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Normalized Tenacity and Normalized Toughness of Graphs

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ABSTRACT

In this paper, we introduce the novel parameters indicating Normalized Tenacity (T_N) and Normalized Toughness (t_N) by a modification on existing Tenacity and Toughness parameters. Using these new parameters enables the graphs with different orders be comparable with each other regarding their vulnerabilities. These parameters are reviewed and discussed for some special graphs as well.

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1 Introduction

The formal definition of the term graph is considered in [16] as $G = (V, E)$. According to this definition, a graph may be used to model objects and relationships among them whenever there is certain relationship or information that needs to be encoded [46]. Graphs

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have several applications in many areas, some examples of them are: Map Coloring, Radio Frequency Assignment, VLSI Floorplanning, Communication Networks, Bioinformatics, Data Flow Analysis [11, 15, 18, 31, 42, 45, 48], etc. Among these examples, communication networks are of special interest. Any communication network can be modeled as graphs with sets of vertices and edges. These networks are exposed to the risks of disruption and malfunction by getting removed some vertices/edges. Such degrees of disruption are considered as network vulnerabilities to decomposition and failure. In addition, the existence of servicing clients in mobile and ad-hoc networks are also among the issues that raise the need to assess the vulnerability of networks.

Hence, networks vulnerability study is crucial and any achievement in this regard would lead to better assessment of computer networks. In computer science domain, networks vulnerability has been studied since the late 80s and several parameters have been proposed to estimate and assess such vulnerabilities. The rest of this paper is organized as follows: In section 2, the vulnerability parameters and their desired properties are discussed. Some famous vulnerability parameters such as connectivity, Tenacity, and Toughness are discussed in this section as well. In section 3, the Normalized Toughness (t_N) and Normalized Tenacity (T_N) parameters are proposed and their properties are reviewed. An overview is provided on how the new parameters would work for certain classes of graphs in section 4, and section 5 will conclude the paper and discuss future works.

2 Graph Vulnerability Parameters

As it is mentioned in section 1, the vulnerability parameters are proposed to study the extent of potential weakness and disruption of communication networks that are considered and modeled as graphs. Numerous classes of graphs have been introduced to decrease the vulnerability, and Harary graphs are one of those proposed [32]. Several efforts have been done to calculate the vulnerability of this types of graphs as well [22, 39, 40]. In other words, the vulnerability parameters help us to assess the network resistance to decomposition and they are designed to map such assessment to a real number. There are several theoretical parameters that show and estimate the vulnerability of a graph, including connectivity, integrity, binding number, Toughness, Tenacity, etc. [47]. In [14] some desirable properties of vulnerability measures are discussed. Some of these properties are:

- **Monotonicity:**The parameter values must be monotone (either increasing or decreasing)
- **Comparability (Ordered Values):** the values given by the parameters to any graph must be comparable
- **Distinguishability:** the measure must be global enough so that its values could distinguish between two networks (graphs)

- **Computational Complexity:** the vulnerability parameter should be computed in polynomial time for any graph
- **Normality:** A graph or network vulnerability parameter desired to be normalized. Being normalized means that the values should be in a finite, normal and bounded range of real numbers (i.e. [0,1]).

Connectivity, Tenacity, and Toughness are amongst the most popular parameters. However, their disadvantage is that they do not meet some of the properties mentioned above. For example, these parameters provide diverse ranges of values that make it difficult to distinguish between two graphs regarding their vulnerability (i.e. connectivity, Tenacity or Toughness). On the other hand, their values are not monotone which makes it difficult to estimate the growth or decrease rate of them. Some examples and formal definitions of these parameters are provided here. Since this paper is focused on Toughness and Tenacity, the formal definition of these two parameters is provided below which will help to clarify the normalization process later. For all cases in this paper, we consider the graphs to be simple and undirected. Table 1 shows the values of some vulnerability parameters for some simple graphs.

2.1 Toughness

The Vertex/Edge Toughness of a graph G, $t(G)$, defined in [17] as follows:

$$t_v(G) = \min_{S \subset V(G)} \{\frac{|S|}{\omega(G-S)}\} \tag{1}$$

$$t_e(G) = \min_{S \subseteq E(G)} \{\frac{|S|}{\omega(G-S)}\} \tag{2}$$

In which the minimum is taken over all vertex/edge cut set S of G if G is non-complete and if G is complete then $t(K_p) = \infty$. For convenience, Vertex Toughness is mentioned as Toughness and these words are used in this paper equivalently. In [29] and [28], in order to deal with the value infinity for complete graphs, the minimum is taken over all $S \subset V(G)$ which leads to $p(G) - 1$ instead of ∞; in which, $p(G)$ is the order of graph. The order of a graph is considered as the number of vertices of the graph [11, 16]. This means that in order to achieve the minimum Toughness, the graph must be decomposed so that there would be maximum disconnected components. In [4, 28, 29] the relationship between Toughness and other measures of vulnerability are reviewed. However, many papers dealing with Toughness have appeared since 1973, aimed mainly at establishing links between the Toughness of a graph and its cycle structure, inspired by conjectures in [17]: It was conjectured that a constant c exists such that $t(G) \geq c$ implies hamiltonicity of G, that $t(G) \geq 3/2$ implies the existence of a two-factor in G and, for any positive integer k such that $k_p(G)$ is even, $t(G) \geq k_p(G)$ implies the existence of a k-factor in G. Some of these conjectures were reviewed in [6], as well. Only the last of these conjectures has been proved [25]. In this paper we merely focus on Tenacity and Toughness, to provide a normalized calculation method for them, thus, we dont need to list all references to progress made in investigating the remaining conjectures. The names of authors who are currently the most distinguished in this field may be found in [9, 28].

2.2 Tenacity

The Tenacity of a graph G, $T(G)$, defined in [21] as follows:

$$T(G) = \min_{S}\{\frac{|S| + \tau(G-S)}{\omega(G-S)}\} \tag{3}$$

In which the minimum is taken over all vertex/edge cut set S of G; $\tau(G-S)$ is the number of vertices/edges in the largest component of the graph induced by $(G-S)$ and $\omega(G-S)$ is the number of components of $G-S$. The formula above can be defined as Vertex-Tenacity, Edge-Tenacity, or Mix-Tenacity. The notation will change slightly while the concept is still the same. So, the Vertex-Tenacity would be:

$$T_v(G) = \min_{S \subset V(G)}\{\frac{|S| + \tau(G-S)}{\omega(G-S)}\} \tag{4}$$

Such that, S is a subset of $V(G)$ the set of vertices of graph G, $\tau(G-S)$ is the number of vertices in the largest component of $(G-S)$ and $\omega(G-S)$ is the number of components of $(G-S)$. Edge-Tenacity is defined as follows [33]:

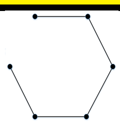
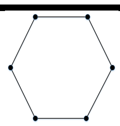
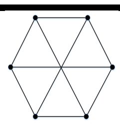
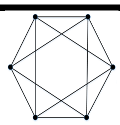
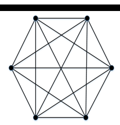
$$T_e(G) = \min_{S \subseteq E(G)}\{\frac{|S| + \tau(G-S)}{\omega(G-S)}\} \tag{5}$$

Such that, S is a subset of $E(G)$ the set of edges of graph G, $\tau(G-S)$ is the number of edges in the largest component of $(G-S)$ and $\omega(G-S)$ is the number of components of $(G-S)$. Finally, Mix-Tenacity is defined as below [33]:

$$T_m(G) = \min_{S \subseteq E(G)}\{\frac{|S| + \tau(G-S)}{\omega(G-S)}\} \tag{6}$$

Such that, S is a subset of $E(G)$ the set of edges of graph G, $\tau(G-S)$ is the number of vertices in the largest component of $(G-S)$ and $\omega(G-S)$ is the number of components of $(G-S)$.

Table 1: Vulnerability parameters for some special graphs with six vertices

Graph		κ	κ_e	t	t_e	T	T_e	T_m
	P_6	1	1	1/2	1/2	4/3	5/6	1
	$H(2,6) = C_6$	2	2	1	1	4/3	1	7/6
	$H(3,6)$	3	3	1	3/2	4/3	3/2	5/3
	$H(4,6)$	4	4	2	2	5/2	2	13/6
	$H(5,6) = K_6$	∞	∞	∞	∞	∞	∞	∞

2.3 Connectivity, Tenacity, and Toughness of Some Special Graphs

In this subsection, some examples of certain classes of graphs are presented to review how connectivity, Tenacity, and Toughness of them are calculated and which desired properties of vulnerability parameters are met by them.

As it is shown in Table 1, all the vulnerability parameters almost meet the Monotonicity property of vulnerability measures. We conclude from Table 1 that, all three measures are almost distinguishable but it is hard to compare two different graphs with different orders (i.e. the number of vertices); however, the new calculation method that is presented in this paper makes the graphs comparable easily with each other regarding their Tenacity and Toughness. In addition, none of the parameters mentioned above can be computed in polynomial time in general [7, 8, 10, 13, 30, 36, 37, 51] and all of them are classified as NP-hard problems. Figure 1 shows the Monotonicity of the parameters.

In [38], the vulnerability parameters including Integrity, Connectivity, Binding number, Toughness and Tenacity of several graph classes are studied and compared. The results achieved in [38], showed that Tenacity is one of the most suitable measures of stability or vulnerability since it can distinguish between graphs that are supposed to have different vulnerabilities. What remains as an issue is the ambiguity of the vulnerability parameters. This means that sometimes these parameters present a dual approach to deal with some special classes of graphs. K_3 is one of the examples; this graph is considered as a cycle graph as well. As it is known, the value of Tenacity and Toughness regarding complete graphs are not available meanwhile they provide definite values for cycles. This is also

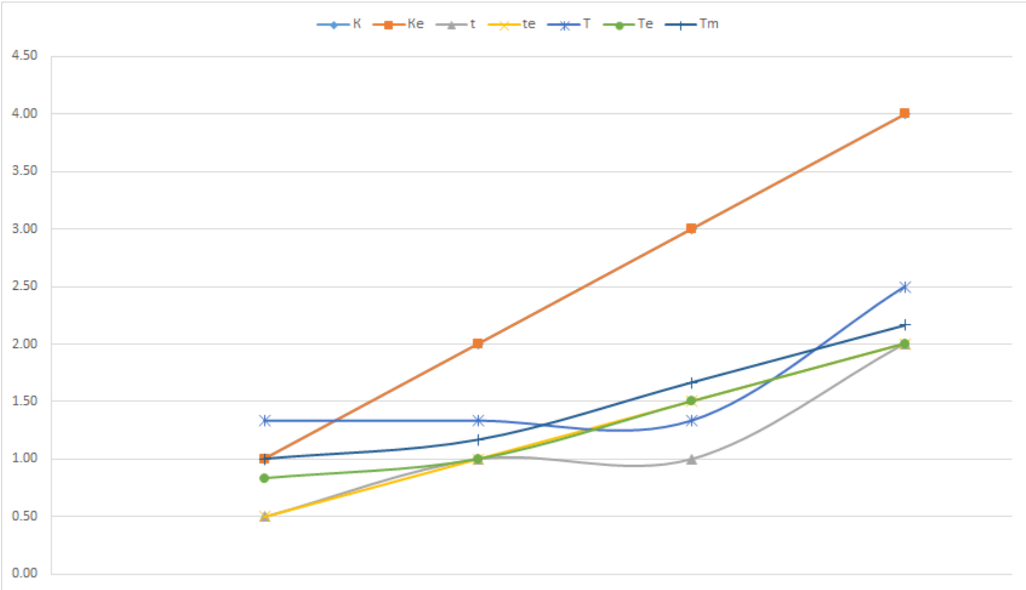


Figure 1: The trend of Monotonicity among Connectivity, Tenacity, and Toughness

one of the issues that this paper is aimed to resolve it.

3 Normalized Toughness and Normalized Tenacity

In this section, we are going to introduce Normalized Tenacity and Normalized Toughness parameters and explain how they are calculated for several types of graphs. The Normalized Tenacity and Normalized Toughness parameters are based on a coefficient which makes their values lie in the closed interval [0,1]. Thus, in this paper the coefficients are presented which are calculated through upper bound achievements both for Tenacity and Toughness. The Tenacity or Toughness value would be multiplied by the reversed coefficient achieved for each. This will results values that are in the closed interval [0,1]. The Normalized Tenacity/Toughness of completely disconnected graphs is considered to be 0 and for complete graphs, they are considered to be 1. The next subsections will introduce Normalized Tenacity and Normalized Toughness as our new parameters.

3.1 Normalized Toughness

In order to normalize the Toughness of graphs so that it would lie in the closed interval [0,1], it is necessary to find a coefficient such that its multiplication to Toughness would results a normal value in this interval. Thus an upper bound should be obtained. Complete graphs with the most connectivity are considered as the upper bound or the hardest case for Toughness. As it discussed before, $t(K_p) = \infty$; so, this cannot be considered as an upper bound. In [28] and [29], $p(G) - 1$ is considered as an upper bound where $p(G)$ is the number of vertices of a graph. If the number of the vertices of the complete graph is considered as n, then $t(K_n) = n-1$. This coefficient is not desirable since it still has ambiguity for calculation of certain graphs such as K_3. K_3 can be considered both as a complete graph (K_3) and a cycle graph (C_3). In order to overcome this ambiguity, another upper bound is proposed. It is proved that the upper bound for the Toughness of all classes of graphs is $2/(n-1)$ using the following lemma.

Lemma 1: For any graph $G = (V, E)$, the upper bound for Toughness is $(n-1)/2$.
Proof: As it is mentioned earlier, the complete graphs are the upper bound or the worst cases for vulnerability assessment. Consider a K_{n-2} graph in which $n-2$ vertices are completely connected to each other and two other vertices are connected to the remaining vertices except each other (K_n having removed one edge). The schema of the graph would be as Figure 2.

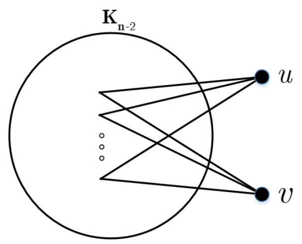


Figure 2: The schema of K_n having removed the edge (u, v)

For the graph in Figure 2, in order to achieve minimum Toughness, $n-2$ vertices should be removed so that 2 components would remain. Replacing the values in formula (1) would results $(n-2)/2$. It is clear that if only one edge between u and v is added, the resulting graph would be K_n. It is also clear that due to the Monotonicity property of Toughness, adding one edge would increase the value of Toughness very little with the value ϵ. Thus clearly:

$$t_v(K_n) > \frac{n-2}{2} \tag{7}$$

Therefore, the upper bound for Toughness can be considered as $(n-1)/2$. According to Lemma 1, the coefficient for Normalized Vertex Toughness would be $2/(n-1)$. This coefficient is also true for Edge Toughness. It can also be proved through the Lemma 2.

Lemma 2: For any graph $G = (V, E)$, the upper bound for Edge Toughness is $(n-1)/2$.
Proof: Based on the method presented in [33], we can conclude that the worst case for Edge Toughness relation, leading to the minimum value, is to remove $n(n-1)/2$ edges to obtain n components. This case provides the minimum value. Thus the value of the Edge Toughness would be:

$$t_e(K_n) = \frac{n(n-1)/2}{n} = \frac{n(n-1)}{2n} = \frac{n-1}{2} \tag{8}$$

So, the upper bound for Edge Toughness would also be $(n-1)/2$.

Therefore, the Normalized (Vertex) Toughness and Edge Toughness would be respectively:

$$t_{v_N}(G) = \frac{2}{n-1} \times \min_{S \subset V(G)} \{\frac{|S|}{\omega(G-S)}\} \tag{9}$$

$$t_{e_N}(G) = \frac{2}{n-1} \times \min_{S \subseteq E(G)} \{\frac{|S|}{\omega(G-S)}\} \tag{10}$$

3.2 Normalized Tenacity

In order to achieve Normalized Tenacity parameters, the equations (4), (5), and (6) must be modified, so that their values can lie in the closed interval [0, 1]. To this purpose, the coefficient must be considered as the upper bound for Tenacity so that its multiplication by Tenacity would meet the constraint above. Clearly, complete graphs are the densest graphs and hence are considered as an upper bound for the Tenacity of graphs.
The proper coefficient for (vertex) Tenacity is proved through the following lemma:

Lemma 3: For any graph $G = (V, E)$, the upper bound for (vertex) Tenacity is n.
Proof: Likewise Lemma 1, consider an almost complete graph K_n. The overall schema of the graph is presented in Figure 2. In order to achieve minimum Tenacity, maximum disconnection is desired. So, $n-2$ vertices must be removed. Thus the Tenacity of the complete graph would be:

$$T(K_n) > \frac{n-2+1}{2} = \frac{n-1}{2}, \tag{11}$$

$$n \geq 0 \Rightarrow n > -1 \Rightarrow 2n > n-1 \Rightarrow n > \frac{n-1}{2} \tag{12}$$

This graph only misses one edge so that it could be a complete graph. Therefore the Tenacity of the complete graph can be considered as:

$$T_v(K_n) = n. \tag{13}$$

This coefficient can be obtained using another method. Consider the graph shown in Figure 3. In this graph, by removing $n-p$ vertices, p components will be generated. Thus, the Tenacity of this graph will be:

$$T_v(G) = \frac{n-p+1}{p}. \tag{14}$$

By setting $p = 1$ the graph will be a complete graph, and we have:

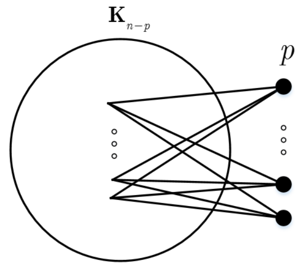


Figure 3: The schema of K_n having removed the edges between p vertices

$$T_v(K_n) = n. \tag{15}$$

So,

$$T_{v_N}(G) = \frac{1}{n} \times \min_{S \subset V(G)} \{ \frac{|S| + \tau(G-S)}{\omega(G-S)} \} \tag{16}$$

In [33], it is shown that the Edge Tenacity for complete graph K_n is as follows:

$$T_e(K_n) = \frac{n-1}{2}. \tag{17}$$

Since this is an upper bound for All Types of Graphs, we consider $1/(T_e(K_n))$ as the coefficient for calculation of Normalized Edge Tenacity. Thus the Normalized Edge Tenacity for the graphs would be:

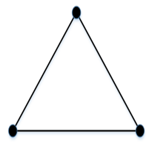
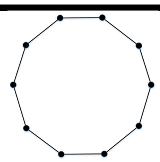
$$T_{e_N}(G) = \frac{2}{n-1} \times \min_{S \subseteq E(G)} \{ \frac{|S| + \tau(G-S)}{\omega(G-S)} \} \tag{18}$$

Lemma 4: For any graph $G = (V, E)$, the upper bound for Mix Tenacity is $(n(n-1) + 2)/2n$.
Proof: This lemma is also proved by considering a complete graph. The complete graph K_n consists of n vertices and $(n(n-1)/2)$ edges. Based on the method which discussed in [33], in order to achieve minimum value, all edges must be removed. Therefore,

$$T_m(K_n) = \frac{\frac{n(n-1)}{2} + 1}{n} = \frac{n(n-1)+2}{2n} \tag{19}$$

Thus, the upper bound for Mix Tenacity of graphs would be $(n(n-1)+2)/2n$. Then,

Table 2: The Toughness and Tenacity and their corresponding normalized parameters for C_3 and C_{10}

Graph		t_v	t_{v_N}	t_e	t_{e_N}	T_v	T_{v_N}	T_e	T_{e_N}
	$K_3 \leftrightarrow C_3$	$\infty \leftrightarrow 1$	1	1	1	$\infty \leftrightarrow 3$	1	1	1
	C_{10}	1	2/9	1	2/9	6/5	6/50	1	2/9

$$T_{m_N}(G) = \frac{2n}{n(n-1)+2} \times \min_{S\subseteq E(G)} \{\frac{|S| + \tau(G-S)}{\omega(G-S)}\} \tag{20}$$

4 The Properties of Normalized Toughness and Tenacity

The Normalized Toughness and Normalized Tenacity parameters that are proposed in this paper, meet some desired properties of the vulnerability parameters. The main properties are reviewed in this section.

4.1 Normalized parameter values and possibility of graph comparisons

The normalized vulnerability parameters values enable one to compare vulnerability of several classes of graphs regardless of their orders (i.e. the number of vertices). As an example the Tenacity and Toughness of consider C_3 and C_{10} along with their corresponding normalized values in Table 2.

As it is shown in Table 2, the Edge Toughness for both graphs is 1, but it is clear that C_{10} is weaker than C_3 and thus their normalized values show this difference better (i.e. $t_{e_N}(C_3) > t_{e_N}(C_{10})$ and $t_{v_N}(C_3) > t_{v_N}(C_{10})$). This is also true for Tenacity except that Tenacity provides different values regarding the vulnerability of these two graphs, but the range and amount of weakness are more comprehensible via normalized values. As a conclusion, normalization of parameters enables us to compare different graphs regardless of their size. Due to this property of Normalized Tenacity and Normalized Toughness parameters, these parameters are also applicable in mobile networks in which the numbers of the nodes vary by time. The sign $\leftrightarrow$ is used to indicate the ambiguity that is faced while dealing with the graph with 3 vertices.

4.2 Overcoming ambiguity

They avoid ambiguity and thus they provide one definite value for any graph. The ambiguity on how to deal with K_3 and C_3 can be mentioned as an example. The fully connected graph with three vertices can be considered both as a cycle and a complete graph. According to the Table 2, while calculating Toughness or Tenacity of this graph, if it is considered as a complete graph, the values for some parameters would not be available; while, the values of the aforementioned parameters are available for cycle graphs. This ambiguity would still remain if the upper bound mentioned in [28, 29] was chosen. That was the reason why new upper bounds were presented in this paper. Table 2 shows this property in a summary for $t_v(K_3 \leftrightarrow C_3)$ and $T_v(K_3 \leftrightarrow C_3)$.

4.3 Bounding normalized parameters to random graphs concepts

Since the values of the normalized parameters lie in the closed interval [0,1], thus these new parameters can be utilized to connect some random graph concepts to vulnerability parameters. This possibility is due to the probabilistic nature of the value of the normalized parameters (i.e. likewise the probability values of the events/variable, the values of the Normalized Toughness and Normalized Tenacity, lie in the closed interval [0,1]), they suite to provide a connection between vulnerability of graphs or networks and randomized graphs concepts.

4.4 Better study of the problems which do not rely on graphs orders

For the case Toughness and Tenacity, the range of possible values would increase by increasing the order of the graphs. However, since the new parameters values are independent of the number of the vertices of the graph. Therfore, they can be considered as better means to study some problems that are conceptually independent of the size of the graphs such as Hamiltonian cycles, etc. As it is discussed, in [17], Chvtal has proposed two conjectures that:

- There exists $t0$ such that every $t0$-tough graph is Hamiltonian and
- Every 2-tough graph is Hamiltonian.

In [5] it is proved that the second conjecture is false, while the first one is still an open problem. By introducing Normalized Toughness, it might be easier to find the threshold t_0 such that the Hamiltonian graphs could be distinguished from the others.
In [5] it is proved that not every 2-tough graph is Hamiltonian by a counterexample. The following theorem has already been proved [5, 7] for graph H and x, y two vertices of H

which are not connected by a Hamilton path of H. If $m \geq 2l+3$, then $G(H,x,y,l,m)$ is non-traceable. It is said that a graph is traceable if it has a path containing all of its vertices. A simple example of the graph along with its schema is provided in [10]. Also, it was shown that for $l \geq 2$ and $m \geq 1$, $\tau(G(L,u,v,l,m)) = (l+4m)/(2m+1)$ where, m is the number of disjoint copies of subgraph H, and l is the number of dominating vertices [5]. Thus, the number of vertices of the graph would be $l+8m$. Considering the coefficient of the Normalized Toughness as $2/(n-1)$, that is, in this case, $2/(l+8m-1)$. In the theorem above a lower bound is obtained for m which is the number of disjoint copies of H. Thus with a constant l, the maximum of the above-mentioned equation is achieved only when m is minimized. (i.e. $m = 2l+3$). Thus:

$$t_{v_N}(G) = \frac{l+4m}{2m+1} \times \frac{2}{l+8m-1} = \frac{2(l+4m)}{(2m+1)(l+8m-1)} \tag{21}$$

It is clear that the value of the equation above would always lie in the interval [0,1] when m is large. But for smaller values of m and l, it must be checked to ensure the constraint remains true, thus the minimum values for m and l must be substituted to see if the value still remains in the interval [0,1]. As the minimum m is substituted in the equation (18):

$$t_{v_N}(H) = \frac{2(l+8l+12)}{(4l+6+1)(l+16l+23)} = \frac{2(9l+12)}{(4l+7)(17l+23)} \tag{22}$$

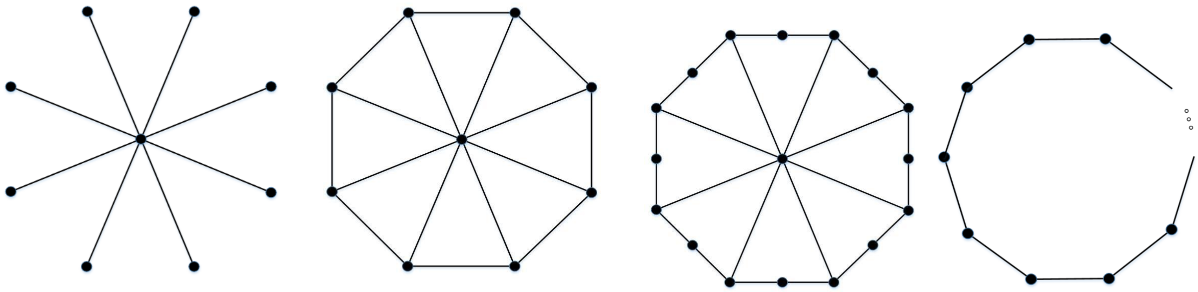
Since $l \geq 2$ [5], by substituting l by its minimum value (i.e. $l = 2$):

$$t_{v_N}(H) = \frac{18l+24}{68l^2+92l+119l+151} = \frac{182+24}{684+922+1192+151} = \frac{60}{845} = \frac{12}{169} \in [0,1] \tag{23}$$

In addition, as m increases, the value becomes even smaller than 12/169.

5 Review and Compare Normalized Toughness and Normalized Tenacity for some special classes of graphs

Now it is useful to review some examples of Toughness and Tenacity for some special classes of graphs to see that the proposed equations remain true and check how they work. In this section Toughness and tenacities of several classes of graphs have already been calculated. The results achieved in previous researchers are presented, here; in addition, the normal Toughness and normal Tenacity are calculated for these classes of graphs. In [2, 3, 5, 12, 20, 22–24, 26–28, 33–35, 39, 40, 43, 44] the Tenacity and Toughness for some classes of graphs are discussed. A brief summary of the observations is displayed



(a) Start Graph ($K_{1,8}$) (b) Wheel Graph (W_8) (c) Gear Graph (G_8) (d) Cycle Graph (C_n)

Figure 4: Examples of Star, Wheel, Gear, and Cycle graphs

in Table 3. It is noteworthy that there are many classes of graphs for which Toughness or Tenacity or both are calculated. Table 3 just presents some important ones as a summary. Before proceeding to Table 3, some definitions of these special classes of the graphs are provided here.

Definition 1 [16]**:** For an integer $n \geq 1$, the path P_n is a graph of order n and size $n+1$ whose vertices can be labeled by $v_1; v_2; ; v_n$ and whose edges are v_iv_{i+1} for $i = 1, 2, , n+1$.

Definition 2 [26]**:** A star graph is a complete bipartite graph denoted by $K_{1,n}$. The number of vertices of the star graph $K_{1,n}$ is $n+1$ and n is its total number of edges. The Figure 4(a) shows $K_{1,8}$.

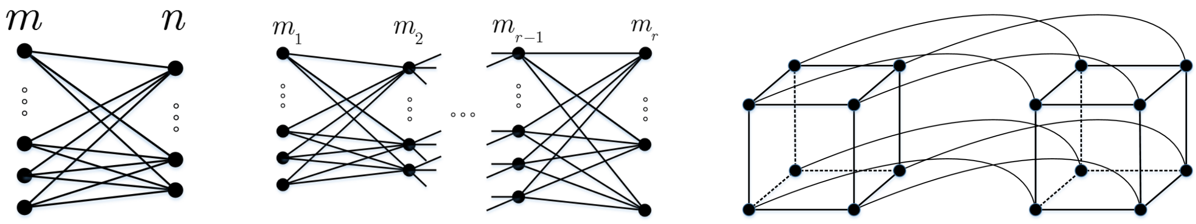
Definition 3 [2]**:** The wheel graph with n spokes, W_n, is the graph that consists of an n-cycle and one additional vertex, say u, that is adjacent to all the vertices of the cycle. The Figure 4(b) shows W_8.

Definition 4 [2, 12]**:** The gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges. The gear graph G_8 is shown in Figure 4(c).

Definition 5 [16]**:** A 3-regular graph is a graph where each vertex has 3 neighbors. A 3-regular graph also called as a cubic graph. One of the best known cubic graphs is the Petersen graph.

Definition 6 [50]**:** The Harary graph $H_{k,n}$ is a particular example of a k-connected graph with n vertices having the smallest possible number of edges.

Definition 7 [2, 35, 49]**:** The Cartesian product of two graphs G_1 and G_2 are conceptually denoted by $G_1 \times G_2$. The total number of vertices of the resulting graphs would be $|V(G_1) \times V(G_2)|$. This definition is applicable to any class of graphs including gear graphs. For which the Cartesian product of G_m and G_n is denoted by $G_m \times G_n$. The Cartesian product of two gear graphs has mn vertices. With the same inspiration, the



(a) $K_{m,n}$ (b) $K_{m_1,m_2,,m_r}$ (c) Q_4

Figure 5: Examples of Complete Bipartite, Complete multipartite, and 4-Dimentional Hypercube

Cartesian product of K_2 (a path/complete graph with 2 vertices) and a gear graph with n vertices ($K_2 \times G_n$) has $4n + 2$ vertices.

Definition 8 [16]**:** Complete Bipartite Graphs are denoted by $K_{m,n}$ where its set of vertices can be partitioned into two partite sets of U and W such that any corresponding edge uw is an edge of G if and only of $u \in U$ and $w \in W$. If $|U| = m$ and $|W| = n$, this graph is denoted by $K_{m,n}$.

Definition 9 [16]**:** A complete multipartite graph is a complete r-partite graph for some integer $r \geq 2$. A complete r-partite graph G is a r-partite graph with the property that two vertices are adjacent in G if and only if the vertices belong to different partite sets. For an integer $r \geq 1$, a graph G is a r-partite graph if $V(G)$ can be partitioned into r subsets $V_1, V_2, , V_r$ (again called partite sets) such that every edge of G joins vertices in two different partite sets. The complete multipartite graphs are denoted by $K_{m_1,m_2,,m_r}$.

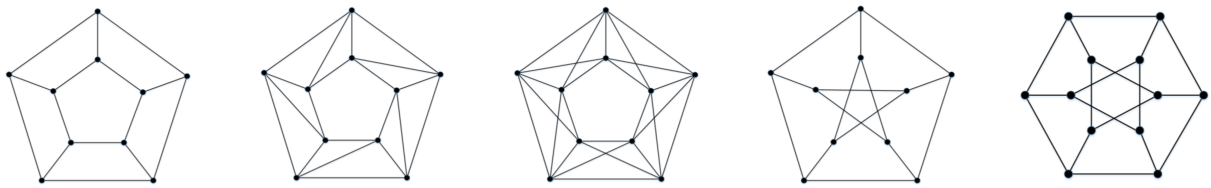
Definition 10 [16]**:** An important class of graphs is defined in terms of Cartesian products. The n-cube Q_n is K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as the Cartesian product $Q_{n-1} \times K_2$ of Q_{n-1} and K_2. The graph Q_n is an n-regular graph of order 2^n.

Definition 11 [23]**:** Generalized prism graph $Z_{n,s}$ has vertex set $\{(i, j)|i = 1, 2$ and $j = 1, 2, , n\}$. Each vertex (i, j) is adjacent to $(i, j \pm 1)$. In addition, each $(1, j)$ is adjacent to $(2, j + \sigma)$ for each σ in $\{\lfloor \frac{s-1}{2} \rfloor, ..., 0, ..., \lfloor \frac{s}{2} \rfloor\}$.

Definition 12 [1]**:** The generalized Petersen graph $GP(n, k), n \geq 2$ and $1 \leq k \leq n - 1$, has vertex-set $\{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 0 \leq i \leq n - 1\}$ with subscripts reduced modulo n.

6 Conclusion

In this paper, Normalized Tenacity and Normalized Toughness are introduced such that the calculated value for any graph would lie in the interval [0,1]. These modified parameters have advantages, including:



(a) $Z(5,1)$ (b) $Z(5,2)$ (c) $Z(5,3)$ (d) Petersen (e) $G(6,2)$

Figure 6: Examples of Prism and Peterson graphs

Table 3: A brief summary of Toughness, Tenacity, Normalized Toughness and Normalized Tenacity values for some special graphs

Graph	ν	t_v	t_{v_N}	T_v	T_{v_N}
C_n	n	1	$\frac{2}{n-1}$		
$C_m \times C_n$	mn	1 : m, n even	$\frac{2}{mn-1}$	$\frac{mn+2}{mn}$	$\frac{mn+2}{mn^2}$
$C_m \times C_n$	mn	$\leq \frac{n}{n-1}$: m even	$\leq \frac{2n}{(n-1)(mn-1)}$		
G_n	$2n+1$	$\frac{n}{n+1}$: $n \geq 4$	$\frac{1}{n+1}$	1	$\frac{1}{2n+1}$
$G_m \times G_n$	$(2m+1)(2n+1)$	$\frac{2mn+m+n}{2mn+m+n+1}$	$\frac{1}{2mn+m+n+1}$	1	$\frac{1}{(2m+1)(2n+1)}$
$K_2 \times G_n$	$4n+2$	1	$\frac{2}{4n+1}$	$\frac{2n+2}{2n+1}$	$\frac{n+1}{(2n+1)^2}$
$K_{1,n}$	$n+1$	$\frac{1}{n}$: $m \leq n, n \geq 2$	$\frac{2}{n^2}$	$\frac{2}{n}$	$\frac{2}{n(n+1)}$
$K_{m,n}$	$m+n$	$\frac{m}{n}$	$\frac{2m}{n(m+n-1)}$	$\frac{m+1}{n}$: $1 \leq m \leq n$	$\frac{m+1}{n(m+n)}$
$K_{m_1,m_2,\ldots,m_r}$	$\sum m_i$	$\frac{m_1+m_2+\ldots+m_{r-1}}{m_r}$	$\frac{2(m_1+m_2+\ldots+m_{r-1})}{m_r(m_1+m_2+\ldots+m_r-1)}$		
$P_m \times P_n$	mn	$\frac{mn-1}{mn+1}$: m, n odd	$\frac{2}{mn+1}$		
$G(n,k)$	$2n$	≥ 1	$\geq \frac{2}{2n-1}$		
$Z_{n,1}$	$2n$	1 : n even	$\frac{2}{2n-1}$		
Q_n	2^n			$\frac{2^n+2}{2^n}$	$\frac{2^n+2}{2^{2n}}$

- Better definitions for Tenacity and Toughness and preventing infinity values for complete graphs
- The calculated values for the above mentioned parameters are independent of n (number of vertices/ edges) and they lie in the interval [0,1]
- Since the formulas give definite value for any graph, it is possible to compare two entirely different graphs with different number of vertices/edges
- It is more convenient to study graphs regardless of their number of edges and vertices and the issues regarding graph structures, such as Hamiltonian cycles can be understood and discussed more conveniently
- Since the Normalized Tenacity and Normalized Toughness lie in interval [0,1], we can apply probabilistic and statistical techniques to analyze compound issues

Although some properties of regular Toughness and Tenacity have been improved, making these two parameters become more useful parameters to apply in graph theory concept, consequently; but, the modifications did not apply any changes to Monotonicity and Computational Complexity of them. The next effort would be to find the point from which the Hamiltonian graphs could be detected. It would be also useful to bring a connection between random graphs concepts and Normalized Toughness and Normalized Tenacity parameters of the graphs, in the future.

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