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Snakes and Caterpillars in Graceful Graphs

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ABSTRACT

Graceful labelings use a prominent place among difference vertex labelings. In this work we present new families of graceful graphs all of them obtained applying a general substitution result. This substitution is applied here to replace some paths with some trees with a more complex structures. Two caterpillars with the same size are said to be *analogous* if the larger stable sets, in both caterpillars, have the same cardinality. We study the conditions that allow us to replace, within a gracefully labeled graph, some snakes (or paths) by analogous caterpillars, to produce a new graceful graph. We present five families of graphs where this replacement is feasible, generalizing in this way some existing results: subdivided trees, first attachment trees, path-like trees, two-point union of paths, and armed crowns.

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1 Introduction

A *graceful labeling* of a graph G of size n is an injective assignment of integers from the set $\{0, 1, ..., n\}$ to the vertices of G such that, when each edge has assigned a *weight*, given the absolute value of the difference of the labels of its end vertices, all the weights are

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distinct. From this definition it is possible to conclude that the order of a graceful graph G of size n, is at most $n+1$. A graceful labeling is called an *α-labeling* when the graph G is bipartite, with stable sets A and B, and the labels assigned to the vertices in A are smaller that the labels assigned to the vertices of B. This last type of labeling have been used to construct graceful or α-labelings of larger graphs. Most of these constructions are based by the method used by Stanton and Zarnke [20], where the authors used an α-labeled tree S and a graceful tree T, with a distinguish vertex w, to form an α-labeled tree G by attaching to every vertex of S the vertex w of a tree that is a copy of T. Several generalizations of this technique are known nowadays, in particular, we must mention here, the results of
Koh et al. [10] (see also [9], [11] and [12]). Burzio and Ferrarese [4] extended some of these results to prove that the subdivision of all edges of a graceful tree is a graceful tree, that is, they replaced every edge of a graceful tree by a path on length k. Sethuraman and Selvaraju [17] replaced every edge of a path by the complete bipartite graph $K_{2,m}$, where m may change from edge to edge; they called this replacement, *supersubdivision*. More general results about super subdivisions can be found in [2], [3], and [18]. In Section 3 we analyze this concept in more detail. There, we introduce a replacement theorem that allows us to replace, within a graceful labeled graph, some specific labeled subgraphs by some analogous graphs.

In Section 4 we present some applications of the replacement theorem, each of these applications produces new families of graceful graphs, extending in this way, the number of known graceful graphs.

The reader interested in graph labelings and, in particular, methods to construct labeled graphs, is refered to Gallian' survey [6]. In this paper we follow the notation and terminology used in [5] and [6].

2 Essential Tools

A *difference vertex labeling*, or simply a labeling, of a graph G of size n is an injective mapping f from $V(G)$ into a set N of nonnegative integers, such that every edge uv of G has assigned a *weight* defined by $|f(u)-f(v)|$. All labelings considered in this work are difference vertex labelings. A labeling is called *graceful* when $N=\{0,1,...,n\}$ and the induced weights are $1,2,...,n$. If G admits such a labeling, then it is called a *graceful graph*.

Let G be a bipartite graph where $\{A_G, B_G\}$ is the natural bipartition of $V(G)$, we refer to A_G and B_G as the *stable sets* of $V(G)$. Without loss of generality, we assume that $|A_G| \leq |B_G|$. A *bipartite labeling* of G is an injection $f: V(G) \rightarrow \{0,1,...,t\}$ for which there is an integer λ, named the *boundary value* of f, such that $f(u) \leq \lambda < f(v)$ for every $(u,v) \in A_G \times B_G$, that induces n different weights. This is an extension of the definition given by Rosa and Širáň in [15]; there, they focussed on bipartite labelings of trees. From the definition we may conclude that $t \geq |E(G)|$, furthermore, the labels assigned by f on the vertices of A_G and B_G are in the integer intervals $[0,\lambda]$ and $[\lambda+1,t]$, respectively.

If $t = n$, the bipartite labeling f is an *α-labeling*. By an *α-graph* we mean a graph that admits an α-labeling. If G is an α-graph, λ is the smaller of the two vertex labels of the edge of weight 1. In the rest of this work, we assume that the vertex labeled λ is in A_G. Thus, if G is an α-tree, then $\lambda = |A_G| - 1$.
Let $f : V(G) \to \{0, 1, ..., t\}$ be a labeling of a graph G of size n, $n \leq t$:

- $\overline{f} : V(G) \to \{0, 1, ..., t\}$, defined for every $v \in V(G)$ as $\overline{f}(v) = t - f(v)$, is the *complementary* labeling of f. If f is graceful, $\overline{f}$ is also graceful. Moreover, if f is an α-labeling with boundary value λ, then $\overline{f}$ is an α-labeling with boundary value $t - \lambda - 1$.
- $g : V(G) \to \{c, c+1, ..., c+t\}$, defined for every $v \in V(G)$ and $c \in \mathbb{Z}$ as $g(v) = c + f(v)$, is the *shifting* of f in c units. Note that this labeling preserves the weights induced by f.
- $h : V(G) \to \{0, \kappa, ..., t\kappa\}$, defined for every $v \in V(G)$ and $\kappa \in \mathbb{Z}^+$ as $h(v) = \kappa f(v)$, is the *amplification* of f in κ units. If w_1, w_2, ..., w_n are the weights induced by f, then the weights induced by h are κw_1, κw_2, ..., κw_n.

Suppose now that $f : V(G) \to \{0, 1, ..., t\}$ is a bipartite labeling with boundary value λ.

- $f_r : V(G) \to \{0, 1, ..., t\}$, defined for every $v \in V(G)$ as, $f_r(v) = \lambda - f(v)$ if $f(v) \leq \lambda$, and $f_r(v) = t + \lambda + 1 - f(v)$ if $f(v) > \lambda$, is the *reverse* labeling of f. Note that if f is an α-labeling, then f_r is also an α-labeling with boundary value λ.
- $f^k : V(G) \to \{0, 1, ..., t + k - 1\}$, defined for every $v \in V(G)$ and $k \in \mathbb{Z}$ as, $f^k(v) = f(v)$ if $f(v) \leq \lambda$ and $f^k(v) = f(v) + k - 1$ if $f(v) > \lambda$, is the *bipartite k-labeling* of G obtained from f. This labeling uses labels from $\{0, 1, ..., \lambda\} \cup \{\lambda + k, \lambda + k + 1, ..., t + k - 1\}$ and induces the weights $k, k+1, ..., t+k-1$. In other terms, this labeling shifts the weights induced by f in $k - 1$ units. Thus, if f is an α-labeling of G and k is a positive constant, then f^k is the, well-known, k-graceful labeling of G.

Let f be an α-labeling of a caterpillar Ω of size n with boundary value λ. Suppose that f is transformed into a k-graceful labeling shifted c units. Then the stable set A_Ω receives the labels $c, c+1, ..., c+\lambda$ and the stable set B_Ω receives the labels $c+\lambda+k, c+\lambda+k+1, ..., c+n+k-1$.

3 The Main Result

Suppose that Γ and Ω are two caterpillars of order n. We say that Γ and Ω are *analogous* if $|A_\Gamma| = |A_\Omega|$ and their diameters have the same parity. Following this definition, in Figure 1 we show the classification of all the caterpillars of order 8. The list of caterpillars is taken from [13].

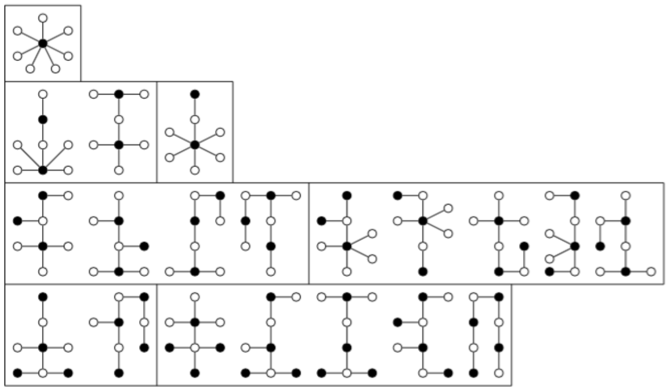

Figure 1: A classification of the caterpillars of size 7.

If U is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle U \rangle$ of G *induced* by U is the subgraph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. A subgraph H of G is called *vertex-induced* or *induced*, denoted $H \prec G$, if $H \cong \langle U \rangle$ for some subset U of $V(G)$. If G is a labeled graph, every element of U is identified with its label in G.
Let Ω be a caterpillar. Suppose that u and v are vertices of Ω such that $d(u,v) = \text{diam } \Omega$. Rosa[14] shown the existence of an α-labeling f of Ω, where $f(u) = 0$ and $f(v) = \lambda$ when diam Ω is even, or $f(v) = \lambda + 1$ when diam Ω is odd. Therefore, for any caterpillar Γ, analogous to Ω, there exists an α-labeling g such that $g(u') = f(u)$ and $g(v') = f(v)$ where $u'v' \in E(\Gamma)$ and $\text{dist}(u',v') = \text{diam } \Gamma$.
Theorem 1. Let G be a gracefully labeled graph of size n. If a caterpillar Ω is an induced subgraph of G which induced labeling is a bipartite k-labeling shifted c units, then the graph G', obtained by replacing Ω by any other caterpillar Γ analogous to Ω, is a graceful graph.

Proof. Since Ω and Γ are analogous caterpillars, the existence of a bipartite k-labeling shifted c units of Ω, implies the existence of a bipartite k-labeling of Γ shifted c units. So, the labelings of Ω and Γ use the same labels and induce the same weights. Therefore, if we delete from G all the edges of Ω and introduce all the edges of Γ, we obtain a graceful graph. □

In Figure 2 we show an example of this result, where two subpaths in C_{20} have been replaced by analogous caterpillars.
In order to see the potential of this theorem, we have identified 14 caterpillars of order 10 analogous to P_{10}, each of them can be used to replace any of the two distinguished copes of P_{10} in C_{20}, producing 196 α-labeled unicyclic graphs.

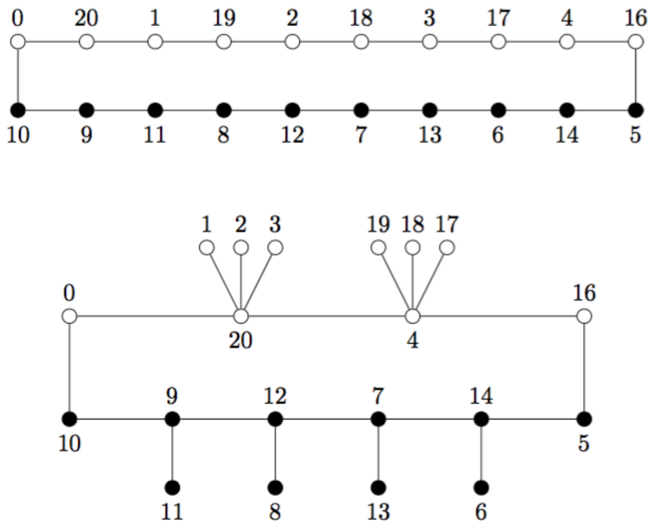

Figure 2: α-labeling of a hairy cycle obtained with the replacement theorem.

4 Using Analogous Caterpillars

In this section we use Theorem 1 to prove the existence of graceful or α-labelings of several families of graphs. We use five families of graceful graphs that have subgraphs isomorphic to a path whose induced labeling satisfies the conditions of the theorem. In this way, the new results correspond to generalizations of the existing ones.

4.1 The Subdivison Tree

Burzio and Ferrarese [4] proved that when every edge of a graceful tree is replaced by a path of length $k > 1$, the resulting graph is a graceful tree.
Proposition Let $S_n(T)$ be the nth subdivision graph of a tree T, i.e., the tree obtained by inserting n new vertices into each edge of T. Then if T is graceful, $S_n(T)$ is also a graceful tree.
Within the proof of this proposition, the authors used as $T(n)$ a path of order n with a graceful labeling that assigns the label 0 on a leaf. (Note that their definition of a graceful labeling uses labels from 1 to n.) There is only one graceful labeling of a path that assigns the label 0 to a leaf, this is the well-known α-labeling given by Rosa [14]. Therefore, the tree $S_n(T)$ contains multiple induced subgraphs isomorphic to a path of length n, whose induced labelings are of the kind described in Theorem 1. Hence, as a consequence of Theorem 1 and Proposition 2, we have the following theorem.
Theorem 2. Let T be a tree of order m with edges e_1, e_2, ..., e_{m-1}, and Γ_1, Γ_2, ..., Γ_{m-1} be a list of caterpillars analogous to P_n. Then the graph obtained by replacing each e_i by Γ_i is a graceful tree.
In Figure 3 (a), we show a graceful labeling of a tree of the form $S_7(T(7))$ constructed using

the directions in [4]. In part (b) we exhibit the graceful labeling obtained by replacing some of the edges of $T(7)$ with caterpillars analogous to the path P_8.

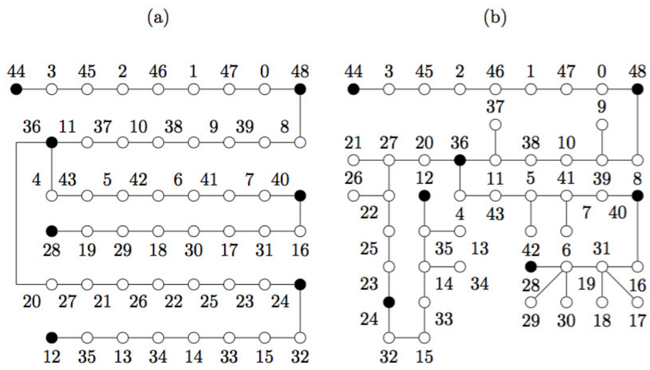

Figure 3: Graceful tree obtained from a subdivided graceful tree.

4.2 The First Attachment Tree

In [19], Sethuraman and Venkatesh used a variation of the Δ-construction to create a new variety of α-trees. Let G be a graph with r vertices of degree at least two and H be any graph with a chosen vertex. Consider r copies of H; $G \oplus H$ denotes the graph obtained by merging (via vertex amalgamation) the chosen vertex of each copy of H with every vertex of degree at least two of G. Let T_0 and T^{A_1} by any two caterpillars. They defined the *first attachment tree* as $T_1 = T_0 \oplus T^{A_1}$. For $i \geq 2$, the *ith attachment tree* T_i, is defined recursively as the tree $T_i = T_{i-1} \oplus T^{A_i}$, here the chosen vertex of T^{A_i} must have eccentricity equal to diam $T^{A_i} - 1$.

Proposition 2. For $i \geq 1$, the ith attachment tree T_i admits an α-labeling.

Within the proof of this proposition, Sethuraman and Venkatesh use a k-graceful labeling of T^{A_i} shifted c units; the values of k and c change from copy to copy, obviously. Therefore, each copy of T^{A_i} can be replaced by any caterpillar Γ analogous to T^{A_i} and the result still holds. Thus, using Theorem 1 and Proposition 4 we can prove the following result.

Theorem 3. Suppose G is a caterpillar and $v_1, v_2, ..., v_r$ are the vertices of degree at least two in G. Let $\Gamma_1, \Gamma_2, ..., \Gamma_r$ be analogous caterpillars, where the chosen vertex of each Γ_i is a vertex with eccentricity diam $\Gamma_i - 1$. Then the graph obtained by merging, for all $1 \leq i \leq r$, the chosen vertex of Γ_i with v_i, is an α-tree.

In Figure 4 we show an example with an α-labeled tree obtained using a caterpillar G of size 6 and $r = 3$, and three analogous caterpillars of size 7.

4.3 The Path-Like Trees

In [1], Barrientos presented another family of α-graphs that uses α-labelings of paths (see also [7]). We embed the path P_n as a subgraph of the two dimensional grid. Figure 5

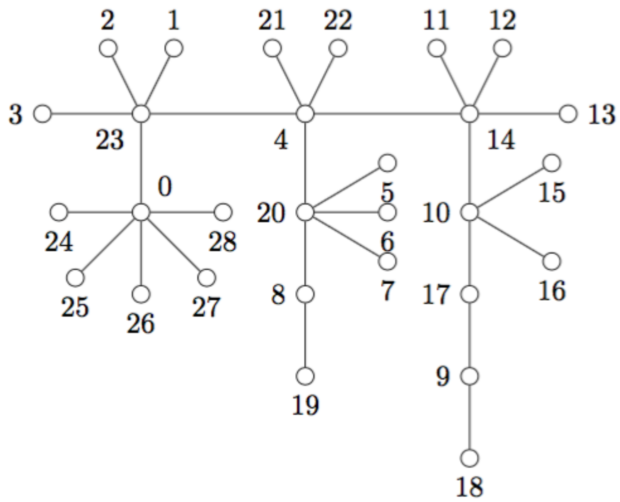

Figure 4: α-labeled tree.

(a) shows such an embedding of P_{36} with an α-labeling. Given such an embedding, we consider the ordered set of subpaths $L_1, L_2, ..., L_k$ which are maximal straight segments in the embedding and such that the end of L_i is the beginning of L_{i+1}. In the example of Figure 5 (a), $L_1 \cong L_3 \cong L_5 \cong L_7 \cong P_9$ and $L_2 \cong L_4 \cong L_6 \cong P_2$. Suppose that for some i, $L_i \cong P_2$, and that some vertex u of L_{i-1} is at distance one, in the grid, from a vertex v in L_{i+1}. An *elementary transformation* of the path P_n consists in replacing the edge of L_i by a new edge uv. We say that a tree T of order n is a *path-like tree* when it can be obtained from some embedding of P_n in the grid, by a set of elementary transformations. Figure 5 (b) shows a path like tree obtained from the embedding of P_{36} in part (a).

Prpposition 3. All path-like trees are graceful.

As a consequence of this result and Theorem 1, we can prove the following theorem.

Theorem 4. If T is a path-like tree, then any group of subpaths that are straight segments in the embedding of T can be replaced by an analogous caterpillar, to obtain a graceful tree T'.

Note that in fact, the labelings of T and the final tree T', are α-labelings. In Figure 5 (c) we show an α-labeling obtained using this procedure.

4.4 The Connected Paths

The following family of graphs was introduced by Kathiresan [8]. Let u and v be two fixed vertices. The graph obtained by connecting u and v by means of b internally disjoint paths, of length a each, is denoted $P_{a,b}$. Kathiresan proved that the graphs $P_{2r,2m+1}$ are graceful for all values of r and m. He conjectured that $P_{a,b}$ is graceful except when $a = 2r + 1$ and $b = 4m + 2$. Note that under these conditions, $P_{a,b}$ is an Eulerian graph with the wrong parity, therefore it is not graceful (see [14]). Sekar [16] studied the cases where $a = 2r + 1$

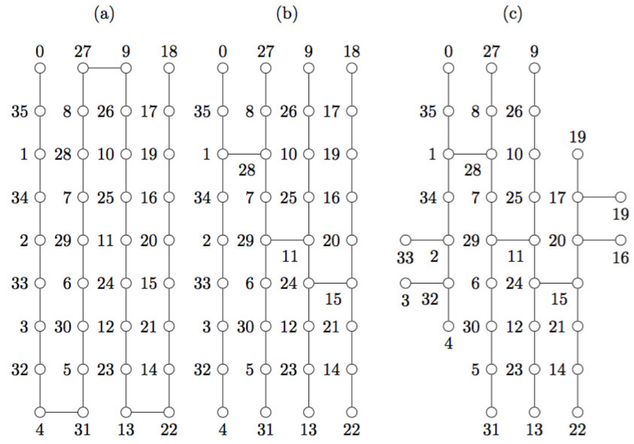

Figure 5: α-tree obtained from a path-like tree.

and $b = 4m + 1$, $a = 4r$ and $b = 2m$, $a = 4r + 2$ and $b = 2m$ for all values of r and m, proving that all these cases correspond to graceful graphs. For $r \geq m$, Sekar [16] proved that $P_{a,b}$ is graceful when $a = 4r + 1$ and $b = 4m$, as well as the case where $a = 4r - 1$ and $b = 4m$.

Proposition 4 $P_{2r,2m+1}$ is graceful for all values of r and m.

Proposition 5 $P_{2r+1,2m+1}$ is graceful for all values of r and m.

The proofs of both results are quite similar. For each $1 \leq i \leq 2m + 1$, let P^i_{a-1} denote the ith copy of the subpath P_{a-1} of length $a - 2$ (the vertices u and v of the definition are not in these paths.) They start with an α-labeling of P^i_{a-1} which is transformed into a 2-graceful labeling. This labeling is amplified by a factor $\kappa = 2m + 1$. Let $Y_{P^i_{a-1}}$ be the stable set of P^i_{a-1} that contains the vertices with the largest labels, from every label of the vertices in $Y_{P^i_{a-1}}$. Kathiresan and Sekar used the same technique, both subtract the constant $d = i-1$. In the case of $P_{2r,2m+1}$, Kathiresan shifted the labeling of P^i_{a-1}, $c = 2i-1$ units and label u and v, 0 and $r(2m + 1)$, respectively. When $P_{a,b} = P_{2r+1,2m+1}$, Sekar shifted the labeling of P^i_{a-1}, $c = m + (i + 1)/2$ units when i is odd, and $c = 2m + 1 + i/2$ units when i is even. In this case, the vertices u and v are labeled 0 and $r(2m + 2)$, respectively.

Since the starting labeling of each P^i_{a-1} is an α-labeling, we can replace each of them by an analogous α-labeled caterpillar and the resulting graph is still graceful. Thus, we have the following result.

Theorem 5. Let $\Gamma_1, \Gamma_2, ..., \Gamma_{2m+1}$ be analogous caterpillars whose stable sets have the same cardinality or differ by one unit. For each $1 \leq i \leq 2m + 1$, let $u_i, v_i \in V(\Gamma_i)$ such that $d(u_i, v_i) = \text{diam}\ \Gamma_i$. Then the graph obtained by connecting all the u_i to a new vertex u and all the v_i to a new vertex v, is a graceful graph.

In all the other cases proved by Sekar, it is possible to replace $b - 1$ of the paths P_{a-1} with analogous caterpillars, the remaining path P_{a-1} contains two subpaths that can be replaced independently.

In Figure 6 we show an example of these replacements on the starting graph $P_{9,7}$.

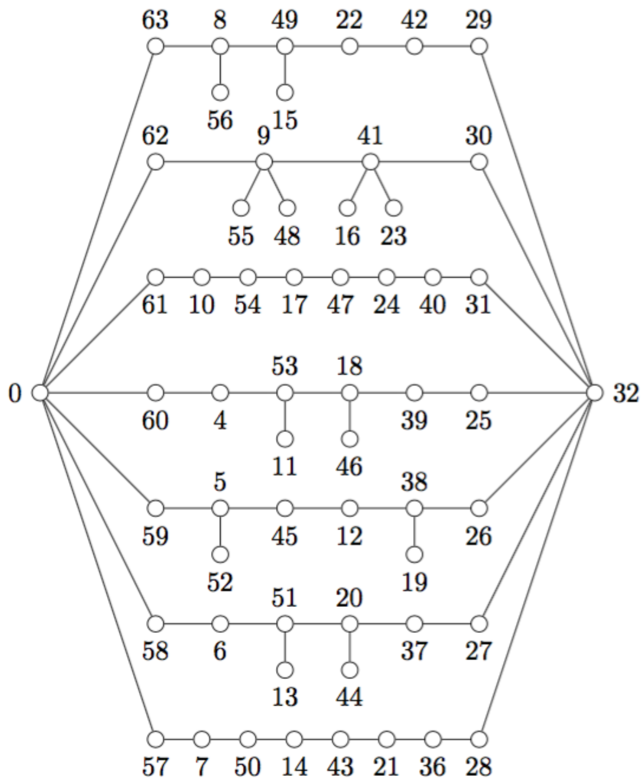

Figure 6: α-graph obtained from a graceful labeling of $P_{9,7}$.

4.5 The Armed Crowns

The last family of graceful graphs that we want to consider in this work, is based in the following result of Sekar [16]. He defined an *armed crown* as a cycle with paths of equal lengths attached at each vertex of the cycle. Sekar used the symbol $C_n \odot P_m$ to denote this graph; instead, we use $C_n \circledast P_m$, to avoid any confusion with the corona of C_n and P_m.

Proposition 6 $C_n \circledast P_m$ is graceful for all m and n.

Once again, Sekar started with α-labelings of the copies of P_m and transformed them into k-graceful labelings shifted c units. When $n \equiv 0, 3 (\text{mod } 4)$, that is, when C_n is graceful, all the n paths are transformed into k-graceful labelings shifted c units, where k and c depend on the copy of P_m used. When $n \equiv 1, 2 (\text{mod } 4)$, Sekar did the same with all the copies of P_m except one, which is treated in a slighly different way. For more details about armed crowns see [16].

Theorem 6 **If $\Gamma_1, \Gamma_2, ..., \Gamma_x$ are caterpillars analogous to P_m, then x copies of P_m in $C_n \circledast P_m$ can be replaced by the Γ_i obtaining a graceful graph, when $x \leq n$ if $n \equiv 0, 3(\mathbf{mod}\ 4)$, or $x \leq n-1$ if $n \equiv 1, 2(\mathbf{mod}\ 4)$.**
In Figure 7 we show an example of a graceful graph obtained in the way described before.

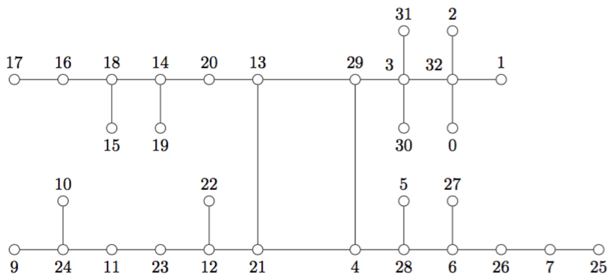

Figure 7: α-labeling of a generalized armed crown.

5 Final Comments

We must mention here that the labeled caterpillar Ω in Theorem 1 can be replaced by any α-tree that satisfies the same conditions that the caterpillar Γ; we used caterpillars because their α-labelings are well-known, which facilitates the understanding of the theorem. We hope that new families of graceful graphs can be obtained applying this theorem.

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