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A generalization of zero-divisor graphs

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ABSTRACT

In this paper, we introduce a family of graphs which is a generalization of zero-divisor graphs and compute an upper-bound for the diameter of such graphs. We also investigate their cycles and cores.

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1 Introduction

For coloring a commutative ring, Beck introduced a version of the zero-divisor graph of a ring in his 1988 paper [8]. Later in 1999, Anderson and Livingston introduced a similar notion which is the by-now standard definition of zero-divisor graphs [4]. This notion has been generalized and investigated for commutative semigroups with zero by DeMeyer et al. [16, 17]. Since then, many authors have investigated the zero-divisor graphs from different perspectives and for a survey on this, one may refer to the papers [2,3]. Similarly, for non-commutative rings, Redmond has introduced a similar notion called zero-divisor (directed) graphs [31].
One of the interesting topics in algebraic combinatorics is to compute invariants of zero-divisor graphs such as their diameters, girths, clique numbers, chromatic numbers, and even "Zagreb indices" [7] and for a survey on the computation of these invariants, one can

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check the paper [13]. For the comparison of these numbers for zero-divisor graphs of a semigroup under Armendariz extension one may see the 2013 paper by Epstein et al. [20] and under polynomial and power series extensions the 2006 paper by Lucas [25]. Section 5 of the 2010 paper [29] is devoted to the comparison of the diameter of zero-divisor graphs under content extensions. One interesting topic for a future project can be to compute the tenacity [14] of zero-divisor graphs.
Our main motivation for this paper was to attribute a graph $RG(M)$ to a module M inspired by zero-divisor graphs of ideals of a ring in the following sense:
Let R be a commutative ring with a nonzero identity and M be a unital R-module. We associate a graph $RG(M)$ to M, which we call residuated graph of M, whose vertices and edges are determined as follows:

1. Let N be a submodule of M. Then N is a vertex of $RG(M)$ if the residuated ideal $[N : M]$ of R is nonzero and there is a submodule $K \neq N$ of M with $[K : M] \neq (0)$ such that
$$[N : M] \cdot [K : M] = (0),$$
where by $[N : M]$, we mean the set of all elements $r \in R$ such that $rM \subseteq N$;

2. Two distinct vertices P and Q of the graph $RG(M)$ are connected if
$$[P : M] \cdot [Q : M] = (0).$$

Surprisingly, similar to the zero-divisor graphs of commutative semigroups [17, Theorem 1.3], the graph $RG(M)$, for any R-module M, is connected and the best upper-bound for $\operatorname{diam} RG(M)$ is 3 if the graph $RG(M)$ is non-empty (see Corollary 3.12). Here we need to recall that the distance between two vertices in a simple graph is the number of edges in a shortest path connecting them. The greatest distance between any two vertices in a graph G is the diameter of G, denoted by $\operatorname{diam}(G)$ [18, p. 8].
Based on our investigations for residuated graphs, in Definition 2.1, we attribute a graph to an arbitrary set which is also a generalization of the notion of zero-divisor graphs of arbitrary commutative semigroups with zero in the following sense:
Let X be a non-empty set, $(S, \cdot, 0)$ a commutative multiplicative semigroup with zero, and f a function from X to S. We attribute a simple graph to X, denoted by $\Gamma_{(S,f)}(X)$, whose vertices and edges are determined as follows:

1. An element $x \in X$ is a vertex of the graph $\Gamma_{(S,f)}(X)$ if $f(x) \neq 0$ and there is a $y \neq x$ in X such that $f(y) \neq 0$ and $f(x) \cdot f(y) = 0$.

2. Let x and y be elements of X. The doubleton $\{x, y\}$ is an edge of the graph $\Gamma_{(S,f)}(X)$ if $x \neq y$, $f(x) \neq 0$, and $f(y) \neq 0$ while $f(x) \cdot f(y) = 0$.

Then, in Section 2, we prove that under some conditions, the graph $\Gamma_{(S,f)}(X)$ is connected with $\operatorname{diam} \Gamma_{(S,f)}(X) \leq 3$ if $\Gamma_{(S,f)}(X)$ is non-empty (see Definition 2.1, Theorem 3.1, and Theorem 3.10).

Note that in the Definition 2.1, if we set $X = S$ and suppose that id_S is the identity map on a commutative semigroup with zero S, then $\Gamma_{(S,\mathrm{id}_S)}(S)$ is nothing but the zero-divisor graph $\Gamma(S)$ defined in [17].
In Section 3, we prove that if S is a commutative semiring with a nonzero identity and the S-semimodule M has the annihilator condition or M is a content S-semimmodule and the content function from M to finitely generated ideals of S is onto, then the graphs $\Gamma_{(\mathrm{Id}(S),\mathrm{Ann})}(M)$ and $\Gamma_{(\mathrm{Id}(S),c)}(M)$ are connected with diameters at most 3 if they are non-empty (see Corollary 3.7 and Corollary 3.9).
We also show that if S is a commutative semiring with a nonzero identity, M is a unital S-semimodule, q is a function from $\mathrm{Sub}(M)$ to $\mathrm{Id}(S)$ with $q(N) = [N : M]$, and the graph $\Gamma_{(\mathrm{Id}(S),q)}(\mathrm{Sub}(M))$ is non-empty, then it is a connected graph whose diameter is at most 3 (see Corollary 3.12).
In Section 4, we discuss the cycles and cores of the graphs defined in Definition 2.1. For example in Theorem 4.3, we prove that if X is a non-empty set, S a commutative semigroup with zero, f a function from X to S, the graph $\Gamma_{(S,f)}(X)$ has at least three vertices, and the function f has this property that for all $x, y \in X$ if $f(x)f(y) \neq 0$ then there exists a $z \in X$ such that $f(z) = f(x)f(y)$, then if $\Gamma_{(S,f)}(X)$ contains a cycle, then the core K of $\Gamma_{(S,f)}(X)$ is a union of triangles and rectangles.
We recall that a trail in a graph G is a walk in which all edges are distinct. A path in the graph G is a trail in which all vertices (except possibly the first and last) are distinct. If $P = x_0 \cdots x_{k-1}$ is a path in G and $k \geq 3$, then the path $C = x_0 \cdots x_{k-1}x_0$ is a cycle in G [18]. We also note that the core of a graph Γ is the largest subgraph of Γ in which every edge is the edge of a cycle in Γ [16].

2 A generalization of zero-divisor graphs for semigroups

One of the interesting areas of research in algebraic combinatorics is to associate a graph $G(A)$ to an algebraic structure A and investigate the interplay between the algebraic properties of the algebra A and the graph-theoretic properties of the graph $G(A)$. One method is to consider the intersection graphs of the substructures of an algebraic structure. For example, in the 2012 paper [1], Akbari et al. investigate the intersection graphs of the submodules of modules over arbitrary commutative rings. Since 1960s, many authors have worked on intersection graphs [9,12,15,30,32,34,35]. Note that all graphs are intersection graphs [19]. In this direction, Malakooti Rad and Nasehpour generalize the notion of intersection graphs and attribute a graph to the bounded semilattices and investigate their properties and compute the invariants of such graphs [26].
In this section, we attribute a graph to an arbitrary set which is on one hand a generalization of the notion of zero-divisor graphs of commutative semigroups and on the other hand is a generalization of the graphs attributed to submodules of a module given in Corollary 3.12.

Definition 2.1. Let X be a non-empty set, $(S, \cdot, 0)$ a commutative multiplicative semigroup with zero, and f a function from X to S. We attribute a graph to X, denoted by $\Gamma_{(S,f)}(X)$, whose vertices and edges are determined as follows:

1. An element $x \in X$ is a vertex of the graph $\Gamma_{(S,f)}(X)$ if $f(x) \neq 0$ and there is a $y \neq x$ in X such that $f(y) \neq 0$ and $f(x) \cdot f(y) = 0$.

2. Let x and y be elements of X. The doubleton $\{x, y\}$ is an edge of the graph $\Gamma_{(S,f)}(X)$ if $x \neq y$, $f(x) \neq 0$, and $f(y) \neq 0$ while $f(x) \cdot f(y) = 0$.

Remark 2.2. Let X be a non-empty set, S a commutative semigroup with zero, and f a function from X to S. The graph $\Gamma_{(S,f)}(X)$ is a generalization of the usual zero-divisor graph $\Gamma(S)$ defined in [17]. In fact, if suppose that S is a commutative semigroup with zero and $X = S$, then $\Gamma_{(S,\mathrm{id}_S)}(S)$ is the zero-divisor graph $\Gamma(S)$, where id_S is the identity map on S.
A graph C is called to be a zero-divisor if these exist non-isomorphic graphs A and B for which $A \times C \cong B \times C$ [23, p. 310]. For examples of these graphs see [24]. And one should not confuse this concept in graph theory with the concept of zero-divisor graphs in [17].

Question 2.3. Let G be an arbitrary graph. Is it possible to find a set X, a commutative semigroup with zero S, and a function f from X to S such that G is isomorphic to the graph $\Gamma_{(S,f)}(X)$?

3 Diameter of Zero-Divisor Graphs and Their Generalizations

Theorem 3.1. *Let X be a non-empty set, S a commutative semigroup with zero, and f a function from X to S with this property that for all $x, y \in X$, if $f(x)f(y) \neq 0$ then there exists a $z \in X$ such that $f(z) = f(x)f(y)$. Then the graph $\Gamma_{(S,f)}(X)$ is connected with* $\mathrm{diam}(\Gamma_{(S,f)}(X)) \leq 3$.

Proof. Let x, y be two distinct vertices of $\Gamma_{(S,f)}(X)$. Therefore, there exists $z, w \in X$ such that $f(z) \neq 0$, $f(w) \neq 0$ and $f(x)f(z) = 0$ and $f(y)f(w) = 0$. Note that by definition, $f(x) \neq 0$ and $f(y) \neq 0$.
Now we show that $d(x, y) \leq 3$. If $f(x)f(y) = 0$, then $d(x, y) = 1$. If $f(x)f(y) \neq 0$, but $f(z)f(w) = 0$, then $x - z - w - y$ is a path in $\Gamma_{(S,f)}(X)$ and therefore, $d(x, y) \leq 3$.
Finally, let $f(x)f(y) \neq 0$ and $f(z)f(w) \neq 0$. Since there exists a $t \in X$ such that $f(t) = f(z)f(w)$, we have $f(x)f(t) = f(t)f(y) = 0$ and $d(x, y) \leq 2$. Therefore, the graph $\Gamma_{(S,f)}(X)$ is connected with diameter at most 3 and the proof is complete. □

Corollary 3.2. *Let S be a commutative semigroup with zero. The zero-divisor graph $\Gamma(S)$ is connected with* $\mathrm{diam}\,\Gamma(S) \leq 3$ *[16, Theorem 1].*

Let X be a non-empty set, S a commutative semigroup with zero, and f a function from X to S. We do not know if the graph $\Gamma_{(S,f)}(X)$ is connected in general. Based on this, the following question arises:

Question 3.3. Let X be a non-empty set, S a commutative semigroup with zero, and f a function from X to S. If the graph $\Gamma_{(S,f)}(X)$ defined in Definition 2.1 is connected, what is the best upper-bound for the diameter of this graph?

Related to the above question, we bring the following remark:

Remark 3.4. Let us recall that if S is a semigroup (not necessarily commutative) with zero, a directed graph $\Gamma(S)$, called zero-divisor graph of S, is attributed to S whose vertices are the proper zero-divisors of S and $s \to t$ is an edge of $\Gamma(S)$ between the vertices s and t if $s \neq t$ and $st = 0$ [10]. The following result from [10, 31], is an interesting generalization of Corollary 3.2 though written in the terminology of the paper [27]:

Theorem 3.5. *Let S be a semigroup with zero. The directed graph $\Gamma(S)$ is connected if and only if S is eversible. Moreover, if $\Gamma(S)$ is connected, then the diameter of the graph $\Gamma(S)$ is at most 3.*

Note that a semigroup with zero S is eversible if every left zero-divisor on S is also a right zero-divisor on S and conversely, i.e., $Z_l(S) = Z_r(S)$ [27, Definition 1.9].

Let us recall that a commutative ring R with an identity has the annihilator condition if for all $a, b \in R$, there is a $c \in R$ such that $\mathrm{Ann}(a, b) = \mathrm{Ann}(c)$ [22]. Inspired by this, we give the following definition for semimodules [21, Chap. 14]:

Definition 3.6. Let S be a commutative semiring with an identity and M be a unital R-semimodule. We say that M has the annihilator condition if for all $x, y \in M$, there is a $z \in M$ such that $\mathrm{Ann}(x, y) = \mathrm{Ann}(z)$, where by $\mathrm{Ann}(N)$, we mean the set of all elements s in S such that $sN = 0$.

Note that we gather all ideals of a semiring S in the set $\mathrm{Id}_S(S)$ and all S-subsemimodules of M in the set $\mathrm{Sub}_S(M)$.

Corollary 3.7. *Let the S-semimodule M have the annihilator condition. Then the graph $\Gamma_{(\mathrm{Id}(S),\mathrm{Ann})}(M)$ is a connected graph whose dimater is at most 3.*

Proof. It is clear that $(\mathrm{Id}(S), \cap)$ is a commutative semigroup and its zero, i.e., its absorbing element, is the zero ideal (0). Consider the function Ann from M to $\mathrm{Id}(S)$. It is straightforward to see that $\mathrm{Ann}(x, y) = \mathrm{Ann}(x) \cap \mathrm{Ann}(y)$ for all $x, y \in M$. Since by assumption the S-semimodule M has the annihilator condition, the proof is complete. □

Let S be a commutative semigroup with zero. A subset I of S is said to be an s-ideal of S, if $0 \in I$ and for all $s \in S$ and $a \in I$, we have $s \cdot a \in I$ [6]. Clearly, the intersection of two s-ideals of a semigroup S is an s-ideal of S. If we denote the set of all s-ideals of S

by $\mathrm{Id}_S(S)$, then $\mathrm{Id}_S(S)$ along with the intersection configures a commutative semigroup with zero and its absorbing element is the s-ideal $\{0\}$.
Let us recall that if S is a semigroup, a set M together with a function $S \times M \to M$, denoted $(s, m) \to sm$, satisfying $(st)x = s(tx)$ for all $s, t \in S$ and $x \in M$ is called a (left) S-act. Also, if M is a S-act and the semigroup S has an absorbing element 0_S and M possesses a distinguished element 0_M such that $s0_M = 0_M$ for all $s \in S$ and $0_S x = 0_M$ for all $x \in M$, then M is called a pointed S-act. Finally, if S is a monoid and 1_S is the neural element for the multiplication of S, then an S-act M is called unital if $1_S m = m$ for all $m \in M$ [33]. Note that if S is a semiring and M is a unital S-semimodule, then obviously M is a unital pointed S-act.
Now, let S be a commutative monoid with zero and M a unital pointed S-act. If $\emptyset \neq N \subseteq M$, we define $\mathrm{Ann}(N)$ to be the set of all elements $s \in S$ such that $sN = \{0_M\}$. One can easily check that $\mathrm{Ann}(N)$ is an s-ideal of the semigroup S and if P and Q are non-empty subsets of M, then $\mathrm{Ann}(P) \cap \mathrm{Ann}(Q) = \mathrm{Ann}(P \cup Q)$. Therefore, we have already showed that the following result is just another example for Theorem 3.1:

Corollary 3.8. *Let S be a commutative monoid with zero and M a unital pointed S-act. If $\mathcal{C}$ is a non-empty class of non-empty subsets of the set M and $(\mathcal{C}, \cup)$ is a semigroup and the graph $\Gamma_{(\mathrm{Id}_S(S), \mathrm{Ann})}(\mathcal{C})$ is non-empty, then it is a connected graph with diameter at most 3.*

Let us recall that if S is a commutative semiring with a nonzero identity and M is a unital S-semimodule, then the content function from M into the ideals $\mathrm{Id}(S)$ of S is defined as follows:
$$c(x) = \bigcap \{I \in \mathrm{Id}(S) : x \in IM\}.$$
An S-semimodule M is called a content semimodule if $x \in c(x)M$ for all $x \in M$. It is straightforward to see that if M is a content S-semimodule, then $c(x)$ is a finitely generated ideal of S for each $x \in M$ [28, Proposition 23]. Now, we give the following corollary:

Corollary 3.9. *Let S be a commutative semiring with a nonzero identity and M a content S-semimmodule. If the content function from M to the set of finitely generated ideals of S is onto and the graph $\Gamma_{(\mathrm{Id}(S), c)}(M)$ is non-empty, then it is a connected graph with a diameter at most 3.*

Proof. Let $x, y \in M$ be vertices of the graph $\Gamma_{(\mathrm{Id}(S), c)}(M)$. Since M is a content S-semimodule, then $c(x)$ and $c(y)$ are both finitely generated ideals of the semiring S [28, Proposition 23]. Clearly, $c(x)c(y)$ is also finitely generated. By assumption, the content function c from M to the set of finitely generated ideals of S is onto. So, there is a $z \in M$ such that $c(z) = c(x)c(y)$. By using Theorem 3.1, the proof is complete. □

Let us recall that a commutative semigroup $(S, \cdot)$ is called positive ordered if S is a semigroup with the zero 0 and there is a partial order $\leq$ on S such that the following conditions are satisfied:

1. The partial order $\leq$ is compatible with the multiplication of the semigroup, i.e. $x \leq y$ implies $xz \leq yz$ for all $x, y, z \in S$,

2. The partial order is positive, i.e. $0 < x$ and $0 < y$ imply that $0 < xy$ for all $x, y \in S$.

Theorem 3.10. *Let X be a non-empty set, S a positive ordered commutative semigroup with zero, and f a function from X to S with this property that for all $w, z \in X$, if $f(w)f(z) \neq 0$, then there exists a $v \in X$ such that $f(w)f(z) \leq f(v)$, $f(v) \leq f(w)$, and $f(v) \leq f(z)$. Then the graph $\Gamma_{(S,f)}(X)$ is connected with $\operatorname{diam}(\Gamma_{(S,f)}(X)) \leq 3$.*

Proof. Let x, y be two distinct vertices of $\Gamma_{(S,f)}(X)$. Therefore, there exists $z, w \in X$ such that $f(z) \neq 0$, $f(w) \neq 0$ and $f(x)f(z) = 0$ and $f(y)f(w) = 0$. Note that $f(x) \neq 0$ and $f(y) \neq 0$. Now we show that $d(x, y) \leq 3$.
The argument for the case $f(x)f(y) = 0$ and the case $f(x)f(y) \neq 0$ while $f(w)f(z) = 0$ is the same as the argument in the proof of Theorem 3.1 and therefore, $d(x, y) \leq 3$.
Now imagine $f(x)f(y) \neq 0$ and $f(z)f(w) \neq 0$. Since by assumption, there exists a $v \in X$ such that $f(z)f(w) \leq f(v)$, $f(v) \leq f(z)$, and $f(v) \leq f(w)$, we have $f(x)f(v) = 0$ and $f(v)f(y) = 0$ and therefore, $d(x, y) \leq 2$ and the proof is complete. □

Let us recall that if M is an S-semimodule and N is an S-subsemimodule of M, $[N : M]$ is defined to be the set of all elements s of the semiring S such that $sM \subseteq N$. The proof of the following proposition is straightforward, but we bring it here only for the sake of reference.

Proposition 3.11. *Let S be a commutative semiring with a nonzero identity and M an S-semimodule. Then the following statements hold:*

1. *If N is an S-subsemimodule of M, then $[N : M]$ is an ideal of S,*

2. *If P and Q are S-subsemimodules of the S-semimodule M, then*

$$[P : M] \cdot [Q : M] \subseteq [P \cap Q : M],$$

3. *If P and Q are S-subsemimodules of the S-semimodule M and $P \subseteq Q$, then*

$$[P : M] \subseteq [Q : M].$$

Corollary 3.12. *Let S be a commutative semiring with a nonzero identity and M be a unital S-semimodule. Assume that q is a function from $\operatorname{Sub}(M)$ to $\operatorname{Id}(R)$ with $q(N) = [N : M]$. If the graph $\Gamma_{(\operatorname{Id}(R),q)}(\operatorname{Sub}(M))$ is non-empty, then it is a connected graph whose dimeter is at most 3.*

Proof. Use Theorem 3.10 and Proposition 3.11. □

Let us recall that if S is an idempotent commutative semigroup, then S can be ordered by the following order: $x \leq y$ if $xy = x$ for all $x, y \in S$. Additionally, if $(S, \cdot, 0, 1)$ is a monoid with the absorbing element 0, then S is called a bounded semilattice [11].

Proposition 3.13. *Let $(S, \cdot, 0, 1)$ be a bounded semilattice and d be the largest element of the poset $S - \{0, 1\}$ such that $d^2 = 0$. If f is a function from a set X to S such that the graph $\Gamma_{(S,f)}(X)$ has at least two vertices, then* $\operatorname{diam}(\Gamma_{(S,f)}(X)) = 1$.

Proof. Let x, y be vertices of the graph $\Gamma_{(S,f)}(X)$. It is clear that $f(x)$ and $f(y)$ are both nonzero and there are two elements $w, z \in X$ such that $f(x)f(w) = 0$ and $f(y)f(z) = 0$. Clearly, these two imply that $f(x) \neq 1$ and $f(y) \neq 1$. Therefore, $f(x) \leq d$ and $f(y) \leq d$, because d is the largest element the poset $S - \{0, 1\}$. On the other hand, $f(x)f(y) \leq d^2 = 0$. Hence, $\{x, y\}$ is an edge of the graph $\Gamma_{(S,f)}(X)$ and the proof is complete. □

Corollary 3.14. *Let S be a commutative semiring with an identity and M be a unital S-semimodule. Also, let q be the function from* $\operatorname{Sub}(M)$ *to $Id(R)$ with $q(N) = [N : M]$. If $\mathfrak{m}$ is the only maximal ideal of the semiring S such that $\mathfrak{m}^2 = 0$ and the graph $\Gamma_{(Id(S),q)}(\operatorname{Sub}(M))$ has at least two vertices, then its diameter is 1.*

4 Cycles and Cores of Zero-Divisor Graphs and Their Generalizations

Now we proceed to discuss the cycles of the graph $\Gamma_{(S,f)}(X)$. Let Γ be a graph. We denote the set of all vertices adjacent to the vertex a of the graph Γ by $N(a)$. In particular, if X is a non-empty set, S a commutative semigroup with zero, and f a function from X to S, then $N(a)$ is the set of all vertices $x \in X - \{a\}$ in the graph $\Gamma_{(S,f)}(X)$ such that $f(x) \neq 0$ and $f(a)f(x) = 0$.

Lemma 4.1. *If $a - x - b$ is a path in a graph Γ, then either $N(a) \cap N(b) = \{x\}$ or $a - x - b$ is contained in a cycle of the length of at most 4.*

Proof. Let $a - x - b$ be a path in the graph Γ. It is obvious that $\{x\} \subseteq N(a) \cap N(b)$. If $N(a) \cap N(b) \neq \{x\}$, then there exists a vertex c such that $c \notin \{x, a, b\}$ and c is adjacent to the both vertices a and b. So, $a - x - b - c - a$ is a path in Γ. Hence, $a - x - b$ is contained in a cycle of the length ≤ 4. □

Theorem 4.2. *Let X be a non-empty set, S a commutative semigroup with zero, and f a function from X to S. Also, let the graph $\Gamma_{(S,f)}(X)$ have at least three vertices such that for all $a, b, x \in X$ if $a - x - b$ is a path in $\Gamma_{(S,f)}(X)$ then $N(a) \cap N(b) \neq \{x\}$. If $\Gamma_{(S,f)}(X)$ is a connected graph with* $\operatorname{diam}(\Gamma_{(S,f)}(X)) \leq 3$, *then any edge in $\Gamma_{(S,f)}(X)$ is contained in a cycle of the length at most 4 and therefore, $\Gamma_{(S,f)}(X)$ is a union of triangles and rectangles.*

Proof. Let $a - x$ be an edge in $\Gamma_{(S,f)}(X)$. Since by assumption $\Gamma_{(S,f)}(X)$ is connected with $\operatorname{diam}(\Gamma_{(S,f)}(X)) \leq 3$ and possesses at least three vertices, there exists a vertex b such that either $a - x - b$ or $x - a - b$ is a path in $\Gamma_{(S,f)}(X)$ and in any case, by Lemma 4.1, $a - x$ is contained in a cycle of the length of at most 4 and, therefore, is an edge of either a triangle or a rectangle. □

Let us recall that the core of a graph Γ is the largest subgraph of Γ in which every edge is the edge of a cycle in Γ [16].

Theorem 4.3. *Let X be a non-empty set, S a commutative semigroup with zero, and f a function from X to S. Also, let the graph $\Gamma_{(S,f)}(X)$ have at least three vertices and the function f have this property that for all $x, y \in X$ if $f(x)f(y) \neq 0$ then there exists a $z \in X$ such that $f(z) = f(x)f(y)$. If $\Gamma_{(S,f)}(X)$ contains a cycle, then the core K of $\Gamma_{(S,f)}(X)$ is a union of triangles and rectangles.*

Proof. Let $a_1 \in K$ and suppose that a_1 is a part of neither a triangle nor a rectangle in $\Gamma_{(S,f)}(X)$. So, a_1 is a part of a cycle

$$C\colon a_1 - a_2 - a_3 - a_4 - \cdots - a_n - a_1,$$

where $n \geq 5$. Without loss of generality, we can suppose that this is the shortest cycle containing a_1 and it follows that $\{a_2, a_4\}$ is not an edge of the graph $\Gamma_{(S,f)}(X)$ and by the definition of the graph $\Gamma_{(S,f)}(X)$, $f(a_2) \cdot f(a_4) \neq 0$. So, by assumption, there exist a $z \in X$ such that $f(z) = f(a_2) \cdot f(a_4)$. Note that $f(a_1) \cdot f(a_2) = f(a_2) \cdot f(a_3) = 0$, so $f(a_1) \cdot f(z) = f(z) \cdot f(a_3) = 0$. Therefore, $a_1 - z - a_3$ is a path in $\Gamma_{(S,f)}(X)$. Since C is the shortest cycle of the graph $\Gamma_{(S,f)}(X)$ containing a_1, $z = a_2$ and we have $f(a_2) = f(a_2) \cdot f(a_4)$. Now consider $0 = f(a_2) \cdot ((fa_4) \cdot f(a_5)) = ((f(a_2) \cdot f(a_4)) \cdot f(a_5) = f(a_2) \cdot f(a_5) \neq 0$, a contradiction. This completes the proof. □

Remark 4.4. Note that Theorem 4.3 is related to Theorem 1.5 in [17].

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