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Tenacity and some other Parameters of Interval Graphs can be computed in polynomial time

Niloofar Vahdat*[1] and Dara Moazzami†[2]

[1]University of Tehran, Department of Algorithms and Computation
[2]University of Tehran, College of Engineering, Faculty of Engineering Science

ABSTRACT

In general, computation of graph vulnerability parameters is NP-complete. In past, some algorithms were introduced to prove that computation of toughness, scattering number, integrity and weighted integrity parameters of interval graphs are polynomial.
In this paper, two different vulnerability parameters of graphs, tenacity and rupture degree are defined.
In general, computing the tenacity of a graph is NP-hard and the rupture degree of a graph is NP-complete, but in this paper, we will show that these parameters can be computed in polynomial time for interval graphs.

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1 Introduction

Study of the reliability of the networks is very critical to obtain efficiency goals. Potential vulnerabilities mean that in an unfriendly external environment, how a system can be sustained in the face of destruction. Graph theoretic approaches are used in this paper. Different models of graph theory, under various assumptions, are presented for the study

*vahdatn@alumni.ut.ac.ir
†Corresponding author: D. Moazzami. Email: dmoazzami@ut.ac.ir

and assessment of network vulnerabilities.

The concept of tenacity of a graph G was introduced in [2], [3], as a useful measure of the "vulnerability" of G. In [3], Cozzens et al. calculated tenacity of the first and second case of the Harary Graphs but they didn't show the complete proof of the third case. In [13] we showed a new and complete proof for case three of the Harary Graphs. In [7], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is the most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability.
In [[7] - [14]] Moazzami studied more about this parameter. Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed. The motivation for studying vulnerability measures is derived from design and analysis of networks under hostile environment. Graph tenacity has been an active area of research since the concept was introduced in 1992. Cozzens et al. in [2], introduced measure of network vulnerability termed the tenacity, $T(G)$.

The tenacity $T(G)$ of a graph G is defined as:

$$T(G) = \min\{|S| + m(G-S)/c(G(V-S))\} \quad S \subseteq V \tag{1}$$

where $c(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$.

In [14], Dadvand and Moazzami proved that computing the tenacity of a graph is NP-hard in general. So, it is an interesting problem to determine tenacity for some special graphs.

In [5], the rupture degree of a noncomplete connected graph G is defined as:

$$r(G) = \max\{c(G(V-S)) - |S| - m(G-S)\} \quad S \subseteq V, c(G(V-S)) \geq 2 \tag{2}$$

where $c(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$ and for a complete graph K_n, we have $r(K_n) = 1-n$. This parameter can be used to measure the vulnerability of a graph. To some extent, it represents a trade-off between the amount of work done to damage the network and how badly the network is damaged. The rupture degree of a graph is NP-complete.

In [4], [15], some algorithms were introduced that proved toughness, scattering number, integrity and weighted integrity parameters of interval graphs are polynomial. In this paper, it is shown that for these graphs, tenacity and rupture degree parameters can also be computed in polynomial time.

Preliminaries:

At first we present several notes about undirected graph[4]:
For a graph G, with n vertices, $G[W]$ is a subgraph of G induced by the vertex set $W \subseteq V$. The number of connected component of G is denoted by $c(G)$, and maximum order of a component of G by $n(G)$. A set $S \subseteq V$ is a separator of graph G if we have at least two components:

$$c(G(V-S)) > 1$$

We denote vertex connectivity of G by $\kappa(G)$ and an independent set of G by $\alpha(G)$.

Information about the fields of struct:

$n_{c,S}(G-S)$ is the largest component for minimum cut S and component number c.
$c_{n,S}(G-S)$ is the number of components for minimum cut S and largest component n.

Tenacity parameter defined in equation 1 can easily be derived from finding the minimum of whole T_Cell fields of our structs:

$$T(G) = \min\{\frac{|S| + n_{c,S}(G-S)}{c_{n,S}(G-S)}\} = \min\{T\_Cell\} \tag{3}$$

Rupture degree parameter defined in equation 2 can easily be derived from finding the maximum of whole R_Cell fields of our structs:

$$r(G) = \max\{c_{n,S}(G-S) - |S| - n_{c,S}(G-S)\} = \max\{R\_Cell\} \tag{4}$$

Interval graphs

Given intervals $[a_i, b_i]$ for $i = 1, \ldots, n$. An intersection graph $G = (V, E)$ is constructed with vertices $v = 1, \ldots, n$ and edges $E = \{(i,j) | [a_i, b_i] \cap [a_j, b_j] \neq \emptyset\}$, it means that if the intervals have intersection with each other, the corresponding vertices will have edges.
a_i is called the left end point and b_i called right end point. We can assume that neither of these two intervals, don‘t have the same end points. For example, $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$ and $a_i \neq b_j$ for all i and j.

Definition 1.1. *A scanline in the interval diagram is any straight vertical line segment with an end point on horizontal line (Real numbers axis) such that this end point doesnt coincide with any end points of intervals.*

Each scanline s generates a set $S(s)$ of vertices of the graph, such that for each vertex v_i in this set, interval $[a_i, b_i]$ has nonempty intersection with scanline s (for $i = 1, \ldots, n$).

$$S(s_1) = \{v_1, v_2, v_5\}$$
$$S(s_2) = \{v_8, v_9\}$$

We say that two scanlines s_1 and s_2 are equivalent together, if we have

$$S(s_1) = S(s_2)$$

Definition 1.2.
Let s_1 and s_2 be non-equivalent scanlines such that s_1 is in left of s_2, piece $\rho(s_1, s_2)$ consists of all vertices v_i such that intervals $[a_i, b_i]$ are between s_1 and s_2. The end points of these intervals $[a_i, b_i]$ are completely between the end points of s_1 and s_2 on horizontal line.

Algorithms for interval graphs:

Observation 1.3.
According to (table1 of [4]), a maximal set of pairwise non-equivalent scanlines of any interval diagram of an interval graph with n vertices, consists of $O(n)$ scanlines.

Finding the toughness, Scattering Number and Integrity parameters of interval graphs:

According to (table1 of [4]), the auxiliary directed graphs $D^c(G)$ and $D^n_t(G)$ have both $O(n)$ vertices and $O(n^2)$ edges for any interval graph G of order n.
Therefore the running time of algorithms computing $(c_i)^n_{i=0}$ and $(n_i)^n_{i=0}$ is $O(n^3)$ and so the graph parameters like: toughness, Scattering Number and Integrity can simply computed in $O(n^3)$.

Finding the Tenacity and rupture degree parameters of interval graphs

Given an interval diagram of an interval graph $G(V, E)$, the algorithms computing $n_{c,S}(G-S)$ and $c_{n,S}(G - S)$ solve suitable shortest path problem on auxiliary directed acyclic graphs whose vertex set is a maximal set of pairwise nonequivalent scanlines in the diagram. Among these scanlines we denote by s_L and s_R the scanline totally to the left and totally to the right, respectively, of all intervals $[a_i, b_i]$ of the interval diagram (for $i = 1, \ldots, n$).

Construction of the auxiliary graphs $D^{nc}_{t_1,t_2}(G)$:

Construct the following auxiliary graphs $D^{nc}_{t_1,t_2}(G)$, $t_2 \in \{1,\ldots,n\}$. The vertex set of $D^{nc}_{t_1,t_2}(G)$ is a maximal set of pairwise nonequivalent scanlines in the diagram. There is an edge directed from s_1 to s_2 in $D^{nc}_{t_1,t_2}(G)$ if $1 <= \rho(s_1,s_2) <= t_2$. The weight of an edge (s_1,s_2) of $D^{nc}_{t_1,t_2}(G)$ is $W(s_1,s_2) = |S(s_1)\backslash S(s_2)|$.

Considering [4], we can conclude the following lemma:

Lemma 1.4. *Let $w^{nc}_{t_2}(t_1)$, for $c(G)+1 <= t_1 <= \alpha(G)$, be the minimum weight of $\sum_{j=1}^{r} w(s_{j-1},s_j)$ of a path $p_{t_1} = (s_0,s_1,\ldots,s_r)$ and $r >= t_1$, $s_0 = s_L$, $s_r = s_R$ among all paths in the graph $D^{nc}_{t_1,t_2}(G)$ from s_L to s_R on at least t_1 edges, then $w^{nc}_{t_2}(t_1) = \min\{|S| : c_{n,S}(G) >= t_1 \quad \&\& \quad n_{c,S}(G) <= t_2\}$ for $c(G)+1 <= t_1 <= \alpha(G)$ and $t_2 \in \{1,2,\ldots,n\}$.*

Proof:

With considering the definition of constructing the auxiliary graphs $D^{nc}_{t_1,t_2}(G)$, we have that the largest component conditions of graphs are satisfied:

$$n_{c,S}(G) <= t_2 \qquad for\ t_2 \in \{1,2,\ldots,n\}$$

Now the paths from s_L to s_R in the auxiliary graphs $D^{nc}_{t_1,t_2}(G)$ have at most $\alpha(G)$ edges. By [4], there is a collection of pairwise nonequivalent scanlines $(s_0,s_1,\ldots,s_r)$, $r \geq t_1$ such that each of $\rho(s_L,s_1)$, $\rho(s_j,s_{j+1})$ for all $j \in \{1,2,\ldots,r-2\}$ and $\rho(s_{r-1},s_R)$ is between 1 and t_2, induces a connected subgraph of G and $S = \bigcup_{j=1}^{r-1} S(s_j)$ is a set of minimum cardinality with $c(G[V\backslash S]) \geq t_1$.
Hence, a shortest (minimum weight) path $p_{t_1} = (s_0,s_1,\ldots,s_r)$ and $r >= t_1$ from s_L to s_R in $D^{nc}_{t_1,t_2}(G)$ determines such a set as $S = \bigcup_{j=1}^{r-1} S(s_j)$. Moreover, $w^{nc}_{t_2}(t_1) = \min\{|S| : c_{n,S}(G) >= t_1 \quad \&\& \quad n_{c,S}(G) <= t_2\}$ for $c(G)+1 <= t_1 <= \alpha(G)$ and $t_2 \in \{1,2,\ldots,n\}$.

Remark 1.5.
There are algorithms for given interval graph G that compute tenacity parameter $T(G)$ and rupture degree parameter $r(G)$ in polynomial time $O(n^4)$.

Proof:

It is not very hard to show that for given interval diagram each of the two auxiliary directed graphs $D^c(G)$ and $D^n_{t_2}(G)$ and the corresponding edge weights can be computed in time $O(n^2)$ as it shown in [4]. Similarly, the auxiliary graphs $D^{nc}_{t_1,t_2}(G)$ and the corresponding edge weights can be computed in polynomial time $O(n^2)$, for $c(G)+1 <= t_1 <= \alpha(G)$ and $t_2 \in \{1,2,\ldots,n\}$.
Given $D^{nc}_{t_1,t_2}(G)$, according to [4] the minimum weight of a path from s_L to s_R on at least t_1 edges for all $c(G)+1 <= t_1 <= \alpha(G)$ can be determined in polynomial time $O(V(D^{nc}_{t_1,t_2}(G)) + n \times E(D^{nc}_{t_1,t_2}(G))) = O(n^3)$ for each $t_2 \in \{1,2,\ldots,n\}$, So total time complexity computed as follow: $O(t_2) \times O(n^3) = O(n) \times O(n^3) = O(n^4)$.

Therefore the minimum weight of a path from s_L to s_R (or minimum cut S) can be computed in polynomial time for all cases. So for all polynomial cases, we fill the fields of our struct like: min cut S, $n_{c,S}$, $c_{n,S}$, T_cell and R_cell.

According to equations 1, 3, Tenacity parameter of an interval graph $T(G)$ can be computed by finding the minimum of T_cell fields of all polynomial cases.

According to equations 2, 4, rupture degree parameter of a trapezoid graph $r(G)$ can be computed by finding the maximum of R_cell fields of all polynomial cases.

Conclusion:

Computing the tenacity of a graph is NP-hard and The rupture degree of a graph is NP-complete, in general. As discussed in above algorithms, the graph vulnerability parameters like: Tenacity and rupture degree are polynomial for interval graphs.

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