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Tenacious Graph is NP-hard

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ABSTRACT

The tenacity of a graph G, $T(G)$, is defined by $T(G) = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph $G-S$, and $\omega(G-S)$ be the number of components of $G-S$. In this paper we consider the relationship between the minimum degree $\delta(G)$ of a graph and the complexity of recognizing if a graph is T-tenacious. Let $T \geq 1$ be a rational number. We first show that if $\delta(G) \geq \frac{Tn}{T+1}$, then G is T-tenacious. On the other hand, for any fixed $\epsilon > 0$, we show that it is NP-hard to determine if G is T-tenacious, even for the class of graphs with $\delta(G) \geq (\frac{T}{T+1} - \epsilon)n$.

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1 Introduction

The concept of tenacity of a graph G was introduced in [2,3], as a useful measure of the "vulnerability" of G. In [3] Cozzens et al. calculated tenacity of the first and second case of the Harary Graphs but they didn't show the complete proof of the third case. In [16] we showed a new and complete proof for case three of the Harary Graphs. In

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[10], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [1 - 38], the authors studied more about this new invariant. We consider only graphs without loops or multiple edges. We use $V(G)$, $\alpha(G)$, and $\omega(G)$ to denote the vertex set, independence number and number of components in a graph G, respectively. We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. We denote by $V(G)$, $E(G)$ and $\mid V(G) \mid$ the set of vertices, the set of edges and the order of G, respectively.
The tenacity of a graph G, $T(G)$, is defined by $T(G) = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph $G-S$, and $\omega(G-S)$ be the number of components of $G-S$. A connected graph G is called T-tenacious if $\mid S \mid + \tau(G-S) \geq T\omega(G-S)$ holds for any subset S of vertices of G with $\omega(G-S) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $S \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|S|+\tau(G-S)}{\omega(G-S)}$.
The Mix-tenacity $T_m(G)$ of a graph G is defined as

$$T_m(G) = \min_{A \subset E(G)} \{\frac{\mid A \mid + \tau(G-A)}{\omega(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of vertices) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$. Cozzens et al. in [2], called this parameter Edge-tenacity, but Moazzami changed the name of this parameter to Mix-tenacity in [14]. It seems Mix-tenacity is a better name for this parameter. $T(G)$ and $T_m(G)$ turn out to have interesting properties.
After the pioneering work of Cozzens, Moazzami, and Stueckle in [2,3], several groups of researchers have investigated tenacity, and its related problems. In [18] and [19] Piazza et al. used the $T_m(G)$ as Edge-tenacity. But this parameter is a combination of cutset $A \subset E(G)$ and the number of vertices of a largest component, $\tau(G-A)$. It may be observed that in the definition of $T_m(G)$, the number of edges removed is added to the number of vertices in a largest component of the remaining graph. Also this parameter didn't seem very satisfactory for Edge-tenacity. Thus Moazzami and Salehian introduced a new measure of vulnerability, the Edge-tenacity, $T_e(G)$, in [14]. The Edge-tenacity $T_e(G)$ of a graph G is defined as

$$T_e = \min_{A \subset E(G)} \{\frac{\mid A \mid + \tau(G-A)}{\omega(G-A)}\}$$

where $\tau(G-A)$ denotes the order (the number of edges) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$. This new measure of vulnerability involves edges only and thus is called the Edge-tenacity. Since 1992 there were several interesting questions. But the question " How difficult is it to recognize T-tenacious graphs?

" has remained an interesting open problem for some time. The question was first raised by Moazzami in [9]. Our purpose in [17] was to show that for any fixed positive rational number T, it is NP-hard to recognize T-tenacious graphs. To prove this we showed that it is NP-hard to recognize T-tenacious graphs by reducing a well-known NP-complete variant of INDEPENDENT SET.

2 Main Results

We begin by considering the following problem. Let $T \geq 1$ be any rational number.

Not T-TENACIOUS

INSTANCE: An undirected graph G.

QUESTION: Does there exist $X \subseteq V(G)$ with $\omega(G-X) > 1$ such that $T\omega(G-X) > |X| + m(G-X)$

Theorem 1. *Not T-TENACIOUS is NP-complete.*

To prove this, we will reduce the following problem, which is known [**?**] to be NP-complete for any fixed β, $0 < \beta < 1$.

INDEPENDENT β-MAJORITY

INSTANCE: An undirected graph G on n vertices.

QUESTION: Is $\alpha(G) \geq \beta n$?

Proof of theorem 1. We reduce INDEPENDENT β-MAJORITY to Not T-TENACIOUS. Let $T = \frac{a}{b} \geq 1$ for positive integers a and b, and fix β where $0 < \beta < 1$. Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and let $k = \lceil \beta n \rceil$. Construct G' from G as follow. First we add a set A includes n complete graphs $A_1, \ldots, A_n$ with

$$|V(A_i)| = h = \lceil Tn \rceil - n + k,\ i = 1 \ldots n,$$

to G and join v_i to any vertex in $A_i, 1 \leq i \leq n$. Then add another set C of br independent vertices to G, where $r > 2$ is an integer. Now add a set B of $ar - 2$ vertices which induces a complete graph, and join each vertex of B to every vertex of $V(G) \cup A \cup C$. It suffices to show that $\alpha(G) \geq k$ if and only if G' is not T-tenacious.

First suppose that G contains an independent set I with $|I| = k$. Define $X' \subseteq V(G')$ by $X' = (V(G) - I) \cup B$. Then

$$\omega(G' - X') = n + |C| = n + br$$

$$|X'| = n - k + |B| = n - k + ar - 2$$

$$m(G' - X') = h + 1 = \lceil Tn \rceil - n + k + 1$$

$$\downarrow$$

$$\begin{aligned} T\omega\left(G'-X'\right)=Tn+ar &> \left(\lceil Tn\rceil-1\right)+ar \\ &= \left(\lceil Tn\rceil-n+k+1\right)+\left(n-k+ar-2\right) \\ &= m\left(G'-X'\right)+\left|X'\right| \end{aligned}$$

Therefore G' is not T-tenacious.

Conversely, suppose G' is not T-tenacious. Then exists $X'\subseteq V\left(G'\right)$ with $\omega\left(G'-X'\right)>1$ such that $T\omega\left(G'-X'\right)>|X'|+m\left(G'-X'\right)$. Clearly $B\subseteq X'$.

Claim 1. $|X'|+m\left(G'-X'\right)\geq|X'-(A\cup C)|+m\left(G'-\left(X'-(A\cup C)\right)\right)$.

Proof. Suppose $X''=X'-(A\cup C)$ and $M\left(G'-X''\right)$ is a largest component of $G'-X''$. Then $M\left(G'-X''\right)-\left(X'-X''\right)$ is a component of $G'-X'$ and

$$\begin{aligned} m\left(G'-X'\right) &\geq \left|M\left(G'-X''\right)-\left(X'-X''\right)\right| \\ &\geq \left|M\left(G'-X''\right)\right|-\left|X'-X''\right| \\ &= m\left(G'-X''\right)-|X'|+|X''| \end{aligned}$$

$$\rightarrow |X'|+m\left(G'-X'\right)\geq|X''|+m\left(G'-X''\right)$$

□

We may also assume $X'\cap(A\cup C)=\phi$; otherwise

$$\begin{aligned} T\omega\left(G'-\left(X'-(A\cup C)\right)\right) &\geq \omega\left(G'-X'\right) \\ &> |X'|+m\left(G'-X'\right) \\ &\geq |X'-(A\cup C)|+m\left(G'-\left(X'-(A\cup C)\right)\right) \end{aligned}$$

And we could use $X'-(A\cup C)$ instead of X'.

Let

$$X=X'\cap V\left(G\right),\quad x=|X|,\quad x'=|X'|$$

$$m'=m\left(G'-X'\right),\quad w=\omega\left(G-X\right),\quad w'=\omega\left(G'-X'\right)$$

Then

$$\begin{aligned} x' &= x+|B|=x+ar-2 \\ w' &= w+x+|C|=w+x+br \\ m' &\geq h+1 \end{aligned}$$

Claim 2. $\lceil Tn\rceil-x-m'+1\geq0$.

Proof.

$$w' \leq n + |C| = n + br$$

$$x' + m' < Tw' \leq T(n + br) = Tn + ar$$

$$\begin{aligned}
&\to \quad Tn + ar - x' - m' > 0 \\
&\to \quad \lceil Tn + ar - x' - m' \rceil \geq 1 \\
&\to \quad \lceil Tn \rceil + ar - x' - m' - 1 \geq 0 \\
&\to \quad \lceil Tn \rceil - x - m' + 1 \geq 0
\end{aligned}$$

□

$$Tw' > x' + m'$$

$$\to \quad Tw + Tx + ar > x + ar - 2 + m'$$

$$\downarrow$$

$$\begin{aligned}
Tw &> x - Tx + m' - 2 \\
&= (T-1)(\lceil Tn \rceil - x - m' + 1) - (T-1)(\lceil Tn \rceil - m' + 1) + m' - 2 \\
&\geq -(T-1)(\lceil Tn \rceil - m' + 1) + m' - 2 \\
&= Tm' - (T-1)\lceil Tn \rceil - T - 1 \\
&\geq T(h+1) - (T-1)\lceil Tn \rceil - T - 1 \\
&= T(\lceil Tn \rceil - n + k + 1) - (T-1)\lceil Tn \rceil - T - 1 \\
&= \lceil Tn \rceil - Tn + Tk - 1 \\
&\geq Tk - 1 \\
\to & \\
w &> k - \frac{1}{T} \\
w &\geq k
\end{aligned}$$

Since it is possible to form an independent set in G by choosing one vertex from each component of $G - X$, we conclude $\alpha(G) \geq k$.

□

Define $\Omega(r)$ to be the class of all graphs with $\delta(G) \geq rn$, where $n = |V(G)|$. We can prove the following two results for any rational number $T \geq 1$.

Theorem 2. *Let G be a graph in $\Omega\left(\frac{T}{T+1}\right)$. Then G is T-tenacious.*

Theorem 3. *For any fixed $\varepsilon > 0$ it is NP-hard to recognize T-tenacious graphs in $\Omega\left(\frac{T}{T+1} - \varepsilon\right)$.*

A decision problem C is NP-complete if C is in NP, and every problem in NP is reducible to C in polynomial time. When a decision version of a combinatorial optimization problem is proved to belong to the class of NP-complete problems, then the optimization version is NP-hard. By theorems 1 and 2 we proved that it is NP-complete to solve decision problem of T-tenacious graphs for any fixed positive rational number T, and therefore finding tenacity of a graph is NP-hard.

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