 12k - 4 \Rightarrow k \leq -1$, which is a contradiction. Maximum number of edges that can get label 1 using $k + 1$ vertices is $(n + 3) + 6k = 10k$. So a necessary condition to get a group
\{1, -1, i, -i\} Cordial labeling is $10k \geq 12k - 4 \Rightarrow k \leq 2$. If $k = 2, n = 5$, and a group
\{1, -1, i, -i\} Cordial labeling of $C_5 + K_5$ is given in table 11.

Case(4): \n
Let $n + 5 = 4k + 3(k \in \mathbb{Z}, k \geq 2)$. Number of edges $= 6n + 10 = 6(4k - 2) + 10 = 24k - 2$. So each edge label appears $12k - 1$ times. If 3 $v_i$’s are given label 1, we get $(n + 4) + (n + 3) + (n + 2) = 3n + 9 = 12k + 3$ edges with label 1. But we need only $12k - 1$ edges with label 1. So at most 2 $v_i$’s are given label 1.

Subcase(i): One $v_i$ is given label 1.

Maximum number of edges that can get label 1 using $k$ vertices is $(n + 4) + (n + 3) + (k - 2)5 = 2(4k - 2) + 7 + 5k - 10 = 13k - 7$. So a necessary condition to get a group \{1, -1, i, -i\} Cordial labeling is $13k - 7 \geq 12k - 1 \Rightarrow k \geq 6$. Maximum number of edges that can get label 1 using $k + 1$ vertices is $(2n + 7) + (k - 1)5 = 2(4k - 2) + 5k - 5 = 13k - 2$. So a necessary condition to get a group \{1, -1, i, -i\} Cordial labeling is $13k - 2 \geq 12k - 1 \Rightarrow k \geq 1$. But for $1 \leq k \leq 5$, we observe that there is no group \{1, -1, i, -i\} Cordial labeling. For $k \geq 6$, label the $k$ vertices $v_1, v_2, u_1, u_3, u_5, u_7, u_8, u_9, \ldots, u_{k+1}$ by 1. Label the remaining vertices arbitrarily so that $k + 1$ of them get label 1, $k + 1$ of them get label $i$ and $k + 1$ of them get label $-i$. Number of edges with label 1 $= 2n + 7 + 20 + (k - 6)4 = 12k - 1$. That this labeling is a group \{1, -1, i, -i\} Cordial labeling of $C_n + K_5(n \geq 8)$ is evident from Table 10.

Subcase(ii): One $v_i$ is given label 1.

Maximum number of edges that can get label 1 using $k$ vertices is $(n + 3) + (k - 1)6 = 10k - 5$. So a necessary condition to get a group \{1, -1, i, -i\} Cordial labeling is $10k - 5 \geq 12k - 1 \Rightarrow k \leq -2$, which is a contradiction. Maximum number of edges that can get label 1 using $k + 1$ vertices is $(n + 3) + 6k = 10k + 1$. So a necessary condition to get a group \{1, -1, i, -i\} Cordial labeling is $10k + 1 \geq 12k - 1 \Rightarrow k \leq 1$, which is a contradiction.

\begin{center}
\begin{tabular}{cccccccccc}
\hline
$n$ & $v_1$ & $v_2$ & $v_3$ & $v_4$ & $v_5$ & $u_1$ & $u_2$ & $u_3$ & $u_4$ & $u_5$ & $u_6$ & $u_7$ & $u_8$ \\
\hline
5 & -1 & -1 & -1 & i & i & 1 & -i & 1 & 1 & -i & 1 & -i & -i \\
8 & 1 & -1 & -1 & 1 & i & 1 & i & 1 & 1 & -i & -i & -i & -i \\
\hline
\end{tabular}
\end{center}

Table 11

References


