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Detour Monophonic Graphoidal Covering Number of Corona Product Graph of Some Standard Graphs with the Wheel

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ABSTRACT

A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ detour monophonic path. A detour monophonic graphoidal cover of a graph G is a collection ψ_{dm} of detour monophonic paths in G such that every vertex of G is an internal vertex of at most one detour monophonic path in ψ_{dm} and every edge of G is in exactly one detour monophonic path in ψ_{dm}. The minimum cardinality of a detour monophonic graphoidal cover of G is called the detour monophonic graphoidal covering number of G and is denoted by $\eta_{dm}(G)$. In this paper, we find the detour monophonic graphoidal covering number of corona product of wheel with some standard graphs.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary[6].

The concept of graphoidal cover was introduced by Acharya and Sampathkumar[2] and further studied in [1, 3 ,7 ,8]. A *graphoidal cover* of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions: (i) Every path in ψ has at least two vertices, (ii) Every vertex of G is an internal vertex of at most one path in ψ, and (iii) Every edge of G is in exactly one path in ψ. The minimum cardinality of a graphoidal cover of G is called the *graphoidal covering number* of G and is denoted by $\eta(G)$. The collection ψ is called an *acyclic graphoidal cover* of G if no member of ψ is cycle; it is called a *geodesic graphoidal cover* if every member of ψ is a shortest path in G. The minimum cardinality of an acyclic (geodesic) graphoidal cover of G is called *the acyclic* (*geodesic*) *graphoidal covering number* of G and is denoted by $\eta_a(\eta_g)$. The acyclic graphoidal covering number and geodesic graphoidal covering number are studied in [4, 5].

A *chord* of a path P is an edge joining any two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ *detour monophonic path.* For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius* is $rad_m(G) = min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter* is $diam_m(G) = max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [10, 11].

A *detour monophonic graphoidal cover* of a graph G is a collection ψ_{dm} of detour monophonic paths in G such that every vertex of G is an internal vertex of at most one detour monophonic path in ψ_{dm} and every edge of G is in exactly one detour monophonic path in ψ_{dm}. The minimum cardinality of a detour monophonic graphoidal cover of G is called the detour monophonic graphoidal covering number of G and is denoted by $\eta_{dm}(G)$. The detour monophonic graphoidal cover was introduced in [13] and further studied in [14,15].

Definition 1.1 *The corona of two graphs G and H is the graph $G \circ H$ formed from one copy of G and $|V(G)|$ copies of H, where the i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H.*

Throughout this paper G denotes a connected graph with at least two vertices.

2 Detour monophonic graphoidal cover on corona product of wheel with standard graphs

Theorem 2.1 *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 5)$,*

$$\eta_{dm}(W_n) = \begin{cases} n & if\ \ n = 5 \\ n+1 & if\ \ n\ \ is\ \ odd\ \ and\ \ n > 5 \\ n+2 & if\ \ n\ \ is\ \ even. \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v\}$ and $V(C_{n-1}) = \{u_1, u_2, \ldots, u_{n-1}\}$.
Case 1. n is odd.
Subcase (i): $n = 5$.

Let $P_1 : u_1, v, u_3$;

$P_2 : u_1, u_2, u_3$;

$P_3 : u_1, u_4, u_3$;

$P_4 : v, u_2$;

$P_5 : v, u_4$.

It is clear that $\psi_{dm} = \{P_1, P_2, P_3, P_4, P_5\}$ is a minimum detour monophonic graphoidal cover of W_n. Hence $\eta_{dm}(W_n) = 5 = n$.
Subcase (ii): $n = 7, 9, \ldots$

Let $P_1 : u_1, u_2, \ldots, u_{\frac{n+1}{2}}$;

$P_2 : u_1, u_{n-1}, u_{n-2}, \ldots, u_{\frac{n+1}{2}}$; and

$P_{i+2} : v, u_i$ $(1 \leq i \leq n-1)$.

Similar to Subcase (i), we have $\eta_{dm}(G) = n + 1$.
Case 2. n is even.

Let $P : u_1, u_2, \ldots, u_{n-2}$. Then $\psi_{dm} = (E(W_n) - E(P)) \cup \{P\}$ is a minimum detour monophonic graphoidal cover of W_n and so $\eta_{dm}(W_n) = 2(n-1)-(n-3)+1 = n+2$. ■

Theorem 2.2 *If $G = P_r \circ W_n$, then*

$$\eta_{dm}(G) = \begin{cases} 10r - 1 & if\ \ n = 5 \\ r(2n+1) - 1 & if\ \ n = 7, 9, 11, \ldots \\ 2r(n+1) - 1 & if\ \ n = 6, 8, 10, \ldots. \end{cases}$$

Proof. Let $P_r : u_1, u_2, \ldots, u_r$ be a path of order r and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$.
Case 1. $n = 5$.

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_r, v_{r,1}$;

$M_{i+1} : v_{i,2}, v_{i,3}, v_{i,4}$ $(1 \leq i \leq r)$;

$M_i' : v_{i,2}, v_{i,5}, v_{i,4}$ $(1 \le i \le r)$;
$M_i'' : v_{i,2}, v_{i,1}, v_{i,4}$ $(1 \le i \le r)$;
$S_1 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{5} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_r, v_{r,1})\}$; and
$S_2 = \bigcup_{i=1}^{r} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\}$.

It is clear that every $M_i (1 \le i \le r+1)$, $M_i' (1 \le i \le r)$ and $M_i'' (1 \le i \le r)$ are detour monophonic paths in G and every element in $S_1 \cup S_2$ is a detour monophonic path in G. Hence $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, \ldots, M_{r+1}, M_1', M_2', \ldots, M_r', M_1'', M_2'', \ldots, M_r''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (3r + 1) + (5r - 2) + 2r = 10r - 1$.

Case 2. $n = 7, 9, 11, \ldots$.

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_r, v_{r,1}$;
$M_{i+1} : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \le i \le r)$;
$M_{i+1}' : v_{i,2}, v_{i,n}, v_{i,n-1}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \le i \le r)$;
$S_1 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_r, v_{r,1})\}$; and
$S_2 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j})$.

It is clear that every M_i $(1 \le i \le r+1)$, and M_{i+1}' $(1 \le i \le r)$ are detour monophonic paths in G and every element in $S_1 \cup S_2$ is a detour monophonic path in G. Hence $\psi_{dm} = S_1 \cup S_2 \cup \left\{M_1, M_2, \ldots, M_{r+1}, M_2', M_3', \ldots, M_{r+1}'\right\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (2r + 1) + (rn - 2) + r(n - 1) = r(2n + 1) - 1$.

Case 3. $n = 6, 8, 10, \ldots$

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_r, v_{r,1}$;
$M_{i+1} : v_{i,2}, v_{i,3}, \ldots, v_{i,n-1}$ $(1 \le i \le r)$;
$M_i' : v_{i,2}, v_{i,n}$ $(1 \le i \le r)$;
$M_i'' : v_{i,n}, v_{i,n-1}$ $(1 \le i \le r)$;
$S_1 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_r, v_{r,1})\}$; and
$S_2 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j})$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, \ldots, M_{r+1}, M_1', M_2', \ldots, M_r', M_1'', M_2'', \ldots, M_r''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (3r + 1) + (nr - 2) + r(n - 1) = 2r(n + 1) - 1$. ∎

Theorem 2.3 *If $G = W_n \circ P_r$, then*

$$\eta_{dm}(G) = \begin{cases} 5(r+1) + 3 & if \;\; n = 5 \\ n(r+2) - 1 & if \;\; n > 5. \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$ and let $P_r : u_1, u_2, \ldots, u_r$ be a path of order r.

Case 1. $n = 5$.

Let $M_1 : u_{2,1}, v_2, v_3, v_4, u_{4,1}$;

$M_2 : v_2, v_5, v_4;$

$M_3 : v_2, v_1, v_4;$

$M_{i+3} : u_{i,1}, u_{i,2}, \ldots, u_{i,r}$ $(1 \leq i \leq 5)$; and

$S = (\bigcup_{i=1}^{5} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_4, u_{4,1})\}) \cup \{(v_1, v_3), (v_1, v_5)\}.$

It is clear that $\psi_{dm} = S \cup \{M_1, M_2, M_3, \ldots, M_{r+3}\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 8 + 5r = 5(r+1) + 3$.

Case 2. $n = 7, 9, 11, \ldots$

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{\frac{n+3}{2}}, u_{\frac{n+3}{2},1};$

$M_2 : v_2, v_n, v_{n-1}, \ldots, v_{\frac{n+3}{2}};$

$M_{i+2} : u_{i,1}, u_{i,2}, \ldots, u_{i,r}$ $(1 \leq i \leq n)$;

$S_1 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \left\{(v_2, u_{2,1}), (v_{\frac{n+3}{2}}, u_{\frac{n+3}{2},1})\right\}$; and

$S_2 = \cup_{i=2}^{n} (v_1, v_i).$

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, \ldots, M_{n+2}\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (n+2) + (nr-2) + (n-1) = n(r+2) - 1$.

Case 3. $n = 6, 8, 10, \ldots$

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n-1}, u_{n-1,1};$

$M_2 : v_{n-1}, v_n, u_{n,1};$

$M_3 : v_n, v_2;$

$M_{i+3} : u_{i,1}, u_{i,2}, \ldots, u_{i,r}$ $(1 \leq i \leq n)$;

$S_1 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1}), (v_n, u_{n,1})\}$; and

$S_2 = \cup_{i=2}^{n} (v_1, v_i).$

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, \ldots, M_{r+3}\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (n+3) + (nr-3) + (n-1) = n(r+2) - 1$. ■

Theorem 2.4 *If $G = C_r \circ W_n$, then*

$$\eta_{dm}(G) = \begin{cases} 10r & if \;\; n = 5 \\ r(2n+1) & if \;\; n = 7, 9, 11, \ldots \\ 2r(n+1) & if \;\; n = 6, 8, 10, \ldots . \end{cases}$$

Proof. Let $C_r : u_1, u_2, \ldots, u_r, u_1$ be a cycle of order r and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$. Let G be the corona product of C_r and W_n.

Case 1. $r = 3$

Subcase (i): $n = 5$.

Let $M_1 : v_{1,1}, u_1, u_2;$

$M_2 : v_{2,1}, u_2, u_3;$

$M_3 : v_{3,1}, u_3, u_1$;
$M_i' : v_{i,2}, v_{i,3}, v_{i,4}$ $(1 \le i \le 3)$;
$M_i'' : v_{i,2}, v_{i,5}, v_{i,4}$ $(1 \le i \le 3)$;
$M_i''' : v_{i,2}, v_{i,1}, v_{i,4}$ $(1 \le i \le 3)$; and
$S = \{\bigcup_{i=1}^{3} \bigcup_{j=2}^{5} (u_i, v_{i,j})\} \cup \{(v_{1,1}, v_{1,3}), (v_{1,1}, v_{1,5}), (v_{2,1}, v_{2,3}), (v_{2,1}, v_{2,5}), (v_{3,1}, v_{3,3}), (v_{3,1}, v_{3,5})\}$.

It is clear that $\psi_{dm} = S \cup \{M_1, M_2, M_3, M_1', M_2', M_3', M_1'', M_2'', M_3'', M_1''', M_2''', M_3'''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 30 = 10r$.

Subcase (ii): $n = 7, 9, 11, \ldots$

Let $M_1 : v_{1,1}, u_1, u_2$;
$M_2 : v_{2,1}, u_2, u_3$;
$M_3 : v_{3,1}, u_3, u_1$;
$M_i' : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \le i \le 3)$;
$M_i'' : v_{i,2}, v_{i,n}, v_{i,n-1}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \le i \le 3)$;
$S_1 = \bigcup_{i=1}^{3} \bigcup_{j=2}^{n} (u_i, v_{i,j})$; and
$S_2 = \bigcup_{i=1}^{3} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j})$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M_1', M_2', M_3', M_1'', M_2'', M_3''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 3(n-1) + 3(n-1) + 9 = 3(2n+1) = r(2n+1)$.

Subcase (iii): $n = 6, 8, 10, \ldots$

Let $M_1 : v_{2,1}, u_1, u_2$;
$M_2 : v_{2,1}, u_2, u_3$;
$M_3 : v_{3,1}, u_3, u_1$;
$M_i' : v_{i,2}, v_{i,3}, \ldots, v_{i,n-1}$ $(1 \le i \le 3)$;
$S_1 = \bigcup_{i=1}^{3} \{(v_{i,2}, v_{i,n}), (v_{i,n}, v_{i,n-1})\}$;
$S_2 = \bigcup_{i=1}^{3} \bigcup_{j=2}^{n} (u_i, v_{i,j})$; and
$S_3 = \bigcup_{i=1}^{3} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j})$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M_1', M_2', M_3'\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 6 + 3(n-1) + 3(n-1) + 6 = 6(n+1) = 2r(n+1)$.

Case 2. $r > 3$ and r is even.

Subcase (i): $n = 5$.

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{\frac{r}{2}+1}, v_{\frac{r}{2}+1,1}$;
$M_2 : u_1, u_r, u_{r-1}, \ldots, u_{\frac{r}{2}+1}$;
$M_i' : v_{i,2}, v_{i,3}, v_{i,4}$ $(1 \le i \le r)$;
$M_i'' : v_{i,2}, v_{i,5}, v_{i,4}$ $(1 \le i \le r)$;

$M_i''' : v_{i,2}, v_{i,1}, v_{i,4}$ $(1 \leq i \leq r)$;
$S_1 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{5} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_{\frac{r}{2}+1}, v_{\frac{r}{2}+1,1})\}$; and
$S_2 = \bigcup_{i=1}^{r} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_1', M_2', \ldots, M_r', M_1'', M_2'', \ldots, M_r'', M_1''', M_2''', \ldots, M_r'''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (3r+2) + (5r-2) + 2r = 10r$.

Subcase (ii): $n = 7, 9, 11, \ldots$.

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{\frac{r}{2}+1}, v_{\frac{r}{2}+1,1}$;
$M_2 : u_1, u_r, u_{r-1}, \ldots, u_{\frac{r}{2}+1}$;
$M_i' : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \leq i \leq r)$;
$M_i'' : v_{i,2}, v_{i,n}, v_{i,n-1}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \leq i \leq r)$;
$S_1 = (\bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (u_i, v_{i,j})) - \{(u_1, v_{1,1}), (u_{\frac{r}{2}+1}, v_{\frac{r}{2}+1,1})\}$; and
$S_2 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j})$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_1', M_2', \ldots, M_r', M_1'', M_2'', \ldots, M_r''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (2r+2) + (nr-2) + r(n-1) = r(2n+1)$.

Subcase (iii): $n = 6, 8, 10, \ldots$.

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{\frac{r}{2}+1}, v_{\frac{r}{2}+1,1}$;
$M_2 : u_1, u_r, u_{r-1}, \ldots, u_{\frac{r}{2}+1}$;
$M_i' : v_{i,2}, v_{i,3}, \ldots, v_{i,n-1}$ $(1 \leq i \leq r)$;
$S_1 = \bigcup_{i=1}^{r} \{(v_{i,2}, v_{i,n}), (v_{i,n}, v_{i,n-1})\}$;
$S_2 = (\bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (u_i, v_{i,j})) - \{(u_1, v_{1,1}), (u_{\frac{r}{2}+1}, v_{\frac{r}{2}+1,1})\}$; and
$S_3 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j})$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_1', M_2', \ldots, M_r'\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (r+2) + 2r + (nr-2) + r(n-1) = 2r(n+1)$.

Case 3. $r > 3$ and r is odd.

Subcase (i): $n = 5$.

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}$;
$M_2 : u_{r-1}, u_r, v_{r,1}$;
$M_3 : u_r, u_1$;
$M_i' : v_{i,2}, v_{i,3}, v_{i,4}$ $(1 \leq i \leq r)$;
$M_i'' : v_{i,2}, v_{i,5}, v_{i,4}$ $(1 \leq i \leq r)$;
$M_i''' : v_{i,2}, v_{i,1}, v_{i,4}$ $(1 \leq i \leq r)$;
$S_1 = (\bigcup_{i=1}^{r} \bigcup_{j=1}^{5} (u_i, v_{i,j})) - \{(u_1, v_{1,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}$; and
$S_2 = \bigcup_{i=1}^{r} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M_1', M_2', \ldots, M_r', M_1'', M_2'', \ldots, M_r'', M_1''', M_2''', \ldots, M_r'''\}$ is a minimum detour monophonic graphoidal cover of G

and so $\eta_{dm}(G) = (3r+3) + (5r-3) + 2r = 10r$.
Subcase (ii): $n = 7, 9, 11, \ldots$.
Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}$;
$M_2 : u_{r-1}, u_r, v_{r,1}$;
$M_3 : u_r, u_1$;
$M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \le i \le r)$;
$M''_i : v_{i,2}, v_{i,n}, v_{i,n-1}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \le i \le r)$;
$S_1 = (\bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (u_i, v_{i,j})) - \{(u_1, v_{1,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}$; and
$S_2 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j})$.
It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_r, M''_1, M''_2, \ldots, M''_r\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (2r+3) + (nr-3) + r(n-1) = r(2n+1)$.
Subcase (iii): $n = 6, 8, 10, \ldots$.
Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}$;
$M_2 : u_{r-1}, u_r, v_{r,1}$;
$M_3 : u_r, u_1$;
$M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,n-1}$ $(1 \le i \le r)$;
$S_1 = \bigcup_{i=1}^{r} \{(v_{i,2}, v_{i,n}), (v_{i,n}, v_{i,n-1})\}$;
$S_2 = (\bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (u_i, v_{i,j})) - \{(u_1, v_{1,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}$; and
$S_3 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j})$.
It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_r\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (r+3) + 2r + (nr-3) + r(n-1) = 2r(n+1)$. ■

Theorem 2.5 *If $G = W_n \circ C_r$, then*

$$\eta_{dm}(G) = \begin{cases} 5r+13 & \text{if } r \text{ is even and } n = 5 \\ 5r+18 & \text{if } r \text{ is odd and } n = 5 \\ n(r+3)-1 & \text{if } r \text{ is even and } n > 5 \\ n(r+4)-1 & \text{if } r \text{ is odd and } n > 5. \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$ and let $C_r : u_1, u_2, \ldots, u_r, u_1$ be a cycle of order r. Let G be the corona product of W_n and C_r.
Case 1. $n = 5$.
Subcase (i): r is even.
Let $M_1 : u_{2,1}, v_2, v_3, v_4, u_{4,1}$;
$M_2 : v_2, v_5, v_4$;

$M_3 : v_2, v_1, v_4$;
$M_i' : u_{i,1}, u_{i,2}, \ldots, u_{i,\frac{r}{2}+1}$ $(1 \leq i \leq 5)$;
$M_i'' : u_{i,1}, u_{i,r}, \ldots, u_{i,\frac{r}{2}+1}$ $(1 \leq i \leq 5)$;
$S_1 = \{(v_1, v_3), (v_1, v_5)\}$; and
$S_2 = \bigcup_{i=1}^{5} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_4, u_{4,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M_1', M_2', M_3', M_4', M_5', M_1'', M_2'', M_3'', M_4'', M_5''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 13 + 2 + (5r - 2) = 5r + 13$.

Subcase (ii): r is odd.

Let $M_1 : u_{2,1}, v_2, v_3, v_4, u_{4,1}$;
$M_2 : v_2, v_5, v_4$;
$M_3 : v_2, v_1, v_4$;
$M_i' : u_{i,1}, u_{i,2}, \ldots, u_{i,r-1}$ $(1 \leq i \leq 5)$;
$S_1 = \bigcup_{i=1}^{5} \{(u_{i,r-1}, u_{i,r}), (u_{i,r}, u_{i,1})\}$;
$S_2 = \{(v_1, v_3), (v_1, v_5)\}$; and
$S_3 = \bigcup_{i=1}^{5} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_4, u_{4,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M_1', M_2', M_3', M_4', M_5'\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 8 + 10 + 2 + (5r - 2) = 5r + 18$.

Case 2. $n = 7, 9, 11, \ldots$

Subcase (i): r is even.

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{\frac{n+3}{2}}, u_{\frac{n+3}{2},1}$;
$M_2 : v_2, v_n, v_{n-1}, \ldots, v_{\frac{n+3}{2}}$;
$M_i' : u_{i,1}, u_{i,2}, \ldots, u_{i,\frac{r}{2}+1}$ $(1 \leq i \leq n)$;
$M_i'' : u_{i,1}, u_{i,r}, \ldots, u_{i,\frac{r}{2}+1}$ $(1 \leq i \leq n)$;
$S_1 = \cup_{i=2}^{n} (v_1, v_i)$; and
$S_2 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \left\{(v_2, u_{2,1}), (v_{\frac{n+3}{2}}, u_{\frac{n+3}{2},1})\right\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_1', M_2', \ldots, M_n', M_1'', M_2'', \ldots, M_n''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (2n + 2) + (n - 1) + (nr - 2) = n(r + 3) - 1$.

Subcase (ii): r is odd.

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{\frac{n+3}{2}}, u_{\frac{n+3}{2},1}$;
$M_2 : v_2, v_n, v_{n-1}, \ldots, v_{\frac{n+3}{2}}$;
$M_i' : u_{i,1}, u_{i,2}, \ldots, u_{i,r-1}$ $(1 \leq i \leq n)$;
$S_1 = \bigcup_{i=1}^{n} \{(u_{i,r-1}, u_{i,r}), (u_{i,r}, u_{i,1})\}$;
$S_2 = \bigcup_{i=2}^{n} (v_1, v_i)$; and
$S_3 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \left\{(v_2, u_{2,1}), (v_{\frac{n+3}{2}}, u_{\frac{n+3}{2},1})\right\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M'_1, M'_2, \ldots, M'_n\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (n+2) + 2n + (n-1) + (nr-2) = n(r+4) - 1$.

Case 3. $n = 6, 8, 10, \ldots$

Subcase (i): r is even.

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n-1}, u_{n-1,1}$;

$M_2 : v_{n-1}, v_n, u_{n,1}$;

$M_3 : v_n, v_2$;

$M'_i : u_{i,1}, u_{i,2}, \ldots, u_{i,\frac{r}{2}+1}$ $(1 \le i \le n)$;

$M''_i : u_{i,1}, u_{i,r}, \ldots, u_{i,\frac{r}{2}+1}$ $(1 \le i \le n)$;

$S_1 = \bigcup_{i=2}^{n} (v_1, v_i)$; and

$S_2 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1}), (v_n, u_{n,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_n, M''_1, M''_2, \ldots, M''_n\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (2n+3) + (n-1) + (nr-3) = n(r+3) - 1$.

Subcase(ii): r is odd.

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n-1}, u_{n-1,1}$;

$M_2 : v_{n-1}, v_n, u_{n,1}$;

$M_3 : v_n, v_2$;

$M'_i : u_{i,1}, u_{i,2}, \ldots, u_{i,r-1}$ $(1 \le i \le n)$;

$S_1 = \bigcup_{i=1}^{n} \{(u_{i,r-1}, u_{i,r}), (u_{i,r}, u_{i,1})\}$;

$S_2 = \bigcup_{i=2}^{n} (v_1, v_i)$; and

$S_3 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1}), (v_n, u_{n,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_n\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (n+3) + 2n + (n-1) + (nr-3) = n(r+4) - 1$. ■

Theorem 2.6 *If $G = K_r \circ W_n$, $r \ge 3$, then*

$$\eta_{dm}(G) = \begin{cases} \frac{r}{2}(r+17) & if \;\; n = 5 \\ \frac{r}{2}(r+4n-1) & if \;\; n = 7, 9, 11, \ldots \\ \frac{r}{2}(r+4n+1) & if \;\; n = 6, 8, 10, \ldots. \end{cases}$$

Proof. Let K_r be the complete graph with the vertex set $\{u_1, u_2, \ldots, u_r\}$, and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$.

Case 1. $n = 5$.

Let $M_i : v_{i,1}, u_i, u_{i+1}$ $(1 \le i \le r-1)$;

$M_r : v_{r,1}, u_r, u_1$;

$M'_i : v_{i,2}, v_{i,3}, v_{i,4}$ $(1 \le i \le r)$;

$M''_i : v_{i,2}, v_{i,5}, v_{i,4}$ $(1 \le i \le r)$;

$M_i''' : v_{i,2}, v_{i,1}, v_{i,4}$ $(1 \leq i \leq r)$;
$S_1 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{5} (u_i, v_{i,j})$;
$S_2 = \bigcup_{i=1}^{r} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\}$; and
$S_3 = E(K_r) - \{(u_1, u_2), (u_2, u_3), \ldots, (u_{r-1}, u_r), (u_r, u_1)\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, \ldots, M_r, M_1', M_2', \ldots, M_r', M_1'', M_2'', \ldots, M_r'', M_1''', M_2''', \ldots, M_r'''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 4r + 4r + 2r + \frac{r(r-1)}{2} - r = \frac{r}{2}(r + 17)$.

Case 2. $n = 7, 9, 11, \ldots$

Let $M_i : v_{i,1}, u_i, u_{i+1}$ $(1 \leq i \leq r - 1)$;
$M_r : v_{r,1}, u_r, u_1$;
$M_i' : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \leq i \leq r)$;
$M_i'' : v_{i,2}, v_{i,n}, v_{i,n-1}, \ldots, v_{i,\frac{n+3}{2}}$ $(1 \leq i \leq r)$;
$S_1 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} (u_i, v_{i,j})$;
$S_2 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} \{(v_{i,1}, v_{i,j})\}$; and
$S_3 = E(K_r) - \{(u_1, u_2), (u_2, u_3), \ldots, (u_{r-1}, u_r), (u_r, u_1)\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, \ldots, M_r, M_1', M_2', \ldots, M_r', M_1'', M_2'', \ldots, M_r''\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 3r + r(n-1) + r(n-1) + \frac{r(r-1)}{2} - r = \frac{r}{2}(r + 4n - 1)$.

Case 3. $n = 6, 8, 10, \ldots$

Let $M_i : v_{i,1}, u_i, u_{i+1}$ $(1 \leq i \leq r - 1)$;
$M_r : v_{r,1}, u_r, u_1$;
$M_i' : v_{i,2}, v_{i,3}, \ldots, v_{i,n-1}$ $(1 \leq i \leq r)$;
$S_1 = \bigcup_{i=1}^{r} \{(v_{i,2}, v_{i,n}), (v_{i,n}, v_{i,n-1})\}$;
$S_2 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} (u_i, v_{i,j})$;
$S_3 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j})$; and
$S_4 = E(K_r) - \{(u_1, u_2), (u_2, u_3), \ldots, (u_{r-1}, u_r), (u_r, u_1)\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{M_1, M_2, \ldots, M_r, M_1', M_2', \ldots, M_r'\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 2r + 2r + r(n-1) + r(n-1) + \frac{r(r-1)}{2} - r = \frac{r}{2}(r + 4n + 1)$. ∎

Theorem 2.7 *If $G = W_n \circ K_r$, then*

$$\eta_{dm}(G) = \begin{cases} \frac{1}{2}(5r^2 + 5r + 6) & if \;\; n = 5 \\ \frac{1}{2}(nr^2 + nr + 2n - 2) & if \;\; n = 7, 9, 11, \ldots \\ \frac{1}{2}(nr^2 + nr + 2n) & if \;\; n = 6, 8, 10, \ldots. \end{cases}$$

Proof. Let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$, and let K_r be the complete graph with the vertex set $\{u_1, u_2, \ldots, u_r\}$.

Case 1. $n = 5$.

Let $M_1 : u_{2,1}, v_2, v_3, v_4, u_{4,1}$;
$M_2 : v_2, v_5, v_4$;
$M_3 : v_2, v_1, v_4$;
$S_1 = \bigcup_{i=1}^{5} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_4, u_{4,1})\}$; and
$S_2 = \bigcup_{i=1}^{5} E(K_r^i) \cup \{(v_1, v_3), (v_1, v_5)\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 3 + (5r - 2) + 5.\frac{r(r-1)}{2} + 2 = \frac{1}{2}(5r^2 + 5r + 6)$.

Case 2. $n = 7, 9, 11, \ldots$.

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{\frac{n+3}{2}}, u_{\frac{n+3}{2},1}$;
$M_2 : v_2, v_n, v_{n-1}, \ldots, v_{\frac{n+3}{2}}$;
$S_1 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \left\{(v_2, u_{2,1}), (v_{\frac{n+3}{2}}, u_{\frac{n+3}{2},1})\right\}$;
$S_2 = \bigcup_{i=1}^{n} E(K_r^i)$; and
$S_3 = \bigcup_{i=2}^{n} (v_1, v_i)$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 2 + (nr - 2) + n.\frac{r(r-1)}{2} + (n - 1) = \frac{1}{2}(nr^2 + nr + 2n - 2)$.

Case 3. $n = 6, 8, 10, \ldots$.

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n-1}, u_{n-1,1}$;
$M_2 : v_{n-1}, v_n, u_{n,1}$;
$M_3 : v_n, v_2$;
$S_1 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1})\}$;
$S_2 = \bigcup_{i=1}^{n} E(K_r^i)$; and
$S_3 = \bigcup_{i=2}^{n} (v_1, v_i)$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3\}$ is a minimum detour monophonic graphoidal cover of G and hence $\eta_{dm}(G) = 3 + (nr - 2) + n.\frac{r(r-1)}{2} + n - 1 = \frac{1}{2}(nr^2 + nr + 2n)$. ■

Theorem 2.8 *If $G = W_r \circ W_s$, then*

$$\eta_{dm}(G) = \begin{cases} 53 & if\ \ r = s = 5 \\ 11r - 1 & if\ \ s = 5\ \ and\ \ r \geq 6 \\ 2(5s + 4) & if\ \ r = 5\ \ and\ \ s = 7, 9, 11, \ldots \\ 10s + 13 & if\ \ r = 5\ \ and\ \ s = 6, 8, 10, \ldots \\ 2r(s + 1) - 1 & if\ \ r \geq 6\ \ and\ \ s = 7, 9, 11, \ldots \\ r(2s + 3) - 1 & if\ \ r \geq 6\ \ and\ \ s = 6, 8, 10, \ldots. \end{cases}$$

Proof. Let $W_r = K_1 + C_{r-1}$ be a wheel with $V(K_1) = \{u_1\}$ and $V(C_{r-1}) =$

$\{u_2, u_3, \ldots, u_r\}$ and let $W_s = K_1 + C_{s-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{s-1}) = \{v_2, v_3, \ldots, v_s\}$.

Case 1. $r = 5$.

Subcase (i): $s = 5$.

Let $M_1 : v_{2,1}, u_2, u_3, u_4, v_{4,1}$;

$M_2 : u_2, u_5, u_4$;

$M_3 : u_2, u_1, u_4$;

$M'_i : v_{i,2}, v_{i,3}, v_{i,4}$ $(1 \leq i \leq 5)$;

$M''_i : v_{i,2}, v_{i,5}, v_{i,4}$ $(1 \leq i \leq 5)$;

$M'''_i : v_{i,2}, v_{i,1}, v_{i,4}$ $(1 \leq i \leq 5)$;

$S_1 = \{(u_1, u_3), (u_1, u_5)\}$;

$S_2 = \bigcup_{i=1}^{5} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\}$; and

$S_3 = \bigcup_{i=1}^{5} \bigcup_{j=1}^{5} (u_i, v_{i,j}) - \{(u_2, v_{2,1}), (u_4, v_{4,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M'_1, M'_2, M'_3, M'_4, M'_5, M''_1, M''_2, M''_3, M''_4, M''_5, M'''_1, M'''_2, M'''_3, M'''_4, M'''_5\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 18 + 12 + 23 = 53$.

Subcase (ii): $s = 7, 9, 11, \ldots$.

Let $M_1 : v_{2,1}, u_2, u_3, u_4, v_{4,1}$;

$M_2 : u_2, u_5, u_4$;

$M_3 : u_2, u_1, u_4$;

$M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{s+3}{2}}$ $(1 \leq i \leq 5)$;

$M''_i : v_{i,2}, v_{i,s}, \ldots, v_{i,\frac{s+3}{2}}$ $(1 \leq i \leq 5)$;

$S_1 = \{(u_1, u_3), (u_1, u_5)\}$;

$S_2 = \bigcup_{i=1}^{5} \bigcup_{j=2}^{s} (v_{i,1}, v_{i,j})$; and

$S_3 = \bigcup_{i=1}^{5} \bigcup_{j=1}^{s} (u_i, v_{i,j}) - \{(u_2, v_{2,1}), (u_4, v_{4,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M'_1, M'_2, M'_3, M'_4, M'_5, M''_1, M''_2, M''_3, M''_4, M''_5\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 13 + 2 + 5(s-1) + (5s-2) = 2(5s+4)$.

Subcase (iii): $s = 6, 8, 10, \ldots$.

Let $M_1 : v_{2,1}, u_2, u_3, u_4, v_{4,1}$;

$M_2 : u_2, u_5, u_4$;

$M_3 : u_2, u_1, u_4$;

$M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,s-1}$ $(1 \leq i \leq 5)$;

$S_1 = \bigcup_{i=1}^{5} \{(v_{i,s-1}, v_{i,s}), (v_{i,s}, v_{i,2})\}$;

$S_2 = \{(u_1, u_3), (u_1, u_5)\}$;

$S_3 = \bigcup_{i=1}^{5} \bigcup_{j=2}^{s} (v_{i,1}, v_{i,j})$; and

$S_4 = \bigcup_{i=1}^{5} \bigcup_{j=1}^{s} (u_i, v_{i,j}) - \{(u_2, v_{2,1}), (u_4, v_{4,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{M_1, M_2, M_3, M'_1, M'_2, M'_3, M'_4, M'_5\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = 8 + 10 + 2 + 5(s-1) + (5s-2) = 10s + 13$.

Case 2. $r = 7, 9, 11, \ldots$.

Subcase (i): $s = 5$.

Let $M_1 : v_{2,1}, u_2, u_3, \ldots, u_{\frac{r+3}{2}}, v_{\frac{r+3}{2},1}$;

$M_2 : u_2, u_r, u_{r-1}, \ldots, u_{\frac{r+3}{2}}$;

$M'_i : v_{i,2}, v_{i,3}, v_{i,4}$ $(1 \le i \le r)$;

$M''_i : v_{i,2}, v_{i,5}, v_{i,4}$ $(1 \le i \le r)$;

$M'''_i : v_{i,2}, v_{i,1}, v_{i,4}$ $(1 \le i \le r)$;

$S_1 = \bigcup_{i=2}^{r} (u_1, u_i)$;

$S_2 = \bigcup_{i=1}^{r} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\}$; and

$S_3 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{5} (u_i, v_{i,j}) - \left\{(u_2, v_{2,1}), (u_{\frac{r+3}{2}}, v_{\frac{r+3}{2},1})\right\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M'_1, M'_2, \ldots, M'_r, M''_1, M''_2, \ldots, M''_r, M'''_1, M'''_2, \ldots, M'''_r\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (3r+2) + (r-1) + 2r + (5r-2) = 11r - 1$.

Subcase (ii): $s = 7, 9, 11, \ldots$.

Let $M_1 : v_{2,1}, u_2, u_3, \ldots, u_{\frac{r+3}{2}}, v_{\frac{r+3}{2},1}$;

$M_2 : u_2, u_r, u_{r-1}, \ldots, u_{\frac{r+3}{2}}$;

$M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{s+3}{2}}$ $(1 \le i \le r)$;

$M''_i : v_{i,2}, v_{i,s}, v_{i,s-1}, \ldots, v_{i,\frac{s+3}{2}}$ $(1 \le i \le r)$;

$S_1 = \bigcup_{i=2}^{r} (u_1, u_i)$;

$S_2 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{s} (v_{i,1}, v_{i,j})$; and

$S_3 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} (u_i, v_{i,j}) - \left\{(u_2, v_{2,1}), (u_{\frac{r+3}{2}}, v_{\frac{r+3}{2},1})\right\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M'_1, M'_2, \ldots, M'_r, M''_1, M''_2, \ldots, M''_r\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (2r+2) + (r-1) + r(s-1) + (rs-2) = 2r(s+1) - 1$.

Subcase (iii): $s = 6, 8, 10, \ldots$.

Let $M_1 : v_{2,1}, u_2, u_3, \ldots, u_{\frac{r+3}{2}}, v_{\frac{r+3}{2},1}$;

$M_2 : u_2, u_r, u_{r-1}, \ldots, u_{\frac{r+3}{2}}$;

$M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,s-1}$ $(1 \le i \le r)$;

$S_1 = \bigcup_{i=1}^{r} \{(v_{i,s-1}, v_{i,s}), (v_{i,s}, v_{i,1})\}$;

$S_2 = \bigcup_{i=2}^{r} (u_1, u_i)$;

$S_3 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{s} (v_{i,1}, v_{i,j})$; and

$S_4 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} (u_i, v_{i,j}) - \left\{(u_2, v_{2,1}), (u_{\frac{r+3}{2}}, v_{\frac{r+3}{2},1})\right\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{M_1, M_2, M'_1, M'_2, \ldots, M'_r\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (r+2) + 2r + (r-1) + r(s-1) + (rs-2) = r(2s+3) - 1$.

Case 3. $r = 6, 8, 10, \ldots$.

Subcase (i): $s = 5$.

Let $M_1 : v_{2,1}, u_2, u_3, \ldots, u_{r-1}, v_{r-1,1}$;

$M_2 : u_{r-1}, u_r, v_{r,1}$;

$M_3 : u_r, u_2$;

$M'_i : v_{i,2}, v_{i,3}, v_{i,4}$ $(1 \leq i \leq r)$;

$M''_i : v_{i,2}, v_{i,5}, v_{i,4}$ $(1 \leq i \leq r)$;

$M'''_i : v_{i,2}, v_{i,1}, v_{i,4}$ $(1 \leq i \leq r)$;

$S_1 = \bigcup_{i=2}^{r} (u_1, u_i)$;

$S_2 = \bigcup_{i=1}^{r} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\}$; and

$S_3 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{5} (u_i, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_r, M''_1, M''_2, \ldots, M''_r, M'''_1, M'''_2, \ldots, M'''_r\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (3r+3) + (r-1) + 2r + (5r-3) = 11r - 1$.

Subcase (ii): $s = 7, 9, 11, \ldots$.

Let $M_1 : v_{2,1}, u_2, u_3, \ldots, u_{r-1}, v_{r-1,1}$;

$M_2 : u_{r-1}, u_r, v_{r,1}$;

$M_3 : u_r, u_2$;

$M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{s+3}{2}}$ $(1 \leq i \leq r)$;

$M''_i : v_{i,2}, v_{i,s}, v_{i,s-1}, \ldots, v_{i,\frac{s+3}{2}}$ $(1 \leq i \leq r)$;

$S_1 = \bigcup_{i=2}^{r} (u_1, u_i)$;

$S_2 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{s} (v_{i,1}, v_{i,j})$; and

$S_3 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} (u_i, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_r, M''_1, M''_2, \ldots, M''_r\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (2r+3) + (r-1) + r(s-1) + (rs-3) = 2r(s+1) - 1$.

Subcase (iii): $s = 6, 8, 10, \ldots$.

Let $M_1 : v_{2,1}, u_2, u_3, \ldots, u_{r-1}, v_{r-1,1}$;

$M_2 : u_{r-1}, u_r, v_{r,1}$;

$M_3 : u_r, u_2$;

$M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,s-1}$ $(1 \leq i \leq r)$;

$S_1 = \bigcup_{i=1}^{r} \{(v_{i,s-1}, v_{i,s}), (v_{i,s}, v_{i,2})\}$;

$S_2 = \bigcup_{i=2}^{r} (u_1, u_i)$;

$S_3 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{s} (v_{i,1}, v_{i,j})$; and

$S_4 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} (u_i, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_r\}$ is a minimum detour monophonic graphoidal cover of G and so $\eta_{dm}(G) = (r+3) + 2r + (r-1) + r(s-1) + rs - 3 = r(2s+3) - 1$. ∎

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