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Pointed Conflict-Free Colouring of Digraphs

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ABSTRACT

In a pointed conflict-free partial (PCFP) colouring of a digraph, each vertex has at least one in-neighbour with unique colour. In this paper, it is proved that PCFP k-colourability of digraphs is NP-complete, for any $k > 0$. Nevertheless for paths and cycles, one can in linear time find a PCFP colouring with a minimum number of colours and for a given tree, one can find a PCFP 2-colouring. In this paper a bipartite digraph whose arcs start from the same part is called a one-way bipartite digraph. It is proved every one-way bipartite planar digraph has a PCFP 6-colouring, every one-way bipartite planar digraph whose each vertex has in-degree zero or greater than one, has a PCFP 5-colouring and every one-way bipartite planar digraph whose each vertex has in-degree zero or greater than two, has a PCFP 2-colouring. Two simple algorithms are proposed for finding a PCFP colouring of a given digraph such that the number of colours used is not more than the maximum out-degree of the vertices. For a digraph with a given PCFP colouring, it is shown how to recolour the vertices after vertex or arc insertion or deletion to obtain a PCFP colouring for the new digraph.

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1 Introduction

Conflict-free colouring of graphs and hypergraphs was defined and studied in [12, 23], motivated by its application in the frequency assignment problem in cellular networks.
A conflict-free k-colouring of a hypergraph assigns one of k different colours to some of the vertices of the hypergraph such that, for every edge, there is a vertex whose colour is unique.
A conflict-free k-colouring of a graph assigns one of k different colours to some of the vertices of the graph such that, for every vertex, there is a vertex in its neighbourhood whose colour is unique.
Most of the research on conflict-free colouring has considered geometric graphs [2, 3, 4, 8, 9, 11, 12, 16, 17, 19, 23] and some others consider abstract graphs and hypergraphs [1, 6, 15, 18, 21].
Pach and Tardos in [21] proved that $O(\log^2 n)$ colours is sufficient for conflict-free colouring of a graph with n vertices. Gargano and Rescigno show that finding a conflict-free colouring with the minimum number of colours for general graphs is NP-complete. In [1], conflict-free colouring of graphs is studied when every vertex is considered as a member of its neighbourhood. By this consideration, for $k = 1, 2$, it is proved that having a conflict-free k-colouring even for planar graphs is NP-complete and it is shown that three colours is always sufficient for a conflict-free colouring of every planar graph. For a more detailed overview, see [1].
With the motivation of using directed networks, in this paper, we have defined pointed conflict-free partial colouring for digraphs. By our definition conflict-free colouring of hypergraphs and also proper colouring of graphs can be reduced to conflict-free colouring of digraphs. Therefore the conflict-free colouring problem for digraphs is not easier than the proper colouring problem of graphs.
After giving preliminaries in the second section, in the third section, we will prove that PCFP k-colourability of one-way bipartite digraphs is an NP-complete problem for any $k > 0$. In the fourth section, we will prove that every one-way bipartite planar digraph D has a PCFP 6-colouring. Furthermore, it is proved that every one-way bipartite planar digraph whose vertices each have in-degree zero or greater than one has a PCFP 5-colouring and every one-way bipartite planar digraph whose vertices each have in-degree zero or greater than two has a PCFP 2-colouring. In the fifth section, for paths and cycles with n vertices, we will show how to find a conflict-free colouring with the minimum number of colours in $O(n)$ time and also how to find a PCFP 2-colouring for a given tree. In the sixth section, two algorithms will be proposed for finding a conflict-free colouring of a digraph such that the number of used colours is not more than the maximum out-degree of the vertices. In the seventh section, PCFP dynamic colouring will be considered and in the last section, the paper will be concluded.

2 Preliminaries

A proper k-colouring of a graph G is a colouring of the vertices of G with k or fewer colours such that no adjacent vertices have the same colour.
A conflict-free partial (CFP) k-colouring of a hypergraph assigns one of k different colours to some of the vertices of the hypergraph such that, for every edge, there is a colour that is assigned to exactly one vertex in the edge.
If v is a vertex of a graph, define $N(v)$ to be the set of all vertices adjacent to v. If v is a vertex of a digraph, define $IN(v)$ to be the set of all in-neighbours of v.
For a graph $G(V, E)$, the hypergraph whose vertex set is V and whose edge set is $\{N(v) \mid v \in V\}$ is denoted by $H^{\cdot}(G)$. A pointed conflict-free partial (PCFP) k-colouring of a graph G is a CFP k-colouring of $H^{\cdot}(G)$.
For a digraph $D(V, A)$, the hypergraph whose vertex set is V and whose edge set is $\{IN(v) \mid v \in V\}$ is denoted by $H^{\cdot}(D)$. A pointed conflict-free partial (PCFP) k-colouring of a digraph D is a CFP k-colouring of $H^{\cdot}(D)$. Considering a PCFP k-colouring of a digraph, a vertex u is a PCFN of a vertex v iff u is an in-neighbour of v with unique colour.
In a digraph a vertex with no out-neighbours and at least one in-neighbour is called an in-vertex and a vertex with no in-neighbours and at least one out-neighbour is called an out-vertex.
A digraph is a bipartite digraph if whose underlying graph is a bipartite graph. A digraph is a one-way bipartite digraph if whose each none isolated vertex is an in-vertex or an out-vertex.
Let H be a hypergraph with vertex set V and edge set E. Let $D(H)$ be the one-way bipartite digraph whose vertex set is $V \cup E$ and there is an arc from vertex x to edge e if e contains x. A PCFP k-colouring of $D(H)$ corresponds to a CFP k-colouring of H.
If D is an arbitrary digraph, $D(H^{\cdot}(D))$ is a one-way bipartite digraph and a PCFP k-colouring of D corresponds to a PCFP k-colouring of $D(H^{\cdot}(D))$.
The number of vertices of $D(H^{\cdot}(D))$ is not less than the number of vertices of D and the number of arcs of $D(H^{\cdot}(D))$ is not more than the number of arcs of D. So $D(H^{\cdot}(D))$ is not denser than D. There are lots of digraphs D which are not planar yet $D(H^{\cdot}(D))$ is planar. For example, consider D_0 be the one-way bipartite digraph whose underlying graph is $K_{3,3}$; after removing isolated vertices of $D(H^{\cdot}(D_0))$, the digraph obtained is a one-way bipartite digraph whose underlying graph is $K_{3,1}$. So $D(H^{\cdot}(D_0))$ is planar while D_0 is not planar.
A digraph D is called semi-planar iff $D(H^{\cdot}(D))$ is planar.
For one of our NP-hardness proofs, we use the POSITIVE PLANAR 1-IN-3-SAT problem which was introduced and proved to be NP-complete in [20]. Let ϕ be a formula in 3-CNF with clause set $C = \{c_1, c_2, ..., c_l\}$ and variable set $X = \{x_1, x_2, ..., x_n\}$. Graph $G(\phi)$ with vertex set $X \cup C$ and edge set $E(\phi) = \{x_i c_j \mid x_i \in X, c_j \in C, x_i$ occurs in $c_j\} \cup \{x_i x_{i+1} \mid i = 1, ..., n\}$ is the associated graph of ϕ. The formula ϕ is called backbone planar if $G(\phi)$ is planar and positive planar iff it is both positive and backbone planar. The problem is whether ϕ is 1-in-3-satisfiable; i.e., is it possible to assign boolean values

to the variables such that each clause contains exactly one true variable.
For simplicity without lost of generality, in this paper we only consider connected digraphs.

3 Complexity results

In this section, we prove some complexity results for the problem of PCFP colouring of digraphs. The first theorem shows that PCFP 1-colourability is an NP-complete problem even for one-way bipartite graphs.

Theorem 1. *It is NP-complete to decide whether a one-way bipartite planar digraph whose in-vertices all have in-degree 3 has a PCFP* 1*-colouring.*

Proof. Membership in NP is obvious. The proof of NP-hardness is done by a reduction from the POSITIVE PLANAR 1-IN-3-SAT problem. Consider a POSITIVE PLANAR 1-IN-3-SAT problem ϕ.
In graph $G(\phi)$, remove all the edges in the set $\{x_i x_{i+1} \mid i = 1, ..., n\}$ and direct every edge $x_i c_j$ from x_i to c_j. The resulting digraph D is a one-way bipartite digraph whose in-vertices all have in-degree 3. Digraph D has a PCFP 1-colouring iff ϕ is 1-in-3-satisfiable. □

The following theorem shows that PCFP $(k-1)$-colourability problem for digraphs is at least as hard as proper k-colourability problem for graphs.

Theorem 2. *For every integer $k > 0$, the proper k-colouring problem for graphs can be reduced to PCFP $(k-1)$-colouring problem for one-way bipartite digraphs.*

Proof. Let G be a graph. Replace each edge uv of G with a vertex w_{uv} and two arcs (u, w_{uv}) and (v, w_{uv}). The resulting digraph is a one-way bipartite digraph D. Consider a proper k-colouring of G and remove the colour of every vertex which is coloured with colour k. Since, for every in-vertex w_{uv}, at least one of u and v is not coloured with colour k, then w_{uv} has a PCFN.
Conversely, considering a PCFP $(k-1)$-colouring Γ of D, colour all the uncoloured vertices with colour k. Since for each edge uv at least one of them is coloured in Γ, so u and v are coloured with different colours. □

The following are suggested by Theorem 2.

Proposition 1. *It is NP-complete to decide whether a one-way bipartite digraph whose in-vertices all have in-degree 2 has a PCFP k-colouring for any $k > 1$.*

Proposition 2. *Every one-way bipartite planar digraph whose in-vertices all have in-degree 2 has a PCFP* 3*-colouring.*

Proof. These follow from Theorem 2 and the fact that every planar graph has a proper 4-colouring [5]. □

Proposition 3. *It is NP-complete to decide whether a one-way bipartite planar digraph whose in-vertices all have in-degree 2 has a PCFP* 2*-colouring.*

Proof. It is NP-complete to decide whether a planar graph has a proper 3-colouring [14]. So by Proposition 2, the proposition is proved. □

Proposition 4. *There are some planar digraphs that need at least 3 colours for a pointed conflict-free colouring.*

Proof. For graph K_4, replace each edge uv of G with a vertex w_{uv} and two arcs (u, w_{uv}) and (v, w_{uv}). The resulting digraph is planar and needs three colours for a PCFP colouring. □

4 PCFP colouring of one-way bipartite planar and semi-planar digraphs

For a plane graph G, a CF k-colouring of G with respect to faces is a k-colouring of G such that for each face f of G, at least one of the vertices on the boundary of f has a unique colour.
In this section we need to use a theorem which is proved in [13] and is about CF k-colouring of a graph with respect to faces.

Theorem 3 ([13])**.** *Every simple plane graph has a 3-colouring with colours black, blue and red such that*
(1) each face is incident with at most one red vertex, and
(2) each face that is not incident with a red vertex is incident with exactly one blue vertex.
(3) the end-vertices of an edge are not both coloured blue.

Note that in [13] property (3) is not in the theorem but in the lemma used for proving the theorem.

Theorem 4. *Every one-way bipartite planar digraph whose in-vertices each have in-degree at least 3 has a PCFP 2-colouring. Therefore, every semi-planar digraph D whose vertices with at least one in-neighbour have in-degree at least 3 has a PCFP 2-colouring.*

Proof. Let D be a one-way bipartite planar digraph whose in-vertices each have in-degree at least 3. Embed D in the plane. Let G be the graph whose vertex set is the set of all out-vertices of D and two vertices are adjacent iff they are in-neighbours of at least one in-vertex in the same face of D. Graph G is a simple plane graph. By Theorem 3, G has a 3-colouring with colours black, blue and red such that
(1) each face is incident with at most one red vertex, and
(2) each face that is not incident with a red vertex is incident with exactly one blue vertex.
This colouring is a PCFP 3-colouring of D such that every in-vertex of D has a PCFN with colour red or blue. Call this colouring Γ. Considering Γ if we remove the colour of every vertex coloured by black, the resulting 2-colouring of D is a PCFP 2-colouring. □

In the next theorem we try to find a PCFP 5-colouring for a one-way bipartite planar digraph whose in-vertices each have at least two in-neighbours.

Theorem 5. *Every one-way bipartite planar digraph whose in-vertices each have in-degree at least 2 has a PCFP 5-colouring. Therefore, every semi-planar digraph D whose vertices with at least one in-neighbour have in-degree at least 2 has a PCFP 5-colouring.*

Proof. Define graph G as in the proof of Theorem 4. Graph G is a simple graph and by Theorem 3, G has a 3-colouring with colours black, blue and red such that
(1) each face is incident with at most one red vertex,
(2) each face that is not incident with a red vertex is incident with exactly one blue vertex, and
(3) two end vertices of an edge are not both coloured with blue.
If we uncolour vertices of D which are coloured with black, every vertex with in-degree more than 2, has a PCFN. Let v be an in-vertex with in-degree 2 and with no PCFN. By (1), the in-neighbours of v are not coloured with red and by (3), they are not coloured with blue. So the in-neighbours of v are uncoloured. Let G' be the subgraph of G inducing on $V' = \{u \mid u \text{ is a vertex of } D \text{ which is an in-neighbour of a vertex with no PCFN}\}$. Find a proper 4-colouring of the plane graph G' with colours black, green, yellow and white. Now, remove the colour of every vertex coloured with white. The resulting 5-colouring is a PCFP 5-colouring of D. □

Proposition 5. *Every one-way bipartite planar digraph D, has a PCFP 6-colouring. Therefore, every semi-planar digraph D has a PCFP 6-colouring.*

Let D be a bipartite digraph. If each of the maximal one-way bipartite subdigraphs of D is planar then D is semi-planar and following is true by Theorems 4 and 5 and Proposition 5.

Proposition 6. *Let D be a bipartite digraph whose maximal one-way bipartite subdigraph are all planar. Then D has a PCFP 6-colouring and also:*
(1) if each vertex with at least one in-neighbour has in-degree at least two, D has a PCFP 5-colouring,
(2) and if each vertex with at least one in-neighbour has in-degree at least three, D has a PCFP 2-colouring.

Since for each hypergraph H, a CF k-colouring of H corresponds to a PCFP k-colouring of digraph $D(H)$, therefore we can propose the following by Theorems 4 and 5 and Proposition 5.

Proposition 7. *Let H be a hypergraph. If $D(H)$ is planar, then H has CFP 6-colouring and also:*
(1) if each edge has cardinality at least two, H has a CFP 5-colouring,
(2) and if each edge has cardinality at least three, H has a CFP 2-colouring.

5 PCFP colouring of simple digraphs

5.1 PCFP colouring of paths

Theorem 6. *There is a linear time algorithm that finds a PCFP colouring with the minimum number of colours for any given path P.*

Proof. If the length of path P is one, colour the out-vertex with colour 1. Otherwise, if P contains a directed subpath with length at least 2, then let v_1, v_2, v_3 be such a directed subpath. Divide P to two subpaths P_1 and P_2 such that $P = P_1 \cup P_2$, $P_1 \cap P_2 = \{v_2\}$, P_1 contains v_1 and P_2 contains v_3. Recursively, find PCFP colourings with minimum number of colours for P_1 and P_2. Colouring v_2 with its colour in P_2 and colouring other vertices as their colour in P_1 and P_2 gives a PCFP colouring of P with the minimum number of colours.
If P contains no directed subpaths with length at least 2, so every vertex of P is an in-vertex or an out-vertex and they appear in P alternately. If one of the end vertices is an out-vertex, let u, v and w be three first vertices from that end of P such that v is an in-vertex, u is an out-vertex and also u is an end vertex of P. Remove u and v, find a PCFP colouring with the minimum number of colours. Then insert the removed vertices. If w is uncoloured, colour u with colour one. In the case that both end vertices of P are in-vertices, let $u_1, ..., u_l$, be out-vertices of P in the order they appear in P. Colour u_{2i-1}, $i = 1, ..., [(l + 1)/2]$, with colour 1. If l is an even number, colour u_l with colour 2. □

5.2 PCFP colouring of cycles

Theorem 7. *There is a linear time algorithm that finds a PCFP colouring with the minimum number of colours for any given cycle C .*

Proof. If the length of cycle C is two, in the case that C is directed, colour both vertices with colour 1 and otherwise colour the out-vertex with colour 1. If the length of cycle C is more than 2 and C contains a directed subpath with length at least 2, then let v_1, v_2, v_3 be such a directed subpath. Remove v_2 and insert vertices v' and v'' and arcs $(v_1, v)'$ and (v'', v_3), find a PCFP colouring with the minimum number of colours for the resulting path. Colour vertices of C as in the path and colour v_2 with the colour of v''.
Otherwise, let $u_1, ..., u_l$, be the out-vertices of C in the order they appear on C. Colour u_{2i-1}, $i = 1, ..., [(l - 1)/2]$, with colour 1. If l is an odd number, colour u_l with colour 2. □

5.3 PCFP colouring of trees

Theorem 8. *There is a linear time algorithm that finds a PCFP 2-colouring for a given tree T.*

Proof. Let v be an out-vertex of T with out-neighbours $v_1, ..., v_l$. Removing v gives l subtrees $T_1, ..., T_l$. Suppose T_i be the subtree including v_i. Find a PCFP 2-colouring of

T_i such that if v_i is not an out-vertex in T_i, then it has a PCFN coloured with colour 1. By colouring v with colour 2 we obtain a PCFP 2-colouring of T. □

6 Algorithms

6.1 Algorithm 1

Consider a digraph D with vertex set $\{v_1, v_2, ..., v_n\}$. Start with v_1 and continue in order. For vertex v_i, if v_i has at least one out-neighbour which does not have any PCFN yet, colour v_i with the least colour c such that no out-neighbour of v_i is labelled by c^*. Label all unlabelled out-neighbours of v_i with label c^*. It is clear that the number of colours used by the algorithm is not more than the maximum out-degree of D and the complexity of the algorithm is $O(m)$.

6.2 Algorithm 2

Let D be a digraph with vertex set V such that D has no vertex with only one in-neighbour. Consider a graph G whose vertex set is V and each of its edges is labelled such that for every three vertices u, v and w of G, there is an edge uv with label w iff $u, v \in IN(w)$.
Choose a maximal independent set I_1 in V and colour its vertices with colour one. Remove all the vertices in I_1 from G. Then for each vertex v in I_1 and every label l of an edge incident to v, remove all edges of G labelled with l. Using a new colour, repeat this procedure until the resulted graph has no edges.
Consider the end of a step. For each vertex v of D which has not yet been removed, vertex v has at least one out-neighbour w in D such that w has at least one in-neighbour which is coloured in this step. So at least one of the out-neighbours of v which had not any PCFN at the start of this step, has a PCFN at the end of the step. Therefore the number of colours used by the algorithm is not more than the maximum out-degree of vertices of D. We expect that the algorithm has even better experimental performance.
If digraph D has some in-vertices with only one in-neighbour, first remove the incoming arcs of these vertices, then find a PCFP colouring of the resulting digraph. Reinsert the removed arcs. Let v be a vertex with exactly one in-neighbour u such that u is not coloured. There is at least one colour c not greater than the out-degree of u such that if u is coloured with c then every out-neighbour of u has at least one PCFN.

7 Dynamic PCFP colouring

In this section, we consider a digraph with a given PCFN colouring and investigate after deletion or insertion of a vertex or an arc how to recolour the digraph to obtain a PCFP colouring.

7.1 Vertex insertion

Let D be a digraph with a given PCFP colouring with colours in the set $C = \{1, ..., k\}$. Suppose a vertex v with some in-neighbours and some out-neighbours is inserted into D. We will recolour some vertices to find a PCFP colouring of the new digraph.

At first we need to define some notations. For a vertex u and an out-neighbour x of u, define $L(u, x) = \{c \in C \mid \text{ if we colour } u \text{ with colour } c, \text{ then} x \text{ still has at least a PCFN}\}$. It is easy to see if x has at least two PCFN other than u then $L(u, x) = C$ and otherwise if x has exactly one PCFN $z(x)$ other than u then $L(u, x) = C - \{\text{the colour of } z(x)\}$ and in the case u is the only PCFN of x then $L(u, x) = \{c \in C \mid c \text{ is not the colour of an in-neighbour of } x \text{ except } u\}$. Note that if $u = v$, the case that v is the PCFN of x does not happen; instead v can be the only in-neighbour of x and in this case $L(v, x) = C$. Define $L(u) = \cap_{u \in IN(x)} L(u, x)$ and $L = \cup_{u \in IN(v)} L(u)$.

For a colour $c \in C$ if there is an in-neighbour of v coloured with c define $V(c) = \{u \in IN(v) \mid u \text{ is coloured with } c\}$.

In part A, we try to find a PCFN for v, if v has at least an in-neighbour and in part B, we try to colour v with a suitable colour in case v has an out-neighbour whose only in-neighbour is v.

Part A: If v has no in-neighbour or has a PCFN, we are done. Otherwise find set L.

If there is a vertex $u \in IN(v)$ such that $L(u)$ contains a colour l^* which is not used for any in-neighbour of v, then colour u with l^*. Otherwise, for each colour c used by at least one in-neighbour of v consider the induced subdigraph $D(c)$ of D containing vertices in $V(c)$ and their out-neighbours except v. A component of $D(c)$ is a good component if there is a colour c' other than c which is in $L(w)$, for every vertex w in the component. Call the colour c' a common colour.

A component of $D(c)$ is a not bad component if there is a colour c' other than c which is in $L(w)$, for every vertex w in the component except for one of them. Call that vertex the bad vertex and call the colour c' the common colour.

If there is a colour l which is used for at least two in-neighbours of v and all of the components of $D(l)$ are good. Choose a vertex u of $V(l)$, and recolour all the vertices of $V(l)$ except u such that not considering u all the vertices of $V(l)$ which are in a same component of $D(l)$ coloured with the same colour and every vertex x is coloured with a colour in $L(x)$.

Suppose there is a colour l which is used for at least two in-neighbour of v and $D(l)$ has exactly one not bad component with bad vertex w and all other components of $D(l)$ are good. For any component C of $D(l)$, if $D(l)$ does not contain w, recolor all the vertices of C with a common color of C. Otherwise, recolor all the vertices of C except w with a common color of C.

If none of the above cases happens, colour one of the in-neighbours of v with colour $k+1$.

Part B: If v has at least one out-neighbour such that v is its only in-neighbour, find $L(v)$. If $L(v)$ is not empty, colour v with a colour in $L(v)$ and otherwise, colour v with colour $k + 1$.

We call the explained algorithm vertex-insertion. This algorithms takes $O(kd(v))$ where

$d(v)$ is the number of inserted arcs.

Theorem 9. *Algorithm vertex insertion recolours some of the vertices of D such that every vertex with at least one in-neighbour has a PCFN.*

Proof. In the cases: (1) vertex v has a PCFN without recolouring, (2) there is a vertex $u \in IN(v)$ such that $L(u)$ contains a colour l^* which is not used for any in-neighbour of v and (3) the algorithm uses colour $k+1$ and the proof is obvious, so we consider the other case.
It is obvious that v has a PCFN. Suppose the algorithm recolours the vertices $u_1, ..., u_t$. Let x be a vertex other than v with at least one in-neighbour in the set $U = \{u_1, ..., u_t\}$. If x has more than one in-neighbour in U, then l is not the colour of a PCFN of x and all in-neighbours of x which are in U are in the same component of $D(l)$ and after recolouring any of them are coloured with l or another colour l'. If x has exactly one PCFN before recolouring so the colour of PCFN of x is not l or l', then after recolouring that vertex is still a PCFN of x and if x has more than one PCFN before recolouring so the colour of at least one of the PCFN of x is not l and l', so after recolouring that vertex is still a PCFN of x.
Finally suppose that x has exactly one in-neighbour w in U. Then since w is coloured with a colour in $L(w, x)$, so x has at least a PCFN after recolouring. □

Using the insertion vertex algorithm, we can present some PCFP colouring algorithms. One idea is starting with one vertex and inserting the other vertices one by one. The other idea is investigating the planarity of $D(H^{\cdot}(D))$ for given digraph D. If $D(H^{\cdot}(D))$ is planar, we find a PCFP colouring of D as in the Theorems 4 and 5 and Proposition 5, otherwise, we remove some in-vertices of $D(H^{\cdot}(D))$ one by one to obtain a planar digraph D'. Then after finding a PCFP colouring of D', we insert the removed vertices one by one.

7.2 Vertex deletion

Consider removing a vertex u of a digraph with a given PCFN colouring. If u is uncoloured we are done. Otherwise for each vertex v with no PCFN except u, do part A of the vertex insertion algorithm.

7.3 Arc insertion

Consider inserting an arc (u, v) of a digraph with a given PCFN colouring. If v has only one PCFN and u is coloured with the PCFN colour of v, do Part A of the vertex insertion algorithm for v.

7.4 Arc deletion

Consider removing an arc (u, v) of a digraph with a given PCFN colouring. If u has been the only PCFN of v, do Part A of the vertex insertion algorithm for v.

8 Conclusion

In this paper, it is proved that PCFP k-colourability even for one-way bipartite digraphs is NP-complete, for any $k > 0$. Since PCFP k-colouring of a digraph D corresponds to PCFP k-colouring of $D(H^{\cdot}(D))$, it is good to investigate about the structure of $D(H^{\cdot}(D))$. Specially it would be interesting to characterize the digraphs D such that $D(H^{\cdot}(D))$ is planar. We know that there are some planar digraphs which need three colours for a PCFP colouring but it would be interesting to know if three colours is always sufficient. Using the two presented algorithm, we proved that every digraph has a PCFP k-colouring such that k is not more than the maximum out-degree. It would be good to find better upper bounds.

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