Detour Monophonic Graphoidal Covering Number of Corona Product Graph of Some Standard Graphs with the Wheel

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ABSTRACT

A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called a monophonic path if it is a chordless path. A longest $x − y$ monophonic path is called an $x − y$ detour monophonic path. A detour monophonic graphoidal cover of a graph $G$ is a collection $\psi_{dm}$ of detour monophonic paths in $G$ such that every vertex of $G$ is an internal vertex of at most one detour monophonic path in $\psi_{dm}$ and every edge of $G$ is in exactly one detour monophonic path in $\psi_{dm}$. The minimum cardinality of a detour monophonic graphoidal cover of $G$ is called the detour monophonic graphoidal covering number of $G$ and is denoted by $\eta_{dm}(G)$. In this paper, we find the detour monophonic graphoidal covering number of corona product of wheel with some standard graphs.

Article history:
Received 5, September 2018
Received in revised form 11, March 2019
Accepted 13 May 2019
Available online 01, June 2019

Keyword: graphoidal cover, monophonic path, detour monophonic graphoidal cover, detour monophonic graphoidal covering number.

AMS subject Classification: 05C78.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology we refer to Harary[6].

The concept of graphoidal cover was introduced by Acharya and Sampathkumar[2] and further studied in [1, 3, 7, 8]. A graphoidal cover of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions: (i) Every path in $\psi$ has at least two vertices, (ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$, and (iii) Every edge of $G$ is in exactly one path in $\psi$. The minimum cardinality of a graphoidal cover of $G$ is called the graphoidal covering number of $G$ and is denoted by $\eta(G)$. The collection $\psi$ is called an acyclic graphoidal cover of $G$ if no member of $\psi$ is cycle; it is called a geodesic graphoidal cover if every member of $\psi$ is a shortest path in $G$. The minimum cardinality of an acyclic (geodesic) graphoidal cover of $G$ is called the acyclic (geodesic) graphoidal covering number of $G$ and is denoted by $\eta_a(G)$ and $\eta_g(G)$. The acyclic graphoidal covering number and geodesic graphoidal covering number are studied in [4, 5].

A chord of a path $P$ is an edge joining any two non-adjacent vertices of $P$. A path $P$ is called a monophonic path if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ detour monophonic path. For any two vertices $u$ and $v$ in a connected graph $G$, the monophonic distance $d_m(u, v)$ from $u$ to $v$ is defined as the length of a longest $u - v$ monophonic path in $G$. The monophonic eccentricity $e_m(v)$ of a vertex $v$ in $G$ is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. The monophonic radius is $\rad_m(G) = \min\{e_m(v) : v \in V(G)\}$ and the monophonic diameter is $\diam_m(G) = \max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [10, 11].

A detour monophonic graphoidal cover of a graph $G$ is a collection $\psi_{dm}$ of detour monophonic paths in $G$ such that every vertex of $G$ is an internal vertex of at most one detour monophonic path in $\psi_{dm}$ and every edge of $G$ is in exactly one detour monophonic path in $\psi_{dm}$. The minimum cardinality of a detour monophonic graphoidal cover of $G$ is called the detour monophonic graphoidal covering number of $G$ and is denoted by $\eta_{dm}(G)$. The detour monophonic graphoidal cover was introduced in [13] and further studied in [14,15].

Definition 1.1 The corona of two graphs $G$ and $H$ is the graph $G \circ H$ formed from one copy of $G$ and $|V(G)|$ copies of $H$, where the $i^{th}$ vertex of $G$ is adjacent to every vertex in the $i^{th}$ copy of $H$.

Throughout this paper $G$ denotes a connected graph with at least two vertices.
2 Detour monophonic graphoidal cover on corona product of wheel with standard graphs

Theorem 2.1  
For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 5)$, 

$$
\eta_{dm}(W_n) = \begin{cases} 
n & \text{if } n = 5 \\
n + 1 & \text{if } n \text{ is odd and } n > 5 \\
n + 2 & \text{if } n \text{ is even.}
\end{cases}
$$

Proof. Let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v\}$ and $V(C_{n-1}) = \{u_1, u_2, \ldots, u_{n-1}\}$.

Case 1. $n$ is odd.

Subcase (i): $n = 5$.

Let $P_1 : u_1, v, u_3$; $P_2 : u_1, u_2, u_3$; $P_3 : u_1, u_4, u_3$; $P_4 : v, u_2$; $P_5 : v, u_4$.

It is clear that $\psi_{dm} = \{P_1, P_2, P_3, P_4, P_5\}$ is a minimum detour monophonic graphoidal cover of $W_n$. Hence $\eta_{dm}(W_n) = 5 = n$.

Subcase (ii): $n = 7, 9, \ldots$

Let $P_1 : u_1, u_2, \ldots, u_{\frac{n+1}{2}}$; $P_2 : u_1, u_{n-1}, u_{n-2}, \ldots, u_{\frac{n+1}{2}}$; and $P_{i+2} : v, u_{i} \ (1 \leq i \leq n-1)$.

Similar to Subcase (i), we have $\eta_{dm}(G) = n + 1$.

Case 2. $n$ is even.

Let $P : u_1, u_2, \ldots, u_{n-2}$. Then $\psi_{dm} = (E(W_n) - E(P)) \cup \{P\}$ is a minimum detour monophonic graphoidal cover of $W_n$ and so $\eta_{dm}(W_n) = 2(n-1)-(n-3)+1 = n+2$.

Theorem 2.2  
If $G = P_r \circ W_n$, then 

$$
\eta_{dm}(G) = \begin{cases} 
10r - 1 & \text{if } n = 5 \\
3r(2n+1) - 1 & \text{if } n = 7, 9, 11, \ldots \\
2r(n+1) - 1 & \text{if } n = 6, 8, 10, \ldots
\end{cases}
$$

Proof. Let $P_r : u_1, u_2, \ldots, u_r$ be a path of order $r$ and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$.

Case 1. $n = 5$.

Let $M_1 : v_{1,1}, u_1, u_2, \ldots, u_r, v_{r,1}$; $M_{i+1} : v_{i,2}, v_{i,3}, v_{i,4} \ (1 \leq i \leq r)$;
$M'_i : v_{i,2}, v_{i,5}, v_{i,4}$ $(1 \leq i \leq r)$;
$M''_i : v_{i,2}, v_{i,1}, v_{i,4}$ $(1 \leq i \leq r)$;
$S_1 = \bigcup_{j=1}^{i} \bigcup_{j=1}^{i} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_r, v_{r,1})\};$ and
$S_2 = \bigcup_{j=1}^{i} (v_{i,1}, v_{i,3}); (v_{i,1}, v_{i,5})$.

It is clear that every $M_i (1 \leq i \leq r+1)$, $M'_i(1 \leq i \leq r)$ and $M''_i(1 \leq i \leq r)$ are detour monophonic paths in $G$ and every element in $S_1 \cup S_2$ is a detour monophonic path in $G$. Hence $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, \ldots, M_{r+1}, M'_1, M'_2, \ldots, M''_r, M''_2, \ldots, M''_r\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (3r+1) + (5r-2) + 2r = 10r - 1$.

Case 2. $n = 7, 9, 11, \ldots$.

Let $M_i : v_{1,1}, u_1, u_2, \ldots, u_r, v_{r,1}$;
$M_{i+1} : v_{1,2}, v_{1,3}, \ldots, v_{1, n-1}, \ldots, v_{1, n}$ $(1 \leq i \leq r)$;
$M'_i : v_{1,2}, v_{1,3}, \ldots, v_{1, n}$ $(1 \leq i \leq r)$;
$M''_i : v_{i,n}, v_{i,n-1}$ $(1 \leq i \leq r)$;
$S_1 = \bigcup_{j=1}^{i} \bigcup_{j=1}^{i} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_r, v_{r,1})\};$ and
$S_2 = \bigcup_{j=1}^{i} \bigcup_{j=2}^{i} (v_{1,1}, v_{i,j})$.

It is clear that every $M_i (1 \leq i \leq r+1)$, and $M'_{i+1} (1 \leq i \leq r)$ are detour monophonic paths in $G$ and every element in $S_1 \cup S_2$ is a detour monophonic path in $G$. Hence $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, \ldots, M_{r+1}, M'_1, M'_2, \ldots, M''_r, M''_2, \ldots, M''_r\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (2r+1) + (rn-2) + r(n-1) = r(2n+1) - 1$.

Case 3. $n = 6, 8, 10, \ldots$.

Let $M_i : v_{1,1}, u_1, u_2, \ldots, u_r, v_{r,1}$;
$M_{i+1} : v_{1,2}, v_{1,3}, \ldots, v_{1, n-1}$ $(1 \leq i \leq r)$;
$M'_i : v_{1,2}, v_{i,n}$ $(1 \leq i \leq r)$;
$M''_i : v_{i,n}, v_{i,n-1}$ $(1 \leq i \leq r)$;
$S_1 = \bigcup_{j=1}^{i} \bigcup_{j=1}^{i} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_r, v_{r,1})\};$ and
$S_2 = \bigcup_{j=1}^{i} \bigcup_{j=2}^{i} (v_{1,1}, v_{i,j})$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, \ldots, M_{r+1}, M'_1, M'_2, \ldots, M''_r, M''_2, \ldots, M''_r\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (3r+1) + (nr-2) + r(n-1) = 2r(n+1) - 1$.

\textbf{Theorem 2.3} \textit{If } $G = W_n \circ P_r$, then

$$\eta_{dm}(G) = \begin{cases} 5(r+1) + 3 & \text{if } n = 5 \\ n(r+2) - 1 & \text{if } n > 5. \end{cases}$$

\textbf{Proof.} Let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$ and let $P_r : u_1, u_2, \ldots, u_r$ be a path of order $r$.

Case 1. $n = 5$.

Let $M_1 : u_{2,1}, v_2, v_3, v_4, u_{4,1}$;
Subcase (i) $n = 5$.

Let $M_1 : v_{1,1}, u_1, u_2;
M_2 : v_{2,1}, u_2, u_3$;
\(M_3 : v_{3,1}, u_3, u_1;\)
\(M'_3 : v_{i,2}, v_{i,3}, v_{i,4} (1 \leq i \leq 3);\)
\(M''_3 : v_{i,2}, v_{i,5}, v_{i,4} (1 \leq i \leq 3);\)
\(M'''_3 : v_{i,2}, v_{i,1}, v_{i,4} (1 \leq i \leq 3);\)
\(S = \{ \bigcup_{i=1}^{3} \bigcup_{j=2}^{5} (u_{i}, v_{i,j}) \} \cup \{(v_{1,1}, v_{1,3}), (v_{1,1}, v_{1,5}), (v_{2,1}, v_{2,3}), (v_{2,1}, v_{2,5}), (v_{3,1}, v_{3,3}), (u_{3,1}, v_{3,5})\}.

It is clear that \(\psi_{dm} = S \cup \{ M_1, M_2, M_3, M'_1, M'_2, M'_3, M''_1, M''_2, M''_3, M'''_1, M'''_2, M'''_3 \}\) is a minimum detour monophonic graphoidal cover of \(G\) and so \(\eta_{dm}(G) = 30 = 10r.\)

**Subcase (ii):** \(n = 7, 9, 11, \ldots\)

Let \(M_1 : v_{1,1}, u_1, u_2;\)
\(M_2 : v_{2,1}, u_2, u_3;\)
\(M_3 : v_{3,1}, u_3, u_1;\)
\(M'_1 : v_{i,2}, v_{i,3}, \ldots, v_{i, \frac{n-1}{3}} (1 \leq i \leq 3);\)
\(M''_1 : v_{i,2}, v_{i,n}, v_{i,n-1}, \ldots, v_{i, \frac{n-1}{3}} (1 \leq i \leq 3);\)
\(S_1 = \bigcup_{i=1}^{3} \bigcup_{j=2}^{n} (u_{i}, v_{i,j});\) and
\(S_2 = \bigcup_{i=1}^{3} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j}).\)

It is clear that \(\psi_{dm} = S_1 \cup S_2 \cup \{ M_1, M_2, M_3, M'_1, M'_2, M'_3, M''_1, M''_2, M''_3 \}\) is a minimum detour monophonic graphoidal cover of \(G\) and so \(\eta_{dm}(G) = 3(n - 1) + 3(n - 1) + 9 = 3(2n + 1) = r(2n + 1).\)

**Subcase (iii):** \(n = 6, 8, 10, \ldots\)

Let \(M_1 : v_{2,1}, u_1, u_2;\)
\(M_2 : v_{2,1}, u_2, u_3;\)
\(M_3 : v_{3,1}, u_3, u_1;\)
\(M'_1 : v_{i,2}, v_{i,3}, \ldots, v_{i,n-1} (1 \leq i \leq 3);\)
\(S_1 = \bigcup_{i=1}^{3} \{ (v_{i,2}, v_{i,n}), (v_{i,n}, v_{i,n-1}) \};\)
\(S_2 = \bigcup_{i=1}^{3} \bigcup_{j=2}^{n} (u_{i}, v_{i,j});\) and
\(S_3 = \bigcup_{i=1}^{3} \bigcup_{j=2}^{n} (v_{i,1}, v_{i,j}).\)

It is clear that \(\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{ M_1, M_2, M_3, M'_1, M'_2, M'_3 \}\) is a minimum detour monophonic graphoidal cover of \(G\) and so \(\eta_{dm}(G) = 6 + 3(n - 1) + 3(n - 1) + 6 = 6(n + 1) = 2r(n + 1).\)

**Case 2.** \(r > 3\) and \(r\) is even.

**Subcase (i):** \(n = 5.\)

Let \(M_1 : v_{1,1}, u_1, u_2, \ldots, u_{\frac{n}{2}}+1, v_{\frac{n}{2}}+1, 1;\)
\(M_2 : u_1, u_r, u_{r-1}, \ldots, u_{\frac{n}{2}}+1;\)
\(M'_1 : v_{i,2}, v_{i,3}, v_{i,4} (1 \leq i \leq r);\)
\(M''_1 : v_{i,2}, v_{i,5}, v_{i,4} (1 \leq i \leq r);\)
\[ M''_i : v_{i,2}, v_{i,1}, v_{i,4} \text{ (} 1 \leq i \leq r \text{)} ; \]
\[ S_1 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{5} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_{2}+1, v_{2}+1,1)\} \text{ and} \]
\[ S_2 = \bigcup_{i=1}^{r} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\}. \]

It is clear that \( \psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M'_1, M'_2, \ldots, M''_1, M''_2, \ldots, M''_r\} \) is a minimum detour monophonic graphoidal cover of \( G \) and so \( \eta_{dm}(G) = (3r + 2) + (5r - 2) + 2r = 10r. \)

Subcase (ii): \( n = 7, 9, 11, \ldots \)

Let \( M_1 : v_{1,1}, u_1, u_2, \ldots, u_{2}+1, v_{2}+1,1; \)
\[ M_2 : u_1, u_r, u_{r-1}, \ldots, u_{2}+1; \]
\[ M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,n} \text{ (} 1 \leq i \leq r \text{)} ; \]
\[ M''_i : v_{i,2}, v_{i,n}, v_{i,n-1}, \ldots, v_{i,2}+1 \text{ (} 1 \leq i \leq r \text{)} ; \]
\[ S_1 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_{2}+1, v_{2}+1,1)\} \text{ and} \]
\[ S_2 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (2(v_{i,1}, v_{i,j})). \]

It is clear that \( \psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M'_1, M'_2, \ldots, M''_1, M''_2, \ldots, M''_r\} \) is a minimum detour monophonic graphoidal cover of \( G \) and so \( \eta_{dm}(G) = (2r + 2) + (nr - 2) + r(n - 1) = r(2n + 1). \)

Subcase (iii): \( n = 6, 8, 10, \ldots \)

Let \( M_1 : v_{1,1}, u_1, u_2, \ldots, u_{2}+1, v_{2}+1,1; \)
\[ M_2 : u_1, u_r, u_{r-1}, \ldots, u_{2}+1; \]
\[ M'_i : v_{i,2}, v_{i,3}, \ldots, v_{i,n-1} \text{ (} 1 \leq i \leq r \text{)} ; \]
\[ S_1 = \bigcup_{i=1}^{r} \{(v_{i,2}, v_{i,n}), (v_{i,n}, v_{i,n-1})\}; \]
\[ S_2 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_{2}+1, v_{2}+1,1)\} \text{ and} \]
\[ S_3 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (2(v_{i,1}, v_{i,j})). \]

It is clear that \( \psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M'_1, M'_2, \ldots, M'_r\} \) is a minimum detour monophonic graphoidal cover of \( G \) and so \( \eta_{dm}(G) = (r + 2) + 2r + (nr - 2) + r(n - 1) = 2r(n + 1). \)

Case 3. \( r > 3 \) and \( r \) is odd.

Subcase (i): \( n = 5. \)

Let \( M_1 : v_{1,1}, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}; \)
\[ M_2 : u_{r-1}, u_r, v_{r,1}; \]
\[ M_3 : u_r, v_{1}; \]
\[ M'_i : v_{i,2}, v_{i,3}, v_{i,4} \text{ (} 1 \leq i \leq r \text{)} ; \]
\[ M''_i : v_{i,2}, v_{i,5}, v_{i,4} \text{ (} 1 \leq i \leq r \text{)} ; \]
\[ M'''_i : v_{i,2}, v_{i,1}, v_{i,4} \text{ (} 1 \leq i \leq r \text{)} ; \]
\[ S_1 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} (u_i, v_{i,j}) - \{(u_1, v_{1,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\} \text{ and} \]
\[ S_2 = \bigcup_{i=1}^{r} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\}. \]

It is clear that \( \psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M''_1, M''_2, \ldots, M'''_r\} \) is a minimum detour monophonic graphoidal cover of \( G \)
and so \( \eta_{dm}(G) = (3r + 3) + (5r - 3) + 2r = 10r. \)

**Subcase (ii):** \( n = 7, 9, 11, \ldots. \)

Let \( M_1 : v_1,1, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}; \)

\( M_2 : u_{r-1}, u_r, v_{r,1}; \)

\( M_3 : u_r, u_1; \)

\( M'_1 : v_{i,2}, v_{i,3}, \ldots, v_{i,n-1} (1 \leq i \leq r); \)

\( M''_1 : v_{i,2}, v_{i,n}, v_{i,n-1}, \ldots, v_{i,n+3} (1 \leq i \leq r); \)

\( S_1 = (\bigcup_{i=1}^{r} \bigcup_{j=1}^{n}(u_{i}, v_{i,j})) - \{(u_1, v_{1,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}; \)

\( S_2 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n}(v_{i,1}, v_{i,j}). \)

It is clear that \( \psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M'_1, M''_1, \ldots, M'_r, M''_r\} \)

is a minimum detour monophonic graphoidal cover of \( G \) and so \( \eta_{dm}(G) = (2r + 3) + (nr - 3) + r(n - 1) = r(2n + 1). \)

**Subcase (iii):** \( n = 6, 8, 10, \ldots. \)

Let \( M_1 : v_1,1, u_1, u_2, \ldots, u_{r-1}, v_{r-1,1}; \)

\( M_2 : u_{r-1}, u_r, v_{r,1}; \)

\( M_3 : u_r, u_1; \)

\( M'_1 : v_{i,2}, v_{i,3}, \ldots, v_{i,n-1} (1 \leq i \leq r); \)

\( S_1 = \bigcup_{i=1}^{r} \{(v_{i,2}, v_{i,n}), (v_{i,n}, v_{i,n-1})\}; \)

\( S_2 = (\bigcup_{i=1}^{r} \bigcup_{j=1}^{n}(u_{i}, v_{i,j})) - \{(u_1, v_{1,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}; \)

\( S_3 = \bigcup_{i=1}^{r} \bigcup_{j=2}^{n}(v_{i,1}, v_{i,j}). \)

It is clear that \( \psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M'_1, M''_1, \ldots, M'_r\} \)

is a minimum detour monophonic graphoidal cover of \( G \) and so \( \eta_{dm}(G) = (r + 3) + 2r + (nr - 3) + r(n - 1) = 2r(n + 1). \)

**Theorem 2.5** If \( G = W_n \circ C_r \), then

\[
\eta_{dm}(G) = \begin{cases} 
5r + 13 & \text{if } r \text{ is even and } n = 5 \\
5r + 18 & \text{if } r \text{ is odd and } n = 5 \\
(nr + 3) - 1 & \text{if } r \text{ is even and } n > 5 \\
(nr + 4) - 1 & \text{if } r \text{ is odd and } n > 5.
\end{cases}
\]

**Proof.** Let \( W_n = K_1 + C_{n-1} \) be a wheel with \( V(K_1) = \{v_1\} \) and \( V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\} \) and let \( C_r : u_1, u_2, \ldots, u_r, u_1 \) be a cycle of order \( r \). Let \( G \) be the corona product of \( W_n \) and \( C_r \).

**Case 1.** \( n = 5. \)

**Subcase (i):** \( r \) is even.

Let \( M_1 : w_{2,1}, v_2, v_3, v_4, u_{4,1}; \)

\( M_2 : v_2, v_5, v_4; \)
Subcase (ii) $r$ is odd.

Let $M_1 : u_{2,1}, v_2, v_3, v_4, u_{4,1};$

$M_2 : v_2, v_5, v_4;$

$M_3 : v_2, v_1, v_4;$

$M'_1 : u_{i,1}, u_{i,2}, \ldots, u_{i,\frac{r}{2}+1} (1 \leq i \leq 5);$

$M''_1 : u_{i,1}, u_{i,r}, \ldots, u_{i,\frac{r}{2}+1} (1 \leq i \leq 5);$

$S_1 = \{(v_2, v_3), (v_1, v_5)\};$ and

$S_2 = \bigcup_{i=1}^{\frac{r}{2}} \bigcup_{j=1}^{r-1} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_4, u_{4,1})\}. $

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M'_1, M''_1, M'_2, M''_2, M'_3, M''_3, M'_4, M''_4\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = 13 + 2 + (5r - 2) = 5r + 13.$

Subcase (i) $r$ is even.

Let $M_1 : u_{2,1}, v_2, v_3, v_4, u_{4,1};$

$M_2 : v_2, v_5, v_4;$

$M_3 : v_2, v_1, v_4;$

$M'_1 : u_{i,1}, u_{i,2}, \ldots, u_{i,\frac{r}{2}+1} (1 \leq i \leq 5);$

$M''_1 : u_{i,1}, u_{i,r}, \ldots, u_{i,\frac{r}{2}+1} (1 \leq i \leq 5);$

$S_1 = \bigcup_{i=1}^{\frac{r}{2}} \{(u_{i,r-1}, u_{i,r}, u_{i,1})\};$

$S_2 = \{(v_2, v_3), (v_1, v_5)\};$ and

$S_3 = \bigcup_{i=1}^{\frac{r}{2}} \bigcup_{j=1}^{r-1} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_4, u_{4,1})\}. $

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M'_1, M''_1, M'_2, M''_2, M'_3, M''_3, M'_4, M''_4\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = 8 + 10 + 2 + (5r - 2) = 5r + 18.$

Case 2. $n = 7, 9, 11, \ldots$
Let $M$.  

Case 1.  

\[ \psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_1', M_2', \ldots, M_n'\} \]

is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (n + 2) + 2n + (n - 1) + (nr - 2) = n(r + 4) - 1$.

**Case 3.** $n = 6, 8, 10, \ldots$

**Subcase (i):** $r$ is even.

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n-1}, u_{n-1,1}$;

$M_2 : v_{n-1}, v_n, u_{n,1}$;

$M_3 : v_n, v_2$;

$M_i' : u_{i,1}, u_{i,2}, \ldots, u_{i, \frac{n}{r} + 1} (1 \leq i \leq n)$;

$M_i'' : u_{i,1}, u_{i,r}, \ldots, u_{i, \frac{n}{r} + 1} (1 \leq i \leq n)$;

$S_1 = \bigcup_{i=2}^{n} (v_1, v_i)$; and

$S_2 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1}), (v_n, u_{n,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3, M_1', M_2', \ldots, M_n', M_1'', M_2'', \ldots, M_n''\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (2n + 3) + (n - 1) + (nr - 3) = n(r + 3) - 1$.

**Subcase (ii):** $r$ is odd.

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n-1}, u_{n-1,1}$;

$M_2 : v_{n-1}, v_n, u_{n,1}$;

$M_3 : v_n, v_2$;

$M_i' : u_{i,1}, u_{i,2}, \ldots, u_{i, r-1} (1 \leq i \leq n)$;

$S_1 = \bigcup_{i=1}^{n} \{(u_{i,r-1}, u_{i,r}), (u_{i,r}, u_{i,1})\}$;

$S_2 = \bigcup_{i=2}^{n} (v_1, v_i)$; and

$S_3 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n-1}, u_{n-1,1}), (v_n, u_{n,1})\}$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M_1', M_2', \ldots, M_n', M_1'', M_2'', \ldots, M_n''\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (n + 3) + 2n + (n - 1) + (nr - 3) = n(r + 4) - 1$.

**Theorem 2.6** If $G = K_r \circ W_n$, $r \geq 3$, then

$$\eta_{dm}(G) = \begin{cases} \frac{r}{2}(r + 17) & \text{if } n = 5 \\ \frac{r}{2}(r + 4n - 1) & \text{if } n = 7, 9, 11, \ldots \\ \frac{r}{2}(r + 4n + 1) & \text{if } n = 6, 8, 10, \ldots \end{cases}$$

**Proof.** Let $K_r$ be the complete graph with the vertex set $\{u_1, u_2, \ldots, u_r\}$, and let $W_n = K_1 + C_{n-1}$ be a wheel with $V(K_1) = \{v_1\}$ and $V(C_{n-1}) = \{v_2, v_3, \ldots, v_n\}$.

**Case 1.** $n = 5$.

Let $M_i : v_{i,1}, u_i, u_{i+1} (1 \leq i \leq r - 1)$;

$M_r : v_{r,1}, u_r, u_1$;

$M_i' : v_{i,2}, v_{i,3}, v_{i,4} (1 \leq i \leq r)$;

$M_i'' : v_{i,2}, v_{i,5}, v_{i,4} (1 \leq i \leq r)$.

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M_1', M_2', \ldots, M_n', M_1'', M_2'', \ldots, M_n''\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (n + 3) + 2n + (n - 1) + (nr - 3) = n(r + 4) - 1$. 


Let $M = \{v_1, v_2, \ldots, v_n\}$ be the complete graph with the vertex set $r$.

Case 1

Let $M = v_{r,1}, u_r, u_1$.

Case 2

Let $M = v_{r,1}, u_r, u_1$.

Case 3

Let $M = v_{r,1}, u_r, u_1$.

\[ \eta_{dm}(G) = \begin{cases} \frac{1}{2}(5r^2 + 5r + 6) & \text{if } n = 5 \\ \frac{1}{2}(nr^2 + nr + 2n - 2) & \text{if } n = 7, 9, 11, \ldots \\ \frac{1}{2}(nr^2 + nr + 2n) & \text{if } n = 6, 8, 10, \ldots \end{cases} \]
Let $M_1 : u_{2,1}, v_2, v_3, v_4, u_{4,1};$

$M_2 : v_2, v_3, v_4;$

$M_3 : v_2, v_1, v_4;$

$S_1 = \bigcup_{i=1}^{5} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_4, u_{4,1})\};$ and

$S_2 = \bigcup_{i=1}^{5} E(K_r) \cup \{(v_1, v_3), (v_1, v_5)\}.$

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup \{M_1, M_2, M_3\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = 3 + (5r - 2) + 5 \cdot \frac{r(r-1)}{2} + 2 = \frac{1}{2}(5r^2 + 5r + 6).

**Case 2.** $n = 7, 9, 11, \ldots$

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n+3}, u_{n+3,1};$

$M_2 : v_2, v_n, v_{n-1}, \ldots, v_{n+3};$

$S_1 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_2, u_{2,1}), (v_{n+3}, u_{n+3,1})\};$

$S_2 = \bigcup_{i=1}^{n} E(K_r);$ and

$S_3 = \bigcup_{i=2}^{n} (v_1, v_i).$

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = 2 + (nr - 2) + n \cdot \frac{r(r-1)}{2} + (n-1) = \frac{1}{2}(nr^2 + nr + 2n - 2).

**Case 3.** $n = 6, 8, 10, \ldots$

Let $M_1 : u_{2,1}, v_2, v_3, \ldots, v_{n-1}, u_{n-1,1};$

$M_2 : v_{n-1}, v_n, u_{n,1};$

$M_3 : v_n, v_2;$

$S_1 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r} (v_i, u_{i,j}) - \{(v_{n-1}, u_{n-1,1})\};$

$S_2 = \bigcup_{i=1}^{n} E(K_r);$ and

$S_3 = \bigcup_{i=2}^{n} (v_1, v_i).$

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3\}$ is a minimum detour monophonic graphoidal cover of $G$ and hence $\eta_{dm}(G) = 3 + (nr - 2) + n \cdot \frac{r(r-1)}{2} + n - 1 = \frac{1}{2}(nr^2 + nr + 2n).$ \hfill \blacksquare

**Theorem 2.8** If $G = W_r \circ W_s$, then

$$\eta_{dm}(G) = \begin{cases} 53 & \text{if } r = s = 5 \\ 11r - 1 & \text{if } s = 5 \text{ and } r \geq 6 \\ 2(5s + 4) & \text{if } r = 5 \text{ and } s = 7, 9, 11, \ldots \\ 10s + 13 & \text{if } r = 5 \text{ and } s = 6, 8, 10, \ldots \\ 2r(s+1) - 1 & \text{if } r \geq 6 \text{ and } s = 7, 9, 11, \ldots \\ r(2s+3) - 1 & \text{if } r \geq 6 \text{ and } s = 6, 8, 10, \ldots \\ \end{cases}$$

**Proof.** Let $W_r = K_1 + C_{r-1}$ be a wheel with $V(K_1) = \{u_1\}$ and $V(C_{r-1}) = \{v_i, u_{i,j}: 1 \leq i \leq n, 1 \leq j \leq r\}$.
\{u_2, u_3, \ldots, u_s\} \text{ and let } W_s = K_1 + C_{s-1} \text{ be a wheel with } V(K_1) = \{v_1\} \text{ and } V(C_{s-1}) = \{v_2, v_3, \ldots, v_s\}.

\textbf{Case 1. } r = 5.

\textbf{Subcase (i): } s = 5.

Let \(M_1: v_{2,1}, u_2, u_3, u_4, v_{4,1};\) 
\(M_2: u_2, u_5, u_4;\) 
\(M_3: u_2, u_1, u_4;\) 
\(M_i': v_{i,2}, v_{i,3}, v_{i,4} \ (1 \leq i \leq 5);\) 
\(M_i'': v_{i,2}, v_{i,5}, v_{i,4} \ (1 \leq i \leq 5);\) 
\(M_i'''': v_{i,2}, v_{i,1}, v_{i,4} \ (1 \leq i \leq 5);\) 
\(S_1 = \{(u_1, u_3), (u_1, u_5)\};\) 
\(S_2 = \bigcup_{i=1}^{5} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\};\) and 
\(S_3 = \bigcup_{i=1}^{5} \bigcup_{j=1}^{5} (v_{i,j}) - \{(u_2, v_{2,1}), (u_4, v_{4,1})\}.\)

It is clear that \(s_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M_i', M_i'', M_i'''', M_{i''}', M_{i''''}, M_{i'''''}, M_{i''''''}, M_{i'''''''}, M_{i'''''''}\} \) is a minimum detour monophonic graphoidal cover of \(G\) and so \(\eta_{dm}(G) = 18 + 12 + 23 = 53.\)

\textbf{Subcase (ii): } s = 7, 9, 11, \ldots.

Let \(M_1: v_{2,1}, u_2, u_3, u_4, v_{4,1};\) 
\(M_2: u_2, u_5, u_4;\) 
\(M_3: u_2, u_1, u_4;\) 
\(M_i': v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{s+3}{2}} \ (1 \leq i \leq 5);\) 
\(M_i'': v_{i,2}, v_{i,s}, \ldots, v_{i,\frac{s+3}{2}} \ (1 \leq i \leq 5);\) 
\(S_1 = \{(u_1, u_3), (u_1, u_5)\};\) 
\(S_2 = \bigcup_{i=1}^{5} \bigcup_{j=2}^{s} (v_{i,j}) - \{(u_2, v_{2,1}), (u_4, v_{4,1})\}.\)

It is clear that \(s_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M_i', M_i'', M_i'''', M_{i''}', M_{i''''}, M_{i'''''}, M_{i''''''}, M_{i'''''''}, M_{i'''''''}\} \) is a minimum detour monophonic graphoidal cover of \(G\) and so \(\eta_{dm}(G) = 13 + 2 + 5(s - 1) + (5s - 2) = 2(5s + 4).\)

\textbf{Subcase (iii): } s = 6, 8, 10, \ldots.

Let \(M_1: v_{2,1}, u_2, u_3, u_4, v_{4,1};\) 
\(M_2: u_2, u_5, u_4;\) 
\(M_3: u_2, u_1, u_4;\) 
\(M_i': v_{i,2}, v_{i,3}, \ldots, v_{i,s-1} \ (1 \leq i \leq 5);\) 
\(S_1 = \bigcup_{i=1}^{5} \{(v_{i,s-1}, v_{i,s}), (v_{i,s}, v_{i,2})\};\) 
\(S_2 = \{(u_1, u_3), (u_1, u_5)\};\) 
\(S_3 = \bigcup_{i=1}^{5} \bigcup_{j=2}^{s} (v_{i,j}) - \{(u_2, v_{2,1}), (u_4, v_{4,1})\}.\)
It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{M_1, M_2, M_3, M'_1, M'_2, M''_1, M''_2\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = 8 + 10 + 2 + 5(s - 1) + (5s - 2) = 10s + 13.$

**Case 2.** $r = 7, 9, 11, \ldots.$

**Subcase (i):** $s = 5.$

Let $M_1 : v_{2,1}, u_2, u_3, \ldots, \frac{u_{r+3}}{2}, \frac{v_{r+3}}{1};$

$M_2 : u_2, u_v, u_{r-1}, \ldots, \frac{u_{r+3}}{2};$

$M'_1 : v_{i,2}, v_{i,3}, v_{i,4} (1 \leq i \leq r);$  

$M''_1 : v_{i,2}, v_{i,5}, v_{i,4} (1 \leq i \leq r);$  

$S_1 = \bigcup_{i=2}^r (u_1, u_i);$

$S_2 = \bigcup_{i=1}^r (v_{i,1}, v_{i,3}, (v_{i,1}, v_{i,5});$  

and

$S_3 = \bigcup_{i=1}^r (u_i, v_{i,j}) - \left\{ (u_2, v_{2,1}), (u_{r+3}, v_{r+3}, 1) \right\}.$

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{M_1, M_2, M_3, M'_1, M'_2, M''_1, M''_2, \ldots, M''_1, M''_2\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (3r + 2) + (r - 1) + 2r + (5r - 2) = 11r - 1.$

**Subcase (ii):** $s = 7, 9, 11, \ldots.$

Let $M_1 : v_{2,1}, u_2, u_3, \ldots, \frac{u_{r+3}}{2}, \frac{v_{r+3}}{1};$

$M_2 : u_2, u_v, u_{r-1}, \ldots, \frac{u_{r+3}}{2};$

$M'_1 : v_{i,2}, v_{i,3}, \ldots, \frac{u_{i+3}}{2} (1 \leq i \leq r);$  

$M''_1 : v_{i,2}, v_{i,s}, v_{i,s-1}, \ldots, \frac{u_{i+3}}{2} (1 \leq i \leq r);$  

$S_1 = \bigcup_{i=2}^r (u_1, u_i);$

$S_2 = \bigcup_{i=1}^r (v_{i,1}, v_{i,j});$  

and

$S_3 = \bigcup_{i=1}^r (u_i, v_{i,j}) - \left\{ (u_2, v_{2,1}), (u_{r+3}, v_{r+3}, 1) \right\}.$

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_{r}, M''_{r}, M''_{r}, \ldots, M''_{r}\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (2r + 2) + (r - 1) + r(s - 1) + (rs - 2) = 2r(s + 1) - 1.$

**Subcase (iii):** $s = 6, 8, 10, \ldots.$

Let $M_1 : v_{2,1}, u_2, u_3, \ldots, \frac{u_{r+3}}{2}, \frac{v_{r+3}}{1};$

$M_2 : u_2, u_v, u_{r-1}, \ldots, \frac{u_{r+3}}{2};$

$M'_1 : v_{i,2}, v_{i,3}, \ldots, v_{i,s-1} (1 \leq i \leq r);$  

$S_1 = \bigcup_{i=1}^r \{(v_{i,s-1}, v_{i,s}, (v_{i,s}, v_{i,1})\);  

$S_2 = \bigcup_{i=1}^r (u_1, u_i);$

$S_3 = \bigcup_{i=1}^r (v_{i,1}, v_{i,j});$  

and

$S_4 = \bigcup_{i=1}^r (u_i, v_{i,j}) - \left\{ (u_2, v_{2,1}), (u_{r+3}, v_{r+3}, 1) \right\}.$

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_{r}\}$ is a
minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (r + 2) + 2r + (r - 1) + r(s - 1) + (rs - 2) = r(2s + 3) - 1$. 

**Case 3.** $r = 6, 8, 10, \ldots$ 

**Subcase (i):** $s = 5$. 

Let $M_1 : v_{2,1}, u_2, v_3, \ldots, u_{r-1}, v_{r-1,1};$ 

$M_2 : u_{r-1}, u_r, v_{r,1};$ 

$M_3 : u_r, u_2; $ 

$M'_1 : v_{i,2}, v_{i,3}, v_{i,4}, \ldots, v_{i,\frac{r+1}{2}} (1 \leq i \leq r);$ 

$M''_1 : v_{i,2}, v_{i,3}, v_{i,4}, \ldots, v_{i,\frac{r+1}{2}} (1 \leq i \leq r);$ 

$S_1 = \bigcup_{i=2}^{r}(u_1, u_i);$ 

$S_2 = \bigcup_{i=1}^{r} \{(v_{i,1}, v_{i,3}), (v_{i,1}, v_{i,5})\};$ and 

$S_3 = \bigcup_{i=1}^{r} \cup_{j=2}^{\frac{r}{2}}(v_{i,1}, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}.$ 

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M'_1, M''_1, M_2, \ldots, M'_m, M''_m\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (3r + 3) + (r - 1) + 2r + (5r - 3) = 11r - 1$. 

**Subcase (ii):** $s = 7, 9, 11, \ldots$ 

Let $M_1 : v_{2,1}, u_2, v_3, \ldots, u_{r-1}, v_{r-1,1}; $ 

$M_2 : u_{r-1}, u_r, v_{r,1};$ 

$M_3 : u_r, u_2; $ 

$M'_1 : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{r+3}{2}} (1 \leq i \leq r);$ 

$M''_1 : v_{i,2}, v_{i,3}, \ldots, v_{i,\frac{r+3}{2}} (1 \leq i \leq r);$ 

$S_1 = \bigcup_{i=2}^{r}(u_1, u_i);$ 

$S_2 = \bigcup_{i=1}^{r} \cup_{j=2}^{\frac{r}{2}}(v_{i,1}, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\};$ and 

$S_3 = \bigcup_{i=1}^{r} \cup_{j=2}^{\frac{r+1}{2}}(v_{i,1}, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}.$ 

It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup \{M_1, M_2, M_3, M'_1, M''_1, M_2, \ldots, M'_m, M''_m\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (2r + 3) + (r - 1) + r(s - 1) + (rs - 3) = 2r(s + 1) - 1$. 

**Subcase (iii):** $s = 6, 8, 10, \ldots$ 

Let $M_1 : v_{2,1}, u_2, v_3, \ldots, u_{r-1}, v_{r-1,1}; $ 

$M_2 : u_{r-1}, u_r, v_{r,1};$ 

$M_3 : u_r, u_2; $ 

$M'_1 : v_{i,2}, v_{i,3}, \ldots, v_{i,s-1} (1 \leq i \leq r);$ 

$S_1 = \bigcup_{i=1}^{r}\{(v_{i,s-1}, v_{i,s}), (v_{i,s}, v_{i,2})\};$ 

$S_2 = \bigcup_{i=1}^{r}(u_1, u_i);$ 

$S_3 = \bigcup_{i=1}^{r} \cup_{j=2}^{s}(v_{i,1}, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\};$ and 

$S_4 = \bigcup_{i=1}^{r} \cup_{j=2}^{s}(u_r, v_{i,j}) - \{(u_2, v_{2,1}), (u_{r-1}, v_{r-1,1}), (u_r, v_{r,1})\}.$
It is clear that $\psi_{dm} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{M_1, M_2, M_3, M'_1, M'_2, \ldots, M'_r\}$ is a minimum detour monophonic graphoidal cover of $G$ and so $\eta_{dm}(G) = (r + 3) + 2r + (r - 1) + r(s - 1) + rs - 3 = r(2s + 3) - 1.$

References


