Edge-Tenacity

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ABSTRACT

The edge-tenacity $T_e(G)$ of a graph G was defined as

$$T_e(G) = \min_{F \subset E(G)} \left\{ \frac{\mid F \mid + \tau(G - F)}{\omega(G - F)} \right\}$$

where the minimum is taken over all edge cutset F of G. We define G-F to be the graph induced by the edges of $E(G) - F$, $\tau(G - F)$ is the number of edges in the largest component of the graph induced by G-F and $\omega(G - F)$ is the number of components of $G - F$. A set $F \subset E(G)$ is said to be a T_e-set of G if

$$T_e(G) = \frac{|F| + \tau(G - F)}{\omega(G - F)}$$

Each component has at least one edge. In this paper we introduce a new invariant edge-tenacity, for graphs. it is another vulnerability measure. we present several properties and bounds on the edge-tenacity. we also compute the edge-tenacity of some classes of graphs.

Keyword: edge-tenacity, cutset, Mix-tenacity, Edge-Integrity, Vulnerability.

AMS subject Classification: 05C78.

ARTICLE INFO

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1   Introduction

The concept of graph tenacity was introduced by Cozzens, Moazzami and Stueckle in [1,2], as a measure of network vulnerability and reliability. Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed. the motivation for studying vulnerability measures is derived from design and analysis of networks under hostile environment. Graph tenacity has been an active area of research since the the concept was introduced in 1992. Cozzens et al. introduced two measures of network vulnerability termed the tenacity,T(G), and the Mix-tenacity, $T_m(G)$, of a graph. The tenacity T(G) of a graph G is defined as

$$T(G) = \min_{A \subset V(G)} \{\frac{\mid A \mid + \tau(G - A)}{\omega(G - A)}\}$$

where $\tau(G - A)$ denotes the order (the number of vertices) of a largest component of G-A and $\omega(G - A)$ is the number of components of G-A.
The Mix-tenacity T_m of a graph G is defined as

$$T_m = \min_{A \subset E(G)} \{\frac{\mid A \mid + \tau(G - A)}{\omega(G - A)}\}$$

where $\tau(G - A)$ denotes the order (the number of vertices) of a largest component of G-A and $\omega(G - A)$ is the number of components of G-A.
we called this parameter Mix-tenacity. It seems Mix-tenacity is a better name for this parameter. T(G) and $T_m(G)$ turn out to have interesting properties. After the pioneering work of Cozzens, Moazzami, and Stueckle several groups of researchers have investigated tenacity, and related problems.
In [31] and [32] Piazza et al. used the above parameter as edge-tenacity. But this parameter is a combination of cutset $A \subset E(G)$ and $\tau(G - A)$ the number of vertices of a largest component. It may be observed that in the definition of T_m, the number of edges removed is added to the number of vertices in a largest component of the remaining graph. This may not seem very satisfactory. This motivated the authors to introduce a new measure of vulnerability in this paper. This new measure of vulnerability involves edges only and thus is called the Edge-Integrity.
The concept of tenacity of a graph G was introduced in [1,2], as a useful measure of the "vulnerability" of G. In [6], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [3-33], they studied more about this new invariant.
In this paper we present results on edge-tenacity, $T_e(G)$ a new invariant, for several classes of graphs. In Section 2 basic properties and some bounds for edge-tenacity, $T_e(G)$, are developed. Edge-tenacity values for various classes of graphs are developed in Section 3. Future work and concluding remarks are summarized in Section 4.

2   Basic Properties and Bounds

In this section we develop basic properties and bounds related to edge-tenacity.
Proposition 1 Let G be a graph, and H, a subgraph of G. Then

$$T_e(H) \leq T_e(G)$$

Proof. Let H be a subgraph of G, and F is a minimum subset of E(G) which achieves $T_e(G)$ then $\omega(H - F) \geq \omega(G - F)$. Thus $\frac{1}{\omega(H-F)} \leq \frac{1}{\omega(G-F)}$. Also we have

$$\mid F \mid +\tau(H - F) \leq \mid F \mid +\tau(G - F)$$

Thus

$$\frac{\mid F \mid +\tau(H - F)}{\omega(H - F)} \leq \frac{\mid F \mid +\tau(G - F)}{\omega(G - F)}$$

Therefore

$$T_e(H) \leq T_e(G)$$

our next result, besides being of interest in its own right is also found to be useful in several proofs later.
Theorem 2.1 If F is a minimum subset of E(G) which achieves $T_e(G)$, then every edge of F connects vertices from different components of G-F.
Proof. Suppose $e = uv$ is an edge in F. If u and v lie in the same component of G-F, then if $F_1 = F - e$, we have $\omega(G - F_1) = \omega(G - F)$ and

$$\begin{aligned}
T_e(G) \quad &\leq \frac{|F_1|+\tau(G-F_1)}{\omega(G-F_1)} \\
&= \frac{|F_1|+\tau(G-F_1)}{\omega(G-F)} \\
&= \frac{|F|-1+\tau(G-F_1)}{\omega(G-F)} \\
&\leq \frac{|F|-1+\tau(G-F)+1}{\omega(G-F)} \\
&= T_e(G)
\end{aligned}$$

This shows that F_1 achieves $T_e(G)$ which is a contradiction since F_1 is a proper subset of F.
Theorem 2.2 Suppose that $\lambda = \lambda(G) \geq 1$ the edge-connectivity of G and $e = \mid E(G) \mid$. Then

$$T_e(G) \geq min(e, \lceil \frac{\lambda}{16e}(8e - \lambda) \rceil)$$

Proof. Suppose that $F \subset E(G)$ is such that G-F has k components. Then $\tau(G - F) \geq \frac{e-|F|}{k}$. For $k = 1$ $\tau(G - F)+ \mid F \mid \geq e- \mid F \mid + \mid F \mid = e$. if $k > 1$, then $\mid F \mid \geq \frac{k\lambda}{2}$, so that

$$\frac{\tau(G - F)+ \mid F \mid}{k} \geq \frac{\frac{e-|F|}{k}+ \mid F \mid}{k} \geq \frac{1}{k}(\frac{e}{k} + \frac{(k - 1)\lambda}{2})$$

The quantity on the right hand side of this inequality is minimized for $k = \frac{4e}{\lambda}$. Hence the result follows.

3   Edge-Tenacity of some Classes of Graphs

In this section we will give results for edge-tenacity of complete graphs and some other classes of graphs. First observe that for Line graph, $L(C_n) = C_n$. Therefore, the values of edge-tenacity for cycles can be derived from those of tenacity of respective graphs [5]. Thus the theorem below follows.

Theorem 3.1 Let C_n be the n-cycle, then

$$T_e(C_n) = \begin{cases} 1 + \frac{2}{n} & \text{if } n \equiv 0 \mod 2 \\ 1 + \frac{4}{n-1} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

It may be noted that if G is a connected graph of order n then $L(G) = G$ if and only if $G = C_n$. This fact leads to the following observation.

Observation 3.1 Let G be a graph of order n. If G is isomorphic to C_n then $T_e(G) = T(G)$.

This raises to the following question: For what graphs G is $T_e(G) = T(G)$?

Next, we develop a formula for the edge-tenacity of a complete graph K_n. It may be noted that the tenacity of a complete graph is easy to establish, but the formula for edge-tenacity is somewhat more involved.

Theorem 3.2 For all $n \geq 1$

$$T_e(K_n) = \frac{\lfloor \frac{n^2}{4} \rfloor + \binom{\lceil \frac{n}{2} \rceil}{2}}{2}$$

Proof. We first introduce a convenient notation. For a partition $n = k_1 + k_2 + \cdots + k_r$ of n where $1 \leq k_1 \leq k_2 \leq \ldots \leq k_r$, we shall denote by $G(k_1, k_2, \ldots, k_r)$ the graph $K_{k_1} \cup \ldots \cup K_{k_r}$. Now let F be a minimum set achieving $T_e(K_n)$. We can see that all the components of $K_n - F$ are complete. Thus there exists a partition $n = k_1 + k_2 + \cdots + k_r$ of n such that $K_n - F = G(k_1, k_2, \ldots, k_r)$. Then

$$T_e(K_n) = \frac{|F| + \binom{k_r}{2}}{r}$$
$$= \frac{\sum_{i \neq j} k_i k_j + \binom{k_r}{2}}{r}$$

We first claim that the sequence $k_1, k_2, \ldots, k_r$ has at most two distinct values. For, if not, then let i be such that $k_1 < k_i < k_r$. Then by considering a set $F' \subset E(K_n)$ such that $G - F' = G(k_1 - 1, k_2, \ldots, k_{i-1}, k_i, k_i + 1, \ldots, k_r)$ we have

$$| F' | = \sum_{j \neq 1, i} (k_1 - 1)k_j + \sum_{j \neq 1, i} (k_1 + 1)k_j +$$
$$(k_1 - 1)(k_1 - 1) + \sum_{\substack{j \neq l \\ j, l \neq 1, i}} k_j k_l$$

while $\tau(G - F') = \begin{pmatrix} k_r \\ 2 \end{pmatrix}$. Hence, upon simplification, we get

$$\frac{|F'| + \tau(G-F')}{k} = \frac{\sum\limits_{j \neq l} k_j k_l + \begin{pmatrix} k_r \\ 2 \end{pmatrix} + (k_1 - k_i)}{k} < T_e(K_n),$$

a contradiction. This proves the claim. We now consider the following case depending on whether the sequence $k_1, k_2, \ldots, k_r$ takes two distinct values.

Suppose that $K_n - F = G(a, \ldots, a, b, \ldots, b)$ where there are α a's and β b's, with $a < b$. Suppose that $\alpha > 1$. Then by considering $G(a - 1, a, \ldots, a, a + 1, b, \ldots, b)$ we get a contradiction as before. Hence $\alpha = 1$. Similarly $\beta = 1$. Thus $K_n - F = G(a, b)$ with $a + b = n$. Hence

$$T_e(K_n) = \frac{ab + \begin{pmatrix} b \\ 2 \end{pmatrix}}{2}$$

If $a + 1 \leq b - 1$, then we can get a contradiction as before by considering $G(a + 1, b - 1)$. Thus $K_n - F = G(a, a + 1)$ so that $n = 2a + 1$, and we are done. If $a = b$, we have the same result.

The converse of Theorem 3.2 holds.

Theorem 3.3 If G is of order n and

$$T_e(K_n) = \frac{\lfloor \frac{n^2}{4} \rfloor + \begin{pmatrix} \lceil \frac{n}{2} \rceil \\ 2 \end{pmatrix}}{2}$$

then $G = K_n$.

Proof. Suppose $G \neq K_n$. Let e be an edge of K_n such that $e \notin E(G)$. Let F be a set of $E(K_n)$ such that $\frac{|F| + \tau(K_n - F)}{\omega(K_n - F)} = T_e(K_n)$. Without loss of generality we assume that $e \in F$. Then, by letting $F_1 = F \cap E(G) - \{e\}$, we have

$$\begin{aligned} T_e(G) &\leq \frac{|F_1| + \tau(G - F_1)}{\omega(G - F_1)} \\ &< \frac{|F| + \tau(K_n - F)}{\omega(K_n - F)} \\ &= T_e(K_n) \\ &= \frac{\lfloor \frac{n^2}{4} \rfloor + \begin{pmatrix} \lceil \frac{n}{2} \rceil \\ 2 \end{pmatrix}}{2} \end{aligned}$$

This proves the theorem.

4   Conclusion

In this paper we have investigated the edge-tenacity of graphs. The relationship between tenacity and edge-tenacity is explored. Edge-tenacity values for various classes of graphs are developed. Work in progress related to this paper includes characterization of graphs

for edge-tenacity and tenacity values are the same, and finding a formula for the edge-tenacity of hypercube, complete bipartite graph, and certain other classes of graphs.

Acknowledgement

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