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One Modulo Three Geometric Mean Graphs

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ABSTRACT

A graph G is said to be one modulo three geometric mean graph if there is an injective function ϕ from the vertex set of G to the set $\{a \mid 1 \leq a \leq 3q-2\}$ and either $a \equiv 0(mod3)$ or $a \equiv 1(mod3)\}$ where q is the number of edges of G and ϕ induces a bijection ϕ^* form the edge set of G to $\{a \mid 1 \leq a \leq 3q-2$ and $a \equiv 1(mod3)\}$ given by $\phi^*(uv) = \left\lceil \sqrt{\phi(u)\phi(v)} \right\rceil$ or $\left\lfloor \sqrt{\phi(u)\phi(v)} \right\rfloor$ and the function ϕ is called one modulo three geometric mean labeling of G. In this paper, we establish that some families of graphs admit one modulo three geometric mean labeling.

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1 Introduction

All graphs considered here are simple, finite, connected and undirected. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. We follow the basic notations and terminology of graph theory as in [2]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and

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a detailed survey of graph labeling can be found in [1]. The concept of mean labeling was introduced by Somasundaram and Ponraj [4]. A graph $G = (p, q)$ with p vertices and q edges is called a mean graph if there is an injective function f that maps $V(G)$ to $\{0, 1, 2, 3, ..., q\}$ such that for each edge uv, is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u) + f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u) + f(v)$ is odd. Jeyanthi and Maheswari introduced the concept of one modulo three mean labeling in [3]. A graph G is called one modulo three mean graph if there is an injective function ϕ from the vertex set of G to the set $\{a \mid 0 \leq a \leq 3q-2$ and either $a \equiv 0(mod3)$ or $a \equiv 1(mod3)\}$ where q is the number of edges of G and ϕ induces a bijection ϕ^* from the edge set of G to $\{a \mid 1 \leq a \leq 3q - 2$ and either $a \equiv 1(mod3)\}$ given by $\phi^*(uv) = \left\lceil \frac{\phi(u)+\phi(v)}{2} \right\rceil$ and the function ϕ is called one modulo three mean labeling of G. The concept of geometric mean labeling was due to Somasundram et al.[5]. A graph $G = (V, E)$ with p vertices and q edges is said to be geometric mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, ..., q + 1$ in such a way that when each edge $e = uv$ is labeled with $f(e = uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$ or $\left\lfloor \sqrt{f(u)f(v)} \right\rfloor$, then the resulting edge labels are all distinct. In this case, the function f is called geometric mean labeling of G.

Motivated by the concepts in [3], [5] we define a new type of labeling called one modulo three geometric mean labeling as follows: A graph G is said to be one modulo three geometric mean graph if there is an injective function ϕ from the vertex set of G to the set $\{a \mid 1 \leq a \leq 3q - 2$ and either $a \equiv 0(mod3)$ or $a \equiv 1(mod3)\}$ where q is the number of edges of G and ϕ induces a bijection ϕ^* from the edge set of G to $\{a \mid 1 \leq a \leq 3q - 2$ and either $a \equiv 1(mod3)\}$ given by $\phi^*(uv) = \left\lceil \sqrt{\phi(u)\phi(v)} \right\rceil$ or $\left\lfloor \sqrt{\phi(u)\phi(v)} \right\rfloor$ and the function ϕ is called one modulo three geometric mean labeling of G.

Remark: If G is a one modulo three geometric mean graph, then $1, 3$ and $3q - 2, 3q - 3$ must be appear as the vertex labels.

We begin with a brief summary of definitions which are necessary for the present study.

Definition 1.1. *The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is defined as a graph obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2.*

Definition 1.2. *A Cartesian product of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ such that its vertex set is a cartesian product of $V(G_1)$ and $V(G_2)$ i.e. $V(G_1 \times G_2) = V(G_1) \times V(G_2) = \{(x, y)/x \in V(G_1), y \in V(G_2)\}$ and its edge set is defined as $E(G_1 \times G_2) = \{((x_1, x_2), (y_1, y_2))/x_1 = y_1$ and $(x_2, y_2) \in E(G_2)$ or $x_2 = y_2$ and $(x_1, y_1) \in E(G_1)\}$.*

Definition 1.3. *The graph $P_n \times P_2$ is called a ladder graph.*

Definition 1.4. *The graph obtained by joining a single pendant edge to each vertex of a path is called a comb graph.*

Definition 1.5. *Let G be a graph. The subdivision graph $S(G)$ is obtained from G by subdividing each edge of G with a vertex.*

2 One modulo three geometric mean graphs

Theorem 2.1. *The path P_n is a one modulo three geometric mean graph.*

Proof. Let the vertex set $V(P_n) = \{u_1, u_2, ..., u_n\}$ and the edge set $E(P_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\}$. Clearly it has n vertices and $n-1$ edges. Define the vertex labeling ϕ as $\phi : V(P_n) \to \{1, 3, ..., 3n-5\}$ by $\phi(u_1) = 1$, $\phi(u_i) = 3(i-1)$ if $2 \leq i \leq n-1$ and $\phi(u_n) = 3n-5$. It can be verified that the induced edge labels of P_n are $1, 4, ..., 3n-5$. Hence ϕ is a one modulo three geometric mean labeling of P_n. Therefore, P_n is a one modulo three geometric mean graph. □

Theorem 2.2. *If $n > 2$, $K_{1,n}$ is not a one modulo three geometric mean graph.*

Proof. Let $n > 2$. Suppose $K_{1,n}$ is a one modulo three geometric mean graph with labeling ϕ. Let (V_1, V_2) be the bipartition of $K_{1,n}$ with $V_1 = \{u\}$ and $V_2 = \{u_1, u_2, ..., u_n\}$. To get the edge label $3q-2$, we must have $3q-2$ and $3q-3$ as the vertex labels of the adjacent vertices. Therefore, either $\phi(u) = 3q-2$ or $\phi(u) = 3q-3$. In both cases, since $q > 2$, there will be no edge whose label is 1. This contradiction proves that $K_{1,n}$ is not a one modulo three geometric mean graph for $n > 2$. □

Theorem 2.3. *The comb graph is a one modulo three geometric mean graph.*

Proof. Let G be a comb graph obtained from the path $u_1, u_2, ..., u_n$ by joining a vertex u_i to v_i, $1 \leq i \leq n$. Now G has $2n$ vertices and $2n-1$ edges. Define the vertex labeling ϕ as $\phi : V(G) \to \{1, 3, ..., 6n-5\}$ by $\phi(u_1) = 3$, $\phi(v_1) = 1$ and $\phi(u_i) = 6i-5$ if $2 \leq i \leq n$, $\phi(v_i) = 6(i-1)$ if $2 \leq i \leq n$. Then the induced edge labels of G are $1, 4, ..., 6n-5$. Hence ϕ is a one modulo three geometric mean labeling of G. □

Theorem 2.4. *The graph G obtained by attaching a path of length two at each vertex of the path P_n, then the graph G is a one modulo three geometric mean graph.*

Proof. Let G be the graph obtained by attaching a path of length of two at each vertex of the path P_n. The vertex set $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_iv_i, v_iw_i : 1 \leq i \leq n\}$. Clearly it has $3n$ vertices and $3n-1$ edges. Define the vertex labeling $\phi : V(G) \to \{1, 3, ..., 9n-5\}$ as follows: $\phi(u_1) = 7$, $\phi(w_1) = 1$, $\phi(w_2) = 4$,

$$\phi(u_i) = \begin{cases} 9i-5 & \text{if } i \text{ is odd}, 2 \leq i \leq n \\ 9i-6 & \text{if } i \text{ is even}, 2 \leq i \leq n, \end{cases}$$

$$\phi(v_i) = \begin{cases} 9i-6 & \text{if } i \text{ is odd}, 1 \leq i \leq n \\ 9i-5 & \text{if } i \text{ is even}, 1 \leq i \leq n, \end{cases}$$

$$\phi(w_i) = \begin{cases} 9(i-1) & \text{if } i \text{ is odd}, 3 \leq i \leq n \\ 9i-11 & \text{if } i \text{ is even}, 4 \leq i \leq n, \end{cases}$$

It can be verified that the induced edge labels of G are $1, 4, ..., 9n-5$. Hence ϕ is a one modulo three geometric mean labeling of G. Thus the graph G is a one modulo three geometric mean graph. □

Theorem 2.5. *The graph $P_n \odot \overline{K_2}$ is a one modulo three geometric mean graph.*

Proof. Let $G = P_n \odot \overline{K_2}$. The vertex set $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and the edge set $E(G) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_iv_i, u_iw_i : 1 \leq i \leq n\}$. Clearly, it has $3n$ vertices and $3n-1$ edges. Define the vertex labeling $\phi : V(G) \to \{1, 3, ..., 9n-5\}$ as follows: $\phi(u_1) = 3$, $\phi(u_2) = 13$, $\phi(u_i) = 9i-6$ if $3 \leq i \leq n$, $\phi(v_1) = 1$, $\phi(v_i) = 9(i-1)$ if $2 \leq i \leq n$,

$$\phi(w_i) = \begin{cases} 6i & \text{if } 1 \leq i \leq 2 \\ 9i-5 & \text{if } 3 \leq i \leq n, \end{cases}.$$

It can be verified that the induced edge labels of G are $1, 4, ..., 9n-5$. Hence ϕ is a one modulo three geometric mean labeling of $P_n \odot \overline{K_2}$. Thus the graph $P_n \odot \overline{K_2}$ is a one modulo three geometric mean graph. □

Theorem 2.6. *The subdivision graph $S(P_n \odot K_1)$ is a one modulo three geometric mean graph.*

Proof. Let $G = S(P_n \odot K_1)$. The vertex set $V(G) = \{v_i, u_i, u_i'; 1 \leq i \leq n\} \bigcup \{v_i' : 1 \leq i \leq$

$n-1\}$ and the edge set $E(G) = \{v_i^{'}v_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_i v_i^{'}, v_i u_i^{'}, u_i^{'} u_i : 1 \leq i \leq n\}$. Clearly it has $4n-1$ vertices and $4n-2$ edges. Define the vertex labeling $\phi : V(G) \rightarrow \{1, 3, ..., 12n-8\}$ as follows: $\phi(u_1) = 1$, $\phi(u_2) = 12$, $\phi(v_1) = 6$, $\phi(u_{2i+1}) = 6(4i-1)$ if $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $\phi(u_{2i+2}) = 6(4i+1)$ if $1 \leq i \leq \left\lceil \frac{n+1}{2} \right\rceil - 2$, $\phi(u_{2i-1}^{'}) = 24i - 21$ if $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, $\phi(u_{2i}^{'}) = 24i - 8$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\phi(v_i^{'}) = 7$, $\phi(v_{2i+1}) = 24i + 4$ if $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $\phi(v_{2i}) = 24i - 9$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\phi(v_{2i+1}^{'}) = 24i + 12$ if $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $\phi(v_{2i}^{'}) = 24i$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. It can be verified that the induced edge labels of G are $1, 4, ..., 12n-8$. Hence ϕ is a one modulo three geometric mean labeling of G. Thus the graph $S(P_n \odot K_1)$ is a one modulo three geometric mean graph. □

Theorem 2.7. *The subdivision graph $S(P_n \odot \overline{K_2})$ is a one modulo three geometric mean graph.*

Proof. Let $G = S(P_n \odot \overline{K_2})$. The vertex set $V(G) = \{u_i, u_{i1}, u_{i2}, u_{i1}^{'}, u_{i2}^{'} : 1 \leq i \leq n\} \bigcup \{u_i^{'} : 1 \leq i \leq n-1\}$ and the edge set $E(G) = \{u_i u_i^{'}, u_i u_{i1}^{'}, u_i u_{i2}^{'}, u_{i1}^{'} u_{i1}, u_{i2}^{'} u_{i2} : 1 \leq i \leq n\} \bigcup \{u_i^{'} u_{i+1} : 1 \leq i \leq n-1\}$. Clearly it has $6n-1$ vertices and $6n-2$ edges. Define the vertex labeling $\phi : V(G) \rightarrow \{1, 3, ..., 18n-8\}$ as follows: $\phi(u_1) = 7$, $\phi(u_2) = 22$, $\phi(u_i) = 3(6i-5)$ if $3 \leq i \leq n$, $\phi(u_i^{'}) = 6(3i+1)$ if $1 \leq i \leq n-1$, $\phi(u_{11}) = 1$, $\phi(u_{12}) = 10$, $\phi(u_{i1}) = 3(6i-7)$ if $2 \leq i \leq n$, $\phi(u_{i2}) = 9(2i-1)$ if $2 \leq i \leq n$, $\phi(u_{11}^{'}) = 3$, $\phi(u_{12}^{'}) = 9$, $\phi(u_{i1}^{'}) = 2(9i-10)$ if $2 \leq i \leq n$, $\phi(u_{i2}^{'}) = 2(9i-4)$ if $2 \leq i \leq n$. It can be verified that the induced edge labels of G are $1, 4, ..., 18n-8$. Hence ϕ is a one modulo three geometric mean labeling of G. Thus the graph $S(P_n \odot \overline{K_2})$ is a one modulo three geometric mean graph.

An example for one modulo three geometric labeling $S(P_5 \odot \overline{K_2})$ is given in Figure 1.

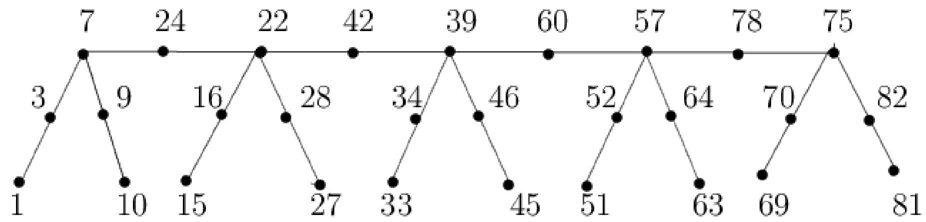

Figure 1:

□

Theorem 2.8. *If G is a graph in which every edge lies on a triangle, then G is not a one modulo three geometric mean graph.*

Proof. Let G be a graph in which every edge is an edge of a triangle. Suppose G is a one modulo three geometric mean graph. To get $3q-2$ on edge label, there must be two adjacent vertices u and v such that $f(u) = 3q-2$ and $f(v) = 3q-3$. Let $uvwu$ be a triangle in which on edge uv lies. To get $3q-5$ on edge label, there must be $f(w) = 3q-6$ or $3q-8$, then uw and vw get the same edge label. This is a contradiction to the fact of one modulo three geometric mean labeling. Hence G is not a one modulo three geometric mean graph. □

Corollary 2.9. *The complete graph K_n where $n \geq 3$, the wheel W_n, the triangular snake, double triangular snake, triangular ladder, flower graph FL_n, fan $P_n + K_1$, $n \geq 2$, double fan $P_n + K_2$, $n \geq 2$, friendship graph C_3^n, windmill K_m^n, $m \succ 3$, square graph $B_{n,n}^2$, total graph $T(P_n)$ and composition graph $P_n[P_2]$ are not one modulo three geometric mean graphs.*

Theorem 2.10. *The cycle C_n is not a one modulo three geometric mean graph for $n = 3, 4$.*

Proof. When $n = 3$. $C_3 = K_3$. By Corollary 2.9, K_3 is not a one modulo three geometric mean graph. Therefore C_3 is not a one modulo three geometric mean graph. When $n = 4$, let $C_4 = u_1u_2u_3u_4$. Suppose C_4 is a one modulo three geometric mean graph. By Remark 2.2, 1,3 and 9,10 there must be the vertex labels of adjacent vertices. Without loss of generality we assume that $\phi(u_1) = 1$, $\phi(u_2) = 3$ and $\phi^*(u_1u_2) = 1$. To get 10 as edge label, we must have either $\phi(u_3) = 9$, $\phi(u_4) = 10$ or $\phi(u_3) = 10$, $\phi(u_4) = 9$. In both cases $\phi^*(u_2u_3) = 5$ or $\phi^*(u_1u_4) = 3$. This is a contradiction to the fact that the edge labels are congruent to one modulo three. Therefore C_4 is not a one modulo three geometric mean graph. □

Theorem 2.11. *The cycle C_n is a one modulo three geometric mean graph for $n \geq 5$.*

Proof. Let C_n be the cycle $u_1u_2, ..., u_n, u_1$. Define the vertex labeling $\phi : V(C_n) \to \{1, 3, ..., 3n-2\}$ by considering the following two cases.

Case(i). n is odd, $n \geq 5$.

$\phi(u_1) = 1$, $\phi(u_i) = 10i - 17$ if $i = 2, 3$,

$$\phi(u_4) = \begin{cases} 19 & \text{if } n = 7 \\ 15 & \text{if } n > 7 \end{cases},$$

$\phi(u_n) = 10$, $\phi(u_{n-2}) = 21$,

$$\phi(u_{n-1}) = \begin{cases} 9 & \text{if } n = 7 \\ 12 & \text{if } n > 7 \end{cases},$$

$\phi(u_{\lceil \frac{n}{2} \rceil + 1}) = 3(n-1)$ and if $n \geq 9$, $\phi(u_i) = 6i-5$ if $5 \leq i \leq \lceil \frac{n}{2} \rceil$, if $n \geq 11$, $\phi(u_{n-i}) = 6i+4$ if $3 \leq i \leq \lceil \frac{n}{2} \rceil - 3$.

Case(ii). n is even. $n \geq 8$.

$\phi(u_1) = 1$, $\phi(u_2) = 3$, $\phi(u_i) = 2i + 7$ if $i = 3, 4$ and $\phi(u_i) = 6i - 8$ if $5 \leq i \leq \frac{n+2}{2}$, $\phi(u_{\frac{n+4}{2}}) = 3(n-1)$, $\phi(u_{n-i+1}) = \begin{cases} 2i + 8 & \text{if } i = 1, 2 \\ 6i + 1 & \text{if } 3 \leq i \leq \frac{n-4}{2} \end{cases}$.

If $n = 6$, we define the labeling as $\phi(u_1) = 1$, $\phi(u_2) = 3$, $\phi(u_3) = 16$, $\phi(u_4) = 15$, $\phi(u_5) = 12$ and $\phi(u_6) = 10$. It can be verified that the induced edge labels of C_n are $1, 4, ...3n-2$. Hence ϕ is a one modulo three geometric mean labeling of C_n. Thus the graph C_n is a one modulo three geometric mean graph. □

Theorem 2.12. *The ladder graph $L_n = P_n \times P_2$ is a one modulo three geometric mean graph.*

Proof. Let the vertex set $V(L_n) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and the edge set $E(L_n) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_iv_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_iv_i : 1 \leq i \leq n\}$. Clearly L_n has

$2n$ vertices and $3n-2$ edges. Define the vertex labeling $\phi : V(L_n) \rightarrow \{1, 3, ..., 9n-8\}$ as follows: $\phi(u_1) = 1$, $\phi(v_1) = 3$. If $n > 3$, $\phi(u_i) = 9i - 6$ if $4 \leq i \leq n-1$ $\phi(v_i) = 9(i-1)$ if $4 \leq i \leq n$, $\phi(u_n) = 9n - 8$, $\phi(u_2) = 10$, $\phi(u_3) = 12$, $\phi(v_i) = \begin{cases} 6i+7 & \text{if } \ i = 2 \\ 6i+6 & \text{if } \ i = 3 \end{cases}$.
If $n = 3$, we define the labeling as $\phi(u_2) = 13$, $\phi(u_3) = 9$, $\phi(v_2) = 18$, $\phi(v_3) = 19$. It can be verified that the induced edge labels of L_n are $1, 4, ..., 9n-8$. Hence ϕ is a one modulo three geometric mean labeling of L_n. Thus the graph L_n is a one modulo three geometric mean graph. □

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