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Defining the Tipping Point Based on Conditionally Convergent Series: Explaining the Indeterminacy

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ABSTRACT

Tipping Point refers to the moment when an adaption or infection sustains itself in network without further external inputs. Until now, studies have mainly focused on the occurrence of the Tipping Point and what it leads to rather than what precedes it. This paper explores the situation leading to the Tipping Point during a process of diffusion in networks. The core of the debate is to manifest that the process can be introduced as an example of conditionally convergent series and that determining the tipping points occurrence is conditional to the arrangement of the series based on Reimann Rearrangement Theorem. Accordingly, the occurrence of curve does not follow a general formulation. That is called indeterminacy since that the predictions about tipping points for any diffusion over the network may include a variety of right answers, although such indeterminacy neither means there is no tipping point nor many.

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1 Introduction

"A Tipping point occurs when a trend, idea or an infection appears to take off on its own" [11]. It can be considered as a sputnik term for Bandwagon effect (or its outcome) which asserts that the probability of individual adoption increases with respect to the proportion of others who have already adopted [6]. The term is rooted in physics and its introduction to Sociology dates back to 1960s; however, the popularization of Tipping Point was not sooner than 2000, when Malcolm Gladwell used it as the name of his book. He applied the idea for explaining a variety of phenomena ranging from crime to advertising, all of which shared a similarity which was their drastic growth after a specific point [3]. Although Gladwell enriched his book by many examples, but -as Phillips observed- the term "remained formally undefined" [9]. Phillips compared Gladwell's Tipping Point with Kurzweil's idea of the "Knee" in a growth curve which was also characterized loosely. "Neither Kurzweil nor Gladwell provided a mathematical definition for a Tipping Point or Knee" [9]. While it seems easy for one to look at the diagram of a diffusion to see where it takes off, providing accurate definition -and respectively prediction- of Tipping Point is indeed challenging. Phillip' paper contains figures which explain what happens if we: A-continue the external input, B-cut it off at the Tipping Point or C- cut it off at some time-sequences sooner than tipping point, but still it doesn't answer why one cannot define and formulate the term based on such various examples. The current paper issues this problem, believing that inability of providing an absolute answer is the result of the nature of conditionally convergent series.

2 Towards a Definition

"The Tipping Point occurs when the adaption or the disease is self-sustaining without any external input" [6]. As a result, self-sustaining and cutting off the inputs is the cornerstone of Tipping Point's definition. Accordingly, before reaching to the Tipping Point, there must be a set of reducing values of inputs. Such inputs give the diffusion a "push", but because the diffusion is not yet self-sustained, it starts to fall back after a while; so there must be another push to lead it towards Tipping Point. At this stage, the diffusion may have become a little bit more self-sustained and the amount of essential attempts for creating the next push would decrease. On that account, in the next stages, new inputs should also have fewer values than their predecessors.

2.1 Inputs and Inverse-Inputs

Generally speaking, an input is any kind of stimulation pushing the trend, idea or infection to go further. It might have an internal or external source. However, not all diffusions produce logistic exponential S-curves. As Gersoki observes, for N potential adopters of a diffused innovation who are influenced from a central source, at time t, $y(t)$ number of them adopt the new diffusion and $y(t)$ is defined as:

$$y(t) = N\left\{1 - e^{-\alpha t}\right\}$$

In which α is the amount of population that the central input reaches to. The above function "is a modified exponential function [in which] the smaller is α, the slower is diffusion what is equally clear is that this particular diffusion process does not produce an S-curve it lacks an initial convex segment" [8]. Valente also differentiates between models of diffusion of innovation based on their internals, external or mixed sources of inputs. According to Valente, diffusions with a cumulative function of:

$$N\left(1 - e^{-at}\right)$$

and a derivative function of:

$$a * [N - y(t)]$$

also produces logistic curves for which the source of diffusion is "awareness" and the type of communication is "mass media" [5]. While diffusions with a cumulative function of:

$$\frac{Ne^{Nat}}{N - 1 + e^{Nat}}$$

and derivative function of:

$$a * y(t)\,[N - y(t)]$$

generate S-curves. [8][1]

The idea of input, in this paper, does not correspond to "non S-curve generating diffusions" because presence of a Knee in a growth curve is essential for defining the tipping point. As there are inputs that push the diffusion forward, there also are inputs that push it back. These inverse-inputs are also the continuous hits applied to the diffusion function but in the opposite direction. They can be the result of resistance, pressure of rival diffusions or anything else, whose attributed-quantities generate an indicator which we call the inverse-input. The inverse inputs share the same literature of inputs and for them too, the source of diffusion is "adoption" and the type of communication is "interpersonal".

2.2 Trajectory of Decreasing Inputs

In this part, we assume that actually there exists a Tipping Point because "though not supplemented by formal mathematical modeling, Gladwell's sociological observations are meticulous and strongly suggest the reality of Tipping Point of some sort in a variety of situations" [9]. Accordingly, there will be an ultimate point where the needed inputs can be cut off. Figure 1 shows the assumed process leading to such a point.

As shown in Figure 1, the first input leads the diffusion to the first extremum, and then the inverse-input (depicted by red vertical line) makes it fall back, so another input is

[1] In all of Valente's functions mentioned above: $Y(t)$ is the cumulative proportion of adopters, N is the population size, a is the rate of diffusion, t is the time period and e is the base of natural logarithm.

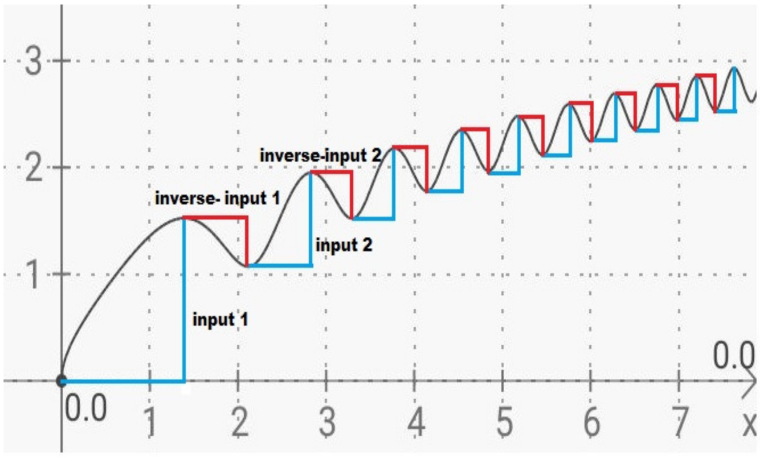

Figure 1: time-input trajectory diagram

needed before the diffusion becomes lost. The continuity of the applied inputs (blue vertical lines) shapes a process in which the amounts of essential inputs decreases and finally at some point equals zero. The below equation defines Figure 1:

$$y = \sqrt{\lambda \cdot sin(x^2) + x} \tag{1}$$

in which λ stands for peak-to-peak amplitude. When λ tends to zero, the diagram is no more jagged; however, increasing λ would make the maxima higher and minima lower. It's important to remember that Figure 1 is different from regular diagrams of diffusions which depict the frequency of adopters per time. Table (1) shows the values matching some of the first extrema in fig (1) when $\lambda = 1$.

The values attributed to Ys show the impact of applying inputs and revesrse inputs. From maxima to minima is where the inverse-inputs are applied and from minima to maxima is where the inputs are applied. Subtracting the values will show the quantities attributed to each one which -in turn- generate a series converging to the Tipping Point. The following table shows the values of inputs and inverse-inputs generated from subtracting the Ys in Table 1:

The input and inverse-input rows in Table 2 have their own reducing pattern; each one tending to zero on its own scale. The point where each of these applied inputs and inverse-inputs equals zero can be estimated by the following method:

Let's assume the sinusoidal function is stitching a logarithmic diagram: The stitches become tighter and tighter and ultimately two diagrams merge together. The following equation defines the blue diagram in fig 3:

$$y = \sqrt{k \cdot ln(x^2) + x} \tag{2}$$

In which "k" is a number very close to zero. As soon as Eq(2) and Eq(2) merge toghether, they will have equal integrals:

$$f(x_1) = \sqrt{\lambda. \cdot sin(x^2) + x}$$

Table 1: The values attributed to the first ten extrema

Point	X	Y
Max 1	1.4	1.525
Min 1	2.1	1.070
Max 2	2.85	1.953
Min 2	3.3	1.518
Max 3	3.8	2.180
Min 3	4.15	1.775
Max 4	4.55	2.350
Min 4	4.85	1.962
Max 5	5.2	2.484
Min 5	5.45	2.112
Max 6	5.75	2.598
Min 6	6	2.238
Max 7	6.28	2.696
Min 7	6.5	2.348
Max 8	6.76	2.784
Min 8	6.98	2.445
Max 9	7.2	2.864
Min 9	7.4	2.532
Max 10	7.62	2.937
Min 10	7.82	2.613

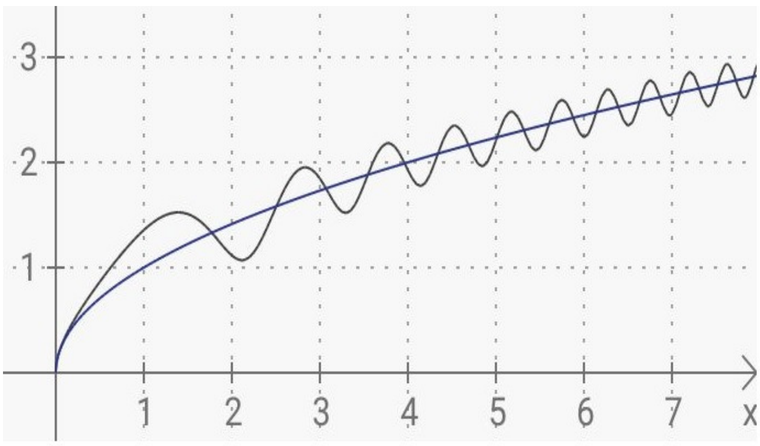

Figure 2: time-input trajectory diagram

Table 2: The values of inputs and inverse-inputs for the first ten extrema

From Max 1 To Min 1	0.445	Inverse-input
From Min 1 To Max2	0.883	Input
From Max 2 To Min 2	0.435	Inverse-input
From Min 2 To Max 3	0.662	Input
From Max 3 To Min 3	0.405	Inverse-input
From Min 3 To Max 4	0.575	Input
From Max 4 To Min 4	0.388	Inverse-input
From Min 4 To Max 5	0.522	Input
From Max 5 To Min 5	0.372	Inverse-input
From Min 5 To Max 6	0.486	Input
From Max 6 To Min 6	0.3605	Inverse-input
From Min 6 To Max 7	0.458	Input
From Max 7 To Min 7	0.348	Inverse-input
From Min 7 To Max 8	0.436	Input
From Max 8 To Min 8	0.339	Inverse-input
From Min 8 To Max 9	0.419	Input
From Max 9 To Min 9	0.332	Inverse-input
From Min 9 To Max 10	0.405	Input
From Max 10 To Min 10	0.324	Inverse-input

$$f(x_1) = \sqrt{k \cdot ln(x^2) + x}$$
$$\int_m^{\infty} f(x_1)dx_1 = \int_m^{\infty} f(x_2)dx_2$$

Tipping point occurs when the integral of Eq(1) equals that of Eq(2). Note1- Eq 2 should not be necessarily a logarithmic function; there are some other types of fuctions which also go through the the extrema in Eq 1. Note 2- Eq 1 provides a harmonic reduction. Not all the applied inputs or inverse inputs have harmonic reductions but their function can also be defined as a form of Eq1.

3 Indeterminacy of Tipping Point

3.1 a Conditional Convergent Series

Formerly, we pointed out to a decreasing sequence or process in table (2):

$$Inputs: \frac{883}{1000} \cdots \frac{622}{1000} \cdots \frac{575}{1000} \cdots \frac{522}{1000} \cdots \frac{486}{1000} \cdots \frac{458}{1000} \cdots \frac{436}{1000} \cdots \frac{419}{1000} \cdots \frac{4.5}{1000} \cdots$$

$$Inverse - inputs: \frac{455}{1000} \cdots \frac{435}{1000} \cdots \frac{405}{1000} \cdots \frac{388}{1000} \cdots \frac{372}{1000} \cdots \frac{360}{1000} \cdots \frac{348}{1000} \cdots \frac{339}{1000} \cdots$$

This sequence can be regarded as a convergent sequence since Knopp defiens it as:
"if (x_n) is a given sequence and if it is related to a finite number ξ in a way that $(x_n - \xi)$ forms a null sequence, then we say that sequence (x_n) converges to ξ or that it is convergent. The number ξ is called the limiting value or limit of this sequence" [7]. Null sequences are the sequences with special limitting value of zero. Summation of the terms in such a sequence (and also in any infinite sequence) turns it into a series. By adding operator $(+)$ for the inputs and operator $(-)$ for the inverse-inputs a sum series will be generated:

$$\begin{aligned} Series(A) &= -0.455 + 0.883 - 0.435 + 0.662 - 0.405 + 0.575 - 0.388 \\ &\quad +0.522 - 0.372 + 0.486 - 0.360 + 0.458 - 0.348 + 0.436 \\ &\quad -0.339 + 0.419 - 0.332 + 0.405 - 0.324 \cdots \end{aligned}$$

Series (A) or similar series generated by Eq(1) with different values of λ may resemble the Madhava-Leibniz series for determining the value of π which was indeed the source of inspiration for this paper. The main idea of Madhava-Leibniz series is to go a little back and a little forth around π, which shapes a series that would finally converge to π [10]. The same idea can be applied for determining the tipping point. In his paper, Philips followed the same method by studying the possibilities of cutting input sooner or later than tipping point but didn't use it in the context of convergent series.

When a series is convergent, the number ξ is the sum of the series which necessarily has a unique and absolute value but if does not have necessarily an absolute value, then the convergent series is conditional. The difference between a conditional and non-conditional convergence is that although in both the limiting value of the sequence or the sum of series exists and is a finite number, but the absolute value of sum of the series in conditional convergent series is infinite. In other words, "if $\sum a_n$ converges but $\sum |a_n|$ diverges, we say that $\sum a_n$ converges non-absolutely" [2]:

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n = k$$

$$\sum_{n=0}^{\infty} |a_n| = \infty$$

Here too, the absolute value for Series (A) equals $+\infty$ because changing minus operators into plus operators would be like changing all minima into maxima and accordingly the sinisoidal diagram in fig (1) would never merge to a straight line.

3.2 Reimann Rearrangement Theorem

The classical Reiman rearrangement theorem declares that the commutative law is no longer true for infinite sums. To be more precise, it says: Let: $\sum_{n=0}^{\infty} x_n$ be a conditionally convergent series of real numbers. Then:

1. For any $s \in \mathbb{R}$ one can find a permutation π such that: $\sum_{n=0}^{\infty} \chi_{\pi_n} = s$
2. One can find a permutation σ such that : $\sum_{n=0}^{\infty} \chi_{\sigma_n} = \infty$
3. One can find a permutation σ such that:$\sum_{n=0}^{\infty} \chi_{\sigma_n} = -\infty$

[1]
The theorem simply discusses that because permutation is a properties of summation then in a conditionally convergent series, it allows the terms to be rearraenged so that respectively the limmiting number would change into another arbitrary real number. As a result, the formation of calculation, or it's better to say arranement of the series, would determine the convergence point. Thus, the value of Tipping Point is conditional to the presupposed approximations about it and its formulation is just bolding one of the preferential permutations.

Discussion

Relevance of the above discusion to the field of diffusion may need further clarifications. Firstly, defining stepts towards Tipping Point based on a infinite sum series may seem far from reality because no diffusion undertakes infinte stepts to reach to tipping point. Here a comparison with one of Zenos paradix may help us clarify more. For Achilles too,

it doesnt takes infinite steps to reach to the tortoise while the distance between them is defined through an infinite divergant sum series. As the same way, tipping points happens after finite numbers of inputs and it welcomes the same proof to avoid paradoxicality.
Secondly, indeterminacy of tipping point neither means that there is no tipping point nor that there are many. Obviously the knee of the curve exits and there is just one. As mentioned above, it just means that determining the Tipping Points is based on our approximations; Like in Madhava-Liebniz series that is predestined to reach to π. Finally, a wrong notion may arise here; one may think that by proposing indeterminacy, we are going to conclude -for example- that a businessman can never be sure when its product has become self-susatined in market and needs no further advertisement. This idea is irrelevant to our discusion because advertisement or any other kind of external inputs with central source do not generate S-curves and have been formerly excluded from our discussion.
Now we can discuss the nature of the inputs better. Diffusion in a network does not occur thorough detached hits of inputs. Indeed any diffusion overcomes the threshold of the nodes when a sufficing number of the neighboring nodes adopt the diffused issue, knowing that "threshold is the number of the propoertion of others who must take one decision before a given factor does so"[2] [4]. It can be proposed in a context of bandwagon effect, network effect, or etc. but all and all, it is the aggregation of former adoptions in a sufficient number that helps the diffusion to move further. So the inputs are the the resultants of attempts towards sufficiency. The attempts are permanently present, but their effect is momentary. This can be regarded as a form of periodic tempred distribution which can relate the frequency-oriented function of diffusion with input-oriented function of tipping point through Fourier transform but then again, this is the subject of another paper. Further more, by regarding it in this way, one can also: quantify inputs based on Dirac's comb function, study the effect of recovery in SIR models in the light of quantifying inverse-inputs and finally provide a better definition and understanding of threshold.

Conclusion

Tipping Point refers to the moment when an adaption or infection sustains itself without external inputs; therefore, knowing when to cut the inputs is the key to define the term. Going back and forth around the Tipping Point can generate a series which converges to the Tipping Point but because the limiting value of such a sum series is not equal with its absolute value, the series converges non-absolutely. According to Reimann's Rearrangement Theorem, permutation of summation allows such series to converge to other numbers as well. This creates a status of indeterminacy which explains the inability of providing a general answer for when or how the Tipping Point should occur. The paper just provides

[2]Author has a critical view towards Granovetters definition for that it just regards the frequency of former adoption in neighboring nodes, while the importance of nodes may be different. This is the same criticism that Metcalfes law has faced (since for homogeneous networks a different topology should be applied than heterogeneous networks), but to avoid losing thematic unity further mathematical discussion on threshold would not be mentioned.

the heuristic grounds for more studies on Tipping Points and diffusions, roughly reaching to a definition whose propositions can help develop more complex methods or inductive nomological models.

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