

On the max - “Fuzzy Or” composition fuzzy inequalities systems

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ABSTRACT

In this paper, a type of fuzzy system is investigated in which the feasible region is formed as the intersection of two types of fuzzy inequalities and “Fuzzy Or” is considered as fuzzy composition. Some theoretical properties are derived and four necessary and sufficient conditions are presented. Moreover, an algorithm is proposed to solve the problem and an example is described to illustrate the algorithm.

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1 Introduction

In this paper, we study the following fuzzy system in which the constraints consist of the intersection of two types fuzzy relational inequalities defined by “Fuzzy Or”

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Operator:

$$\begin{aligned} A \nabla x &\leq b^1 \\ D \nabla x &\geq b^2 \\ x &\in [0, 1]^n \end{aligned} \quad (1)$$

where $I_1 = \{1, 2, \dots, m_1\}$, $I_2 = \{m_1 + 1, m_1 + 2, \dots, m_1 + m_2\}$ and $J = \{1, 2, \dots, n\}$. $A = (a_{ij})_{m_1 \times n}$ and $D = (d_{ij})_{m_2 \times n}$ are fuzzy matrices such that $0 \leq a_{ij} \leq 1$ ($\forall i \in I_1$ and $\forall j \in J$) and $0 \leq d_{ij} \leq 1$ ($\forall i \in I_2$ and $\forall j \in J$). $b^1 = (b_i^1)_{m_1 \times 1}$ is an m_1 -dimensional fuzzy vector in $[0, 1]^{m_1}$ (i.e., $0 \leq b_i^1 \leq 1, \forall i \in I_1$) and $b^2 = (b_i^2)_{m_2 \times 1}$ is an m_2 -dimensional fuzzy vector in $[0, 1]^{m_2}$ (i.e., $0 \leq b_i^2 \leq 1, \forall i \in I_2$). Moreover, “ ∇ ” is the max- ∇ composition where ∇ is “Fuzzy Or” Operator, that is,

$$\Delta(x, y) = \gamma \max\{x, y\} + \frac{(1 - \gamma)(x + y)}{2}$$

in which $\gamma \in [0, 1]$. Furthermore, let $S(A, b^1)$ and $S(D, b^2)$ denote the feasible solutions sets of inequalities type1 $A \nabla x \leq b^1$ and type2 $D \nabla x \geq b^2$, respectively, that is, $S(A, b^1) = \{x \in [0, 1]^n : A \nabla x \leq b^1\}$ and $S(D, b^2) = \{x \in [0, 1]^n : D \nabla x \geq b^2\}$. Also, let $S(A, D, b^1, b^2)$ denote the feasible solutions set of problem (1). Based on the foregoing notations, it is clear that $S(A, D, b^1, b^2) = S(A, b^1) \cap S(D, b^2)$.

By these notations, problem (1) can be also expressed as follows:

$$\begin{aligned} \max_{j \in J} \{\nabla(a_{ij}, x_j)\} &\leq b_i^1, \quad i \in I_1 \\ \max_{j \in J} \{\nabla(d_{ij}, x_j)\} &\geq b_i^2, \quad i \in I_2 \\ x &\in [0, 1]^n \end{aligned} \quad (2)$$

Especially, by setting $A = D$ and $b^1 = b^2$, the above problem is converted to max-“Fuzzy Or” fuzzy relational equations.

The theory of fuzzy relational equations (FRE) was firstly proposed by Sanchez and applied in problems of the medical diagnosis [54]. Nowadays, it is well known that many issues associated with a body knowledge can be treated as FRE problems [50]. In addition to the preceding applications, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [40,48]. The solvability determination and the finding of solutions set are the primary (and the most fundamental) subject concerning with FRE problems. Actually, The solution set of FRE is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [5]. This non-convexity property is one of two bottlenecks making major contribution to the increase of complexity in problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. The other bottleneck

is concerned with detecting the minimal solutions for FREs [2]. Markovskii showed that solving max-product FRE is closely related to the covering problem which is an NP-hard problem [47]. In fact, the same result holds true for a more general t-norms instead of the minimum and product operators [2,3,12,13,22 – 30,43,44,47].

Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [22–30,49,51,52,55,57,58,60,63,66]. Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI)[12,13,15 –20,21,31,32,41,65].

The problem of optimization subject to FRE and FRI is one of the most interesting and on-going research topic among the problems related to FRE and FRI theory [1,8,9,11 – 30,38,42,45,53,56,59,61,65]. The topic of the linear optimization problem was also investigated with max-product operation [11,34,46]. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [14,37,61], max-Discontinuous t-norms composition [29], max-monotone operators composition [30] and max-t-norm composition [15 – 20, 22 – 28,35,42,56].

Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas [4,6,10,22 – 28,32,39,45,62].

The optimization problem subjected to various versions of FRI could be found in the literature as well [12,13,15 – 21,29 – 32,64,65]. Yang [64] applied the pseudo-minimal index algorithm for solving the minimization of linear objective function subject to FRI with addition-min composition. Xiao et al. [65] introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities. Ghodousian et al. [12] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition.

It is well – known that for any membership values $\mu_A(x)$ and $\mu_B(x)$ of arbitrary fuzzy sets A and B , the membership value of their union $A \cup B$ (defined by any S-norm).

lies in the interval $[\max\{\mu_A(x), \mu_B(x)\}, S_{ds}\{\mu_A(x), \mu_B(x)\}]$. Similarly, the membership value of the intersection $A \cap B$ (defined by any T-norm) lies in the interval

$[T_{dp}\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(x), \mu_B(x)\}]$. Therefore, the union and intersection operators

cannot cover the interval between $\min\{\mu_A(x), \mu_B(x)\}$ and $\max\{\mu_A(x), \mu_B(x)\}$. The operators that cover the interval $[\min\{\mu_A(x), \mu_B(x)\}, \max\{\mu_A(x), \mu_B(x)\}]$ are called averaging operators.

Similar to the S-norms and T-norms, an averaging operator is a function from $[0, 1] \times [0, 1]$ to $[0, 1]$. Many averaging operators were proposed in the literature [7].

In this paper, problem (1) was investigated where ∇ is “Fuzzy Or” Operator. Clearly, the “Fuzzy Or” covers the range from $(\mu_A(x) + \mu_B(x))/2$ to $\max\{\mu_A(x), \mu_B(x)\}$ as the parameter γ changes from 0 to 1.

The remainder of the paper is organized as follows. In section 2, some basic properties and the shape of the feasible solutions set of the type1 “Fuzzy Or”-Inequalities have

been attained. It is proved that the set is formed by a unique minimum and a unique maximum solution. Also, two necessary and sufficient conditions for the feasibility of this type of fuzzy systems are presented. The shape of the feasible region of the type2 “Fuzzy Or”-Inequalities is investigated in section 3. It is shown that this region is determined as a union of the finite number of minimal solutions and a unique maximum solution. Moreover, two necessary and sufficient conditions for the feasibility of this type of fuzzy systems are presented. In section4, the intersection of these two fuzzy systems is studied. A necessary and sufficient condition is proposed to determine the feasibility of the main problem and an algorithm is presented to resolve Problem (1). Finally, in section 5 an example is described to illustrate.

2. Basic properties of type1 “Fuzzy Or” – Inequalities

This section describes the structural properties concerning system $A \nabla x \leq b^1$. This fuzzy system consists of m_1 inequalities $\max_{j \in J} \{\nabla(a_{ij}, x_j)\} \leq b_i^1$ ($\forall i \in I_1$). For this purpose,

we firstly investigate corresponding partial inequalities $\nabla(a_{ij}, x_j) \leq b_i^1$, $i \in I_1$ and $j \in J$.

As before, for each $i \in I_1$, let $S(a_i, b_i^1) = \left\{ x \in [0, 1]^n : \max_{j \in J} \{\nabla(a_{ij}, x_j)\} \leq b_i^1 \right\}$. Similarly,

let $S(a_{ij}, b_i^1) = \{x_j \in [0, 1] : \nabla(a_{ij}, x_j) \leq b_i^1\}$ that is, set $S(a_{ij}, b_i^1)$ includes all solutions $x_j \in [0, 1]$ such that

$$\nabla(a_{ij}, x_j) = \gamma \max\{a_{ij}, x_j\} + \frac{(1-\gamma)(a_{ij} + x_j)}{2} \leq b_i^1, \quad i \in I_1, j \in J$$

Definition 1. For each $i \in I_1$ and each $j \in J$, define

$$\underline{W}_{ij}^1 = \frac{2b_i^1 - (1+\gamma)a_{ij}}{1-\gamma} \quad \text{and} \quad \underline{W}_{ij}^2 = \frac{2b_i^1 - (1-\gamma)a_{ij}}{1+\gamma}$$

The following four lemmas are easily verified for each $i \in I_1$ and each $j \in J$, and are very useful for some next proofs.

Lemma 1. Suppose that $\gamma < 1$. Then, $a_{ij} \leq b_i^1 \Leftrightarrow a_{ij} \leq \underline{W}_{ij}^1$.

Lemma 2. $a_{ij} \leq b_i^1 \Leftrightarrow a_{ij} \leq \underline{W}_{ij}^2$.

Also, Lemmas 1 and 2 are true if “ \leq ” is replaced by “ $<$ ”, “ \geq ” or “ $>$ ”.

Lemma 3. Suppose that $\gamma < 1$. Then, $\underline{W}_{ij}^1 \geq 0 \Leftrightarrow a_{ij} = 0$ or $0 \leq \gamma \leq \frac{2b_i^1 - a_{ij}}{a_{ij}}$.

Lemma 4. (a) $\underline{W}_{ij}^2 \geq 0 \Leftrightarrow a_{ij} = 0$ or $\frac{a_{ij} - 2b_i^1}{a_{ij}} \leq \gamma \leq 1$.

(b) $\underline{W}_{ij}^2 \leq 1 \Leftrightarrow a_{ij} = 1$ or $\frac{2b_i^1 - a_{ij} - 1}{1 - a_{ij}} \leq \gamma \leq 1$.

Lemma 5 below determines set $S(a_{ij}, b_i^1)$ where $a_{ij} \leq b_i^1$.

Lemma 5. Suppose that $a_{ij} \leq b_i^1$. Then, $S(a_{ij}, b_i^1) = [0, \min\{\underline{W}_{ij}^2, 1\}]$.

Proof. Since $a_{ij} \leq b_i^1$, Lemma2 implies $0 \leq a_{ij} \leq \underline{W}_{ij}^2$. Thus, $\underline{W}_{ij}^2 \geq 0$. Now, assume that $x_j \in [0, \min\{\underline{W}_{ij}^2, 1\}]$. If $a_{ij} = 1$ or $\gamma \geq (2b_i^1 - a_{ij} - 1)/(1 - a_{ij})$, then by Lemma4(b), $x_j \in [0, \min\{\underline{W}_{ij}^2, 1\}]$

means $x_j \in [0, \underline{W}_{ij}^2]$.

Therefore, in this case we have $\nabla(a_{ij}, x_j) \leq \nabla(a_{ij}, \underline{W}_{ij}^2) = \gamma \underline{W}_{ij}^2 + (1 - \gamma)(a_{ij} + \underline{W}_{ij}^2)/2 = b_i^1$, i.e., $x_j \in S(a_{ij}, b_i^1)$. If $a_{ij} < 1$ and $\gamma < (2b_i^1 - a_{ij} - 1)/(1 - a_{ij})$, then by Lemma4(b), $x_j \in [0, \min\{\underline{W}_{ij}^2, 1\}]$ means $x_j \in [0, 1]$. In this case, we have

$$\begin{aligned} \nabla(a_{ij}, x_j) &\leq \nabla(a_{ij}, 1) = \gamma + \frac{(1-\gamma)(a_{ij}+1)}{2} = \left(\frac{1-a_{ij}}{2}\right)\gamma + \frac{a_{ij}+1}{2} \\ &< \left(\frac{1-a_{ij}}{2}\right)\left(\frac{2b_i^1-a_{ij}-1}{1-a_{ij}}\right) + \frac{a_{ij}+1}{2} = b_i^1 \end{aligned}$$

Thus, $x_j \in S(a_{ij}, b_i^1)$. On the other hand, if $x_j < 0$, then clearly $x_j \notin S(a_{ij}, b_i^1)$. If $x_j > \min\{\underline{W}_{ij}^2, 1\} = 1$, then obviously $x_j \notin S(a_{ij}, b_i^1)$. Finally, if $x_j > \min\{\underline{W}_{ij}^2, 1\} = \underline{W}_{ij}^2$, then we have $b_i^1 = \nabla(a_{ij}, \underline{W}_{ij}^2) < \nabla(a_{ij}, x_j)$ that implies $x_j \notin S(a_{ij}, b_i^1)$. ?

Lemma 6 below determines set $S(a_{ij}, b_i^1)$ where $a_{ij} > b_i^1$.

Lemma 6. Suppose that $a_{ij} > b_i^1$. Then,

$$S(a_{ij}, b_i^1) = \begin{cases} [0, \frac{W_{ij}^1}{a_{ij}}] & , 0 \leq \gamma \leq (2b_i^1 - a_{ij})/a_{ij} \\ \emptyset & , otherwise \end{cases}$$

Proof. Note that in this case we have $a_{ij} > 0$ and $(2b_i^1 - a_{ij})/a_{ij} < 1$. Since $a_{ij} > b_i^1$ and $\gamma < 1$, Lemma1 implies that $\underline{W}_{ij}^1 < a_{ij} \leq 1$. Thus, $\underline{W}_{ij}^1 < 1$. Also, by $\gamma < 1$ and Lemma3 we have $\underline{W}_{ij}^1 \geq 0$. Now, assume that $0 \leq \gamma \leq (2b_i^1 - a_{ij})/a_{ij}$ and $x_j \in [0, \underline{W}_{ij}^1]$. Hence, $\nabla(a_{ij}, x_j) \leq \nabla(a_{ij}, \underline{W}_{ij}^1) = \gamma a_{ij} + (1 - \gamma)(a_{ij} + \underline{W}_{ij}^1)/2 = b_i^1$ that means $x_j \in S(a_{ij}, b_i^1)$. On the other hand, if $x_j < 0$, then $x_j \notin S(a_{ij}, b_i^1)$. If $0 \leq \gamma \leq (2b_i^1 - a_{ij})/a_{ij}$ and $x_j > \underline{W}_{ij}^1$, then $b_i^1 = \nabla(a_{ij}, \underline{W}_{ij}^1) < \nabla(a_{ij}, x_j)$, i.e., $x_j \notin S(a_{ij}, b_i^1)$. Finally, if $\gamma > (2b_i^1 - a_{ij})/a_{ij}$, then we have

$$\nabla(a_{ij}, x_j) \geq \nabla(a_{ij}, 0) = \frac{\gamma a_{ij}}{2} + \frac{a_{ij}}{2} > \left(\frac{2b_i^1 - a_{ij}}{a_{ij}}\right) \frac{a_{ij}}{2} + \frac{a_{ij}}{2} = b_i^1$$

that implies $x_j \notin S(a_{ij}, b_i^1)$. ?

Corollary 1. For each $i \in I_1$ and each $j \in J$,

$$S(a_{ij}, b_i^1) = \begin{cases} [0, \min\{\underline{W}_{ij}^2, 1\}] & , a_{ij} \leq b_i^1 \\ [0, \frac{W_{ij}^1}{a_{ij}}] & , a_{ij} > b_i^1, 0 \leq \gamma \leq (2b_i^1 - a_{ij})/a_{ij} \\ \emptyset & , a_{ij} > b_i^1, \gamma > (2b_i^1 - a_{ij})/a_{ij} \end{cases}$$

The following theorem gives a necessary and sufficient condition for the feasibility of inequality.

Theorem 1. Let $i \in I_1$. $S(a_i, b_i^1) \neq \emptyset$ iff either $a_{ij} \leq b_i^1$ or $\gamma \leq (2b_i^1 - a_{ij})/a_{ij}, \forall j \in J$.

Proof. For an arbitrary $x \in [0,1]^n$, $x \in S(a_i, b_i^1)$ if and only if $\max_{j \in J} \{\nabla(a_{ij}, x_j)\} \leq b_i^1$.

Also, the last inequality holds true iff $\nabla(a_{ij}, x_j) \leq b_i^1, \forall j \in J$. Therefore, $S(a_i, b_i^1) \neq \emptyset$ iff $S(a_{ij}, b_i^1) \neq \emptyset, \forall j \in J$. Now, the result follows from Corollary1. ?

Definition 2. Suppose that $S(a_i, b_i^1) \neq \emptyset$.

We define $\bar{X}(i) = [\bar{X}(i)_1, \bar{X}(i)_2, \dots, \bar{X}(i)_n]$ where

$$\bar{X}(i)_j = \begin{cases} \min\{W_{ij}^2, 1\}, & a_{ij} \leq b_i^1 \\ W_{ij}^1, & a_{ij} > b_i^1, 0 \leq \gamma \leq (2b_i^1 - a_{ij})/a_{ij} \end{cases}$$

By Theorem2 below, the solutions set $S(a_i, b_i^1)$ is completely determined. The theorem shows that $S(a_i, b_i^1)$ has actually the unique maximum solution, $\bar{X}(i)$, and the unique minimum solution, $\mathbf{0}$, where $\mathbf{0}$ is an n -dimensional zero vector.

Theorem 2. Suppose that $S(a_i, b_i^1) \neq \emptyset$. Then, $S(a_i, b_i^1) = [\mathbf{0}, \bar{X}(i)], \forall i \in I_1$.

Proof. Similar to the proof of Theorem1, for each $x \in [0,1]^n$, $x \in S(a_i, b_i^1)$ iff $x_j \in S(a_{ij}, b_i^1), \forall j \in J$. Thus, from Corollary1 and Definition2, for each $j \in J$ we have $x_j \in [0, \bar{X}(i)_j]$. Therefore, $x \in [0, \bar{X}(i)_1] \times [0, \bar{X}(i)_2] \times \dots \times [0, \bar{X}(i)_n] = [\mathbf{0}, \bar{X}(i)]$. ?

Definition 3. Let $\bar{X}(i)$ be as in Definition2, $\forall i \in I_1$. We define $\bar{X} = \min_{i \in I_1} \{\bar{X}(i)\}$.

According to Theorem2 and the fact that $S(A, b^1) = \bigcap_{i \in I_1} S(a_i, b_i^1)$, the following theorem is attained.

Theorem 3. Suppose that $S(a_i, b_i^1) \neq \emptyset, \forall i \in I_1$. Then, $S(A, b^1) = [\mathbf{0}, \bar{X}]$.

Proof. by Theorem2, we have $S(A, b^1) = \bigcap_{i \in I_1} S(a_i, b_i^1) = \bigcap_{i \in I_1} [\mathbf{0}, \bar{X}(i)] = [\mathbf{0}, \min_{i \in I_1} \{\bar{X}(i)\}]$.

Now, the result is obtained from Definition3. ?

Theorem3 determines the solutions set $S(A, b^1)$ as an n -dimensional interval $[\mathbf{0}, \bar{X}]$ with $\mathbf{0}$ as the unique minimum and \bar{X} as the unique maximum solutions. The following Corollary gives a necessary and sufficient condition for the feasibility of general inequalities $A \nabla x \leq b^1$.

Corollary 2. $S(A, b^1) \neq \emptyset$ iff $\mathbf{0} \in S(A, b^1)$.

3. Basic properties of type2 “Fuzzy Or” – Inequalities

In this section, the properties of system $D \nabla x \geq b^2$ are investigated. This fuzzy system consists of m_2 inequalities $\max_{j \in J} \{\nabla(d_{ij}, x_j)\} \geq b_i^2 (\forall i \in I_2)$. As the previous section, we

firstly investigate corresponding partial inequalities $\nabla(d_{ij}, x_j) \geq b_i^2, i \in I_2$ and $j \in J$.

For each $i \in I_2$, let $S(d_i, b_i^2) = \{x \in [0,1]^n : \max_{j \in J} \{\nabla(d_{ij}, x_j)\} \geq b_i^2\}$. Also, let $S(d_{ij}, b_i^2) = \{x_j \in [0,1] : \nabla(d_{ij}, x_j) \geq b_i^2\}$.

Definition 4. For each $i \in I_2$ and each $j \in J$, define

$$\overline{W}_{ij}^1 = \frac{2b_i^2 - (1+\gamma)d_{ij}}{1-\gamma} \text{ and } \overline{W}_{ij}^2 = \frac{2b_i^2 - (1-\gamma)d_{ij}}{1+\gamma}$$

The following four lemmas are easily verified for each $i \in I_2$ and each $j \in J$, and are very useful for some next proofs.

Lemma 7. Suppose that $\gamma < 1$. Then, $d_{ij} \leq b_i^2 \Leftrightarrow d_{ij} \leq \overline{W}_{ij}^1$.

Lemma 8. $d_{ij} \leq b_i^2 \Leftrightarrow d_{ij} \leq \overline{W}_{ij}^2$.

Also, Lemmas 7 and 8 are true if " \leq " is replaced by " $<$ ", " \geq " or " $>$ ".

Lemma 9. Suppose that $\gamma < 1$. Then, $\overline{W}_{ij}^1 \geq 0 \Leftrightarrow d_{ij} = 0$ or $0 \leq \gamma \leq \frac{2b_i^2 - d_{ij}}{d_{ij}}$.

Lemma 10. (a) $\overline{W}_{ij}^2 \geq 0 \Leftrightarrow d_{ij} = 0$ or $\frac{d_{ij} - 2b_i^2}{d_{ij}} \leq \gamma \leq 1$.

(b) $\overline{W}_{ij}^2 \leq 1 \Leftrightarrow d_{ij} = 1$ or $\frac{2b_i^2 - d_{ij} - 1}{1 - d_{ij}} \leq \gamma \leq 1$.

Lemma 11 below determines set $S(d_{ij}, b_i^2)$ where $d_{ij} < b_i^2$.

Lemma 11. Suppose that $d_{ij} < b_i^2$. Then,

$$S(d_{ij}, b_i^2) = \begin{cases} \left[\overline{W}_{ij}^2, 1 \right], & (2b_i^2 - d_{ij} - 1)/(1 - d_{ij}) \leq \gamma \leq 1 \\ \emptyset, & \text{otherwise} \end{cases}$$

Proof. It is easy to verify that $d_{ij} < 1$ and $(2b_i^2 - d_{ij} - 1)/(1 - d_{ij}) < 1$. Also, by $d_{ij} < b_i^2$ and Lemma 8 we have $0 \leq d_{ij} < \overline{W}_{ij}^2$. Thus, $\overline{W}_{ij}^2 \geq 0$. Additionally, Lemma 10(b) implies $\overline{W}_{ij}^2 \leq 1$.

Now, assume that $(2b_i^2 - d_{ij} - 1)/(1 - d_{ij}) \leq \gamma \leq 1$ and $x_j \in [\overline{W}_{ij}^2, 1]$. So, $b_i^2 = \nabla(d_{ij}, \overline{W}_{ij}^2) \leq \nabla(d_{ij}, x_j)$, i.e., $x_j \in S(d_{ij}, b_i^2)$. On the other hand, if $x_j > 1$, then x_j does not clearly belong to $S(d_{ij}, b_i^2)$.

If $(2b_i^2 - d_{ij} - 1)/(1 - d_{ij}) \leq \gamma \leq 1$ and $x_j < \overline{W}_{ij}^2$, then it can be easily calculated $\nabla(d_{ij}, x_j) < \nabla(d_{ij}, \overline{W}_{ij}^2) = \gamma \overline{W}_{ij}^2 + (1 - \gamma)(d_{ij} + \overline{W}_{ij}^2)/2 = b_i^2$ that implies $x_j \notin S(d_{ij}, b_i^2)$. Moreover, if $\gamma < (2b_i^2 - d_{ij} - 1)/(1 - d_{ij})$, then

$$\begin{aligned} \nabla(d_{ij}, x_j) &\leq \nabla(d_{ij}, 1) = \gamma + (1 - \gamma)(d_{ij} + 1)/2 = ((1 - d_{ij})\gamma + d_{ij} + 1)/2 \\ &< ((1 - d_{ij})/2)((2b_i^2 - d_{ij} - 1)/(1 - d_{ij})) + (d_{ij} + 1)/2 = b_i^2 \end{aligned}$$

, that is, $x_j \notin S(d_{ij}, b_i^2)$. ?

Lemma 12 below determines set $S(d_{ij}, b_i^2)$ where $d_{ij} \geq b_i^2$.

Lemma 12. Suppose that $d_{ij} \geq b_i^2$. Then,

$$S(d_{ij}, b_i^2) = \begin{cases} \left[\max \left\{ 0, \overline{W}_{ij}^1 \right\}, 1 \right], & 0 \leq \gamma < 1 \\ [0, 1], & \gamma = 1 \end{cases}$$

Proof. Since $d_{ij} \geq b_i^2$ and $\gamma < 1$, Lemma7 implies that $\overline{w}_{ij}^1 \leq d_{ij} \leq 1$. Thus, $\overline{w}_{ij}^1 \leq 1$. Assume that $x_j \in \left[\max \left\{ 0, \overline{w}_{ij}^1 \right\}, 1 \right]$. If either $d_{ij} = 0$ or $0 \leq \gamma \leq (2b_i^2 - d_{ij})/d_{ij}$, then by Lemma9, $x_j \in \left[\max \left\{ 0, \overline{w}_{ij}^1 \right\}, 1 \right]$ means $x_j \in \left[\overline{w}_{ij}^1, 1 \right]$. In this case, we have $\nabla(d_{ij}, x_j) \geq \nabla(d_{ij}, \overline{w}_{ij}^1) = \gamma d_{ij} + (1 - \gamma)(d_{ij} + \overline{w}_{ij}^1)/2 = b_i^2$ that means $x_j \in S(d_{ij}, b_i^2)$. Furthermore, if $d_{ij} > 0$ and $\gamma > (2b_i^2 - d_{ij})/d_{ij}$, $x_j \in \left[\max \left\{ 0, \overline{w}_{ij}^1 \right\}, 1 \right]$ means $x_j \in [0, 1]$ from Lemma9. In this case, we have

$$\begin{aligned} \nabla(d_{ij}, x_j) &\geq \nabla(d_{ij}, 0) = \gamma d_{ij} + (1 - \gamma)d_{ij}/2 = (1 + \gamma)d_{ij}/2 \\ &> \left((2b_i^2 - d_{ij})/d_{ij} \right) (d_{ij}/2) + (d_{ij}/2) = b_i^2 \end{aligned}$$

, that is, $x_j \in S(d_{ij}, b_i^2)$. On the other hand, if $x_j > 1$ or $x_j < \max \left\{ 0, \overline{w}_{ij}^1 \right\} = 0$, then obviously $x_j \notin S(d_{ij}, b_i^2)$. If $x_j < \max \left\{ 0, \overline{w}_{ij}^1 \right\} = \overline{w}_{ij}^1$, then $\nabla(d_{ij}, x_j) < \nabla(d_{ij}, \overline{w}_{ij}^1) = b_i^2$, i.e., $x_j \notin S(d_{ij}, b_i^2)$. Moreover, if $\gamma = 1$, then $\nabla(d_{ij}, x_j) \geq b_i^2$ is converted into $\max \left\{ d_{ij}, x_j \right\} \geq b_i^2$. In this case, we have trivially $x_j \in S(d_{ij}, b_i^2), \forall x_j \in [0, 1]$. ?

Corollary 3. For each $i \in I_2$ and each $j \in J$,

$$S(d_{ij}, b_i^2) = \begin{cases} \left[\max \left\{ 0, \overline{w}_{ij}^1 \right\}, 1 \right] & , d_{ij} \geq b_i^2, 0 \leq \gamma < 1 \\ [0, 1] & , d_{ij} \geq b_i^2, \gamma = 1 \\ \left[\overline{w}_{ij}^1, 1 \right] & , d_{ij} < b_i^2, (2b_i^2 - d_{ij} - 1)/(1 - d_{ij}) \leq \gamma \leq 1 \\ \emptyset & , d_{ij} < b_i^2, \gamma < (2b_i^2 - d_{ij} - 1)/(1 - d_{ij}) \end{cases}$$

The following theorem gives a necessary and sufficient condition for the feasibility of inequality.

Theorem 4. Let $i \in I_2$. $S(d_i, b_i^2) \neq \emptyset$ iff there exists some $j \in J$ such that either $d_{ij} \geq b_i^2$ or $(2b_i^2 - d_{ij} - 1)/(1 - d_{ij}) \leq \gamma \leq 1$.

Proof. For an arbitrary $x \in [0, 1]^n$, $x \in S(d_i, b_i^2)$ if and only if $\max_{j \in J} \{ \nabla(d_{ij}, x_j) \} \geq b_i^2$.

Therefore, $x \in S(d_i, b_i^2)$ iff $\nabla(d_{ij}, x_j) \geq b_i^2$, for some $j \in J$. Therefore, $S(d_i, b_i^2) \neq \emptyset$ iff $S(d_{ij}, b_i^2) \neq \emptyset$, for some $j \in J$. Now, the result follows from Corollary3. ?

Definition 5. Suppose that $S(d_i, b_i^2) \neq \emptyset$.

Let $J_1 = \{ j \in J : d_{ij} \geq b_i^2, \gamma < 1 \}$, $J_2 = \{ j \in J : d_{ij} \geq b_i^2, \gamma = 1 \}$ and

$J_3 = \{ j \in J : d_{ij} < b_i^2, \gamma \geq (2b_i^2 - d_{ij} - 1)/(1 - d_{ij}) \}$

Definition 6. Suppose that $S(d_i, b_i^2) \neq \emptyset$. For each $j \in J_1 \cup J_2 \cup J_3$, we define $\underline{X}(i, j) =$

$[\underline{X}(i, j)_1, \underline{X}(i, j)_2, \dots, \underline{X}(i, j)_n]$ where

$$\underline{X}(i, j)_k = \begin{cases} \max \left\{ 0, \overline{W}_{ij}^1 \right\} & , k = j, j \in J_1 \\ 0 & , k = j, j \in J_2 \\ \overline{W}_{ij}^2 & , k = j, j \in J_3 \\ 0 & , otherwise \end{cases}$$

By Theorem5 below, the solutions set $S(d_i, b_i^2)$ is completely determined. The theorem shows that $S(d_i, b_i^2)$ has actually the finite number of minimal solutions, $\underline{X}(i, j)$, and the unique maximum solution, $\mathbf{1}$, where $\mathbf{1}$ is an n -dimensional unite vector.

Theorem 5. Suppose that $S(d_i, b_i^2) \neq \emptyset$.

Then, $S(d_i, b_i^2) = \bigcup_{j \in J_1 \cup J_2 \cup J_3} [\underline{X}(i, j), \mathbf{1}]$, $\forall i \in I_2$.

Proof. According to the proof of Theorem4, for each $x \in [0, 1]^n$, $x \in S(d_i, b_i^2)$ iff $x_j \in S(d_{ij}, b_i^2)$, for some $j \in J$. Therefore, $S(d_i, b_i^2) = \bigcup_{j \in J} S(d_{ij}, b_i^2)$. Thus, from Corollary3 and Definition5, we have $S(d_i, b_i^2) = \bigcup_{j \in J_1 \cup J_2 \cup J_3} S(d_{ij}, b_i^2)$. Now, the result is attained from Corollary3 and Definition6. ?

Definition 7. Let $e : I_2 \rightarrow J_1 \cup J_2 \cup J_3$ so that $e(i) = j \in J_1 \cup J_2 \cup J_3$, $\forall i \in I_2$, and let E_D be the set of all vectors e . For the sake of convenience, we represent each $e \in E_D$ as an m_2 -dimensional vector $e = [j_1, j_2, \dots, j_{m_2}]$ in which $j_k = e(k)$, $k = 1, 2, \dots, m_2$.

Definition 8. Let $e = [j_1, j_2, \dots, j_{m_2}] \in E_D$. Let $\underline{X}(e) = [\underline{X}(e)_1, \underline{X}(e)_2, \dots, \underline{X}(e)_n]$, where $\underline{X}(e)_j = \max_{i \in I_2} \{ \underline{X}(i, e(i))_j \} = \max_{i \in I_2} \{ \underline{X}(i, j)_j \}$, $\forall j \in J$.

Based on Theorem 5 and Definition8, we have the following theorem characterizing the feasible region of the general inequalities $D \nabla x \geq b^2$.

Theorem 6. Suppose that $S(d_i, b_i^2) \neq \emptyset$, $\forall i \in I_2$. Then, $S(D, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), \mathbf{1}]$.

Proof. Since $S(D, b^2) = \bigcap_{i \in I_2} S(d_i, b_i^2)$, Theorem5 implies that

$S(D, b^2) = \bigcap_{i \in I_2} \bigcup_{j \in J_1 \cup J_2 \cup J_3} [\underline{X}(i, j), \mathbf{1}]$. Therefore, we have

$$S(D, b^2) = \bigcup_{j \in J_1 \cup J_2 \cup J_3} \bigcap_{i \in I_2} [\underline{X}(i, j), \mathbf{1}] = \bigcup_{e \in E_D} \bigcap_{i \in I_2} [\underline{X}(i, e(i)), \mathbf{1}] = \bigcup_{e \in E_D} \left[\max_{i \in I_2} \{ \underline{X}(i, e(i)) \}, \mathbf{1} \right]$$

Now, the result follows from Definition8. ?

Theorem6 determines the solutions set $S(D, b^2)$ as the union of the finite number of n -dimensional interval $[\underline{X}(e), \mathbf{1}]$ with $\underline{X}(e)$ as the minimal and $\mathbf{1}$ as the unique maximum solutions. The following Corollary gives a necessary and sufficient condition for the feasibility of general inequalities $D \nabla x \geq b^2$.

Corollary 4. $S(D, b^2) \neq \emptyset$ iff $\mathbf{1} \in S(D, b^2)$.

4. The resolution of Problem (1)

In this section, a necessary and sufficient condition is derived to determine the feasibility of the main problem. As is shown, the feasible region is completely found by the finite number of closed convex cells.

Lemma 13. $S(A, D, b^1, b^2) \neq \emptyset$ iff there exists some $e \in E_D$ such that $[\mathbf{0}, \bar{X}] \cap [\underline{X}(e), \mathbf{1}] \neq \emptyset$.

Proof. Since $S(A, D, b^1, b^2) = S(A, b^1) \cap S(D, b^2)$, from Theorems 3 and 6 we have

$$S(A, D, b^1, b^2) = [\mathbf{0}, \bar{X}] \cap \bigcup_{e \in E_D} [\underline{X}(e), \mathbf{1}] = \bigcup_{e \in E_D} ([\mathbf{0}, \bar{X}] \cap [\underline{X}(e), \mathbf{1}])$$

This completes the proof. ?

The following Corollary gives a necessary and sufficient condition for the feasibility of the intersection of general inequalities $A \nabla x \leq b^1$ and $D \nabla x \geq b^2$.

Corollary 5. Assume that $S(A, b^1) \neq \emptyset$ and $S(D, b^2) \neq \emptyset$. Then, $S(A, D, b^1, b^2) \neq \emptyset$ iff $\bar{X} \in S(D, b^2)$.

Proof. According to Lemma13, $S(A, D, b^1, b^2) \neq \emptyset$ iff $[\mathbf{0}, \bar{X}] \cap [\underline{X}(e'), \mathbf{1}] \neq \emptyset$ for some $e' \in E_D$. Thus, $S(A, D, b^1, b^2) \neq \emptyset$ iff $\bar{X} \in [\underline{X}(e'), \mathbf{1}]$ that means $\bar{X} \in \bigcup_{e \in E_D} [\underline{X}(e), \mathbf{1}]$. Therefore, $S(A, D, b^1, b^2) \neq \emptyset$ iff $\bar{X} \in S(D, b^2)$, from Theorem6. ?

The following theorem characterizes the feasible region of Problem (1). The theorem determines the solutions set $S(A, D, b^1, b^2)$ as the union of the finite number of closed convex intervals.

Theorem 7. Suppose that $S(A, D, b^1, b^2) \neq \emptyset$. Then $S(A, D, b^1, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), \bar{X}]$.

Proof.

According to the proof of Lemma13, we have $S(A, D, b^1, b^2) = \bigcup_{e \in E_D} ([\mathbf{0}, \bar{X}] \cap [\underline{X}(e), \mathbf{1}])$. Now, the required equality is resulted from Corollary5. ?

We now summarize the preceding discussion as an algorithm.

Algorithm 1 (solution of problem (1))

Given problem (1):

1. If for some $i \in I_1$ and $j \in J$, $a_{ij} > b_i^1$ and $\gamma > (2b_i^1 - a_{ij})/a_{ij}$, then stop; $S(a_i, b_i^1)$ is infeasible (Theorem1).
2. If $\mathbf{0} \notin S(A, b^1)$, then stop; $S(A, b^1)$ is infeasible (Corollary2).
3. If for some $i \in I_2$ and each $j \in J$, $d_{ij} < b_i^2$ and $\gamma < (2b_i^2 - d_{ij} - 1)/(1 - d_{ij})$, then stop; $S(d_i, b_i^2)$ is infeasible (Theorem4).
4. If $\mathbf{1} \notin S(D, b^2)$, then stop; $S(D, b^2)$ is infeasible (Corollary4).
5. Compute vectors $\bar{X}(i)$ ($\forall i \in I_1$) from Definition2, and then vector \bar{X} from Definition 3.
6. If $\bar{X} \notin S(D, b^2)$, then stop; $S(A, D, b^1, b^2)$ is infeasible (Corollary5).
7. Compute vectors $\underline{X}(e)$ ($\forall e \in E_D$) from Definition8.
8. Find the feasible solutions set $S(A, D, b^1, b^2)$ as $\bigcup_{e \in E_D} [\underline{X}(e), \bar{X}]$ (Theorem7).

5. Numerical example

Consider the following problem formed as the intersection of two fuzzy systems defined by “Fuzzy Or”-Inequalities:

$$\begin{array}{c}
 \left[\begin{array}{ccc} 0.4 & 0.8 & 0.4 \\ 0.7 & 0.4 & 0.5 \\ 0.5 & 0.5 & 0.3 \\ 0.8 & 0.8 & 0.7 \\ 0.6 & 0.2 & 0.9 \\ 0.2 & 0.5 & 0.3 \end{array} \right] \nabla x \leq \left[\begin{array}{c} 0.8 \\ 0.7 \\ 0.4 \\ 0.2 \\ 0.3 \\ 0.4 \end{array} \right] \\
 \left[\begin{array}{ccc} 0.4 & 0.8 & 0.4 \\ 0.7 & 0.4 & 0.5 \\ 0.5 & 0.5 & 0.3 \\ 0.8 & 0.8 & 0.7 \\ 0.6 & 0.2 & 0.9 \\ 0.2 & 0.5 & 0.3 \end{array} \right] \nabla x \geq \left[\begin{array}{c} 0.8 \\ 0.7 \\ 0.4 \\ 0.2 \\ 0.3 \\ 0.4 \end{array} \right] \\
 x \in [0, 1]^n
 \end{array}$$

Step1: for $i = 1, 2$ and $j = 1, 2, 3$, we have $a_{ij} \leq b_i^1$. Then, from Theorem1 $S(a_1, b_1^1) \neq \emptyset$ and $S(a_2, b_2^1) \neq \emptyset$. Also, $0.5 = \gamma \leq (2b_3^1 - a_{31})/a_{31} = 0.6$, $0.5 = \gamma \leq (2b_3^1 - a_{32})/a_{32} = 0.6$ and $a_{33} \leq b_3^1$ that imply $S(a_3, b_3^1) \neq \emptyset$.

Step2: The following calculation shows that $\mathbf{0} \in S(A, b^1)$.

$$\begin{bmatrix} 0.4 & 0.8 & 0.4 \\ 0.7 & 0.4 & 0.5 \\ 0.5 & 0.5 & 0.3 \end{bmatrix} \nabla \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6000 \\ 0.5250 \\ 0.3750 \end{bmatrix} \leq \begin{bmatrix} 0.8 \\ 0.7 \\ 0.4 \end{bmatrix}$$

Therefore, $S(A, b^1) \neq \emptyset$, from Corollary2.

Step3: Since $d_{1j} \geq b_1^2$ for each $j \in J$, then $S(d_1, b_1^2) \neq \emptyset$ from Theorem4. Also, $d_{21} \geq b_2^2 d_{23} \geq b_2^2$, and

$-0.75 = (2b_2^2 - d_{22} - 1)/(1 - d_{22}) \leq \gamma = 0.5$ that imply $S(d_2, b_2^2) \neq \emptyset$. Finally, since

$-0.5 = (2b_3^2 - d_{31} - 1)/(1 - d_{31}) \leq \gamma = 0.5$, $-0.7143 = (2b_3^2 - d_{33} - 1)/(1 - d_{33}) \leq \gamma = 0.5$ and $d_{32} \geq b_3^2$, then $S(d_3, b_3^2) \neq \emptyset$.

Step4: According to the calculation below, $\mathbf{1} \in S(D, b^2)$. Hence, from Corollary4, $S(D, b^2) \neq \emptyset$.

$$\begin{bmatrix} 0.8 & 0.8 & 0.7 \\ 0.6 & 0.2 & 0.9 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \nabla \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9500 \\ 0.9750 \\ 0.8750 \end{bmatrix} \geq \begin{bmatrix} 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

Step5: From Definition2, we have

$$\begin{aligned} \bar{X}(1) &= [0.9333 \quad 0.8000 \quad 0.9333] \\ \bar{X}(2) &= [0.7000 \quad 0.8000 \quad 0.7667] \\ \bar{X}(3) &= [0.1000 \quad 0.1000 \quad 0.4333] \end{aligned}$$

Therefore, from Definition3, we attain $\bar{X} = [0.1000 \quad 0.1000 \quad 0.4333]$.

Step6: From Corollary5, since $\bar{X} \in S(D, b^2)$, then $S(A, D, b^1, b^2) \neq \emptyset$. It can be easily verified as follows:

$$\begin{bmatrix} 0.8 & 0.8 & 0.7 \\ 0.6 & 0.2 & 0.9 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \nabla \begin{bmatrix} 0.1000 \\ 0.1000 \\ 0.4333 \end{bmatrix} = \begin{bmatrix} 0.6333 \\ 0.7833 \\ 0.4000 \end{bmatrix} \geq \begin{bmatrix} 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

Step7: From Definition8, the feasible vectors $\underline{X}(e)$ (i.e., $\underline{X}(e) \leq \bar{X}$) are resulted as follows:

$$\begin{aligned} e_1 = [1 \quad 1 \quad 2] &\Rightarrow \underline{X}(e_1) = [0 \quad 0.1 \quad 0] \\ e_2 = [1 \quad 1 \quad 3] &\Rightarrow \underline{X}(e_2) = [0 \quad 0 \quad 0.4333] \end{aligned}$$

Vectors $\underline{X}(e_1)$ and $\underline{X}(e_2)$ are actually minimal solutions of the problem.

Step8: From Theorem7, we attain $S(A, D, b^1, b^2) = [\underline{X}(e_1), \bar{X}] \cup [\underline{X}(e_2), \bar{X}]$.

Conclusion

In this paper, we proposed an algorithm to solve the intersection of two types of fuzzy relational inequalities defined by "Fuzzy Or" operator. The feasible solutions set of each type of these fuzzy systems was obtained. Based on the foregoing results, the feasible region of the problem is completely resolved and four necessary and sufficient conditions were presented to determine the feasibility of the problem. As future works, we aim at testing our algorithm in other type of fuzzy systems and linear optimization problems whose constraints are defined as FRI with other averaging operators.

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