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Linear optimization constrained by fuzzy inequalities defined by Max-Min averaging operator

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ABSTRACT

In this paper, optimization of a linear objective function with fuzzy relational inequality constraints is investigated whereby the feasible region is formed as the intersection of two inequality fuzzy systems and "Fuzzy Max-Min" averaging operator is considered as fuzzy composition. It is shown that a lower bound is always attainable for the optimal objective value. Also, it is proved that the optimal solution of the problem is always resulted from the unique maximum solution and a minimal solution of the feasible region. An algorithm is presented to solve the problem and an example is described to illustrate the algorithm.

Keyword: Fuzzy relation, fuzzy relational inequality, linear programming, fuzzy compositions and fuzzy averaging operator graph.

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1 Introduction

In this paper, we study the following linear problem in which the constraints are formed as the intersection of two fuzzy systems of relational inequalities defined by "Fuzzy Max-Min" averaging operator:

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$$\begin{aligned} \min\ & Z = c^T x \\ & A \diamond x \leq b^1 \qquad (1) \\ & D \diamond x \geq b^2 \\ & x \in [0,1]^n \end{aligned}$$

Where $I_1 = \{1, 2, \ldots, m_1\}, I_2 = \{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\}$ and $J = 1, 2, \ldots, n$. $A = (a_{ij})_{m_1 \times n}$ and $D = (d_{ij})_{m_2 \times n}$ are fuzzy matrices such that $0 \leq a_{ij} \leq 1 (\forall i \in I_1$ and $\forall j \in J)$ and $0 \leq d_{ij} \leq 1$ ($\forall i \in I_2$ and $\forall j \in J$). $b^1 = (b_i^1)_{m_1 \times 1}$ is an m_1–dimensional fuzzy vector in $[0,1]^{m_1}$ (i.e., $0 \leq b_i^1 \leq 1, \forall i \in I_1$) and $b^2 = (b_i^2)_{m_2 \times 1}$ is an m_2–dimensional fuzzy vector in $[0,1]^{m_2}$ (i.e., $0 \leq b_i^2 \leq 1, \forall i \in I_2$), and c is a vector in $\mathbb{R}^n$. Moreover,"$\diamond$" is the $\max - \diamond$ composition where $\diamond$ is "Fuzzy Max-Min" Operator, that is,

$$\diamond(x, y) = \lambda \min\{x, y\} + (1 - \lambda) \max\{x, y\}$$

in which $\lambda \in [0, 1]$. Furthermore, let $S(A, b^1)$ and $S(D, b^2)$ denote the feasible solutions sets of inequalities type1 $A \diamond x \leq b^1$ and type2 $D \diamond x \geq b^2$,respectively, that is, $S(A, b^1) = \{x \in [0, 1]^n \colon A \diamond x \leq b^1\}$ and $S(D, b^2) = \{x \in [0, 1]^n \colon D \diamond x \geq b^2\}$. Also, let $S(A, D, b^1, b^2)$ denote the feasible solutions set of problem (1). Based on the foregoing notations, it is clear that $S(A, D, b^1, b^2) = S(A, b^1) \bigcap S(D, b^2)$.

By these notations, problem (1) can be also expressed as follows:

$$\begin{aligned} \min\ & Z = c^T x \\ & \max_{j \in J} \{\diamond(a_{ij}, x_j)\} \leq b_i^1, i \in I_1 \\ & \max_{j \in J} \{\diamond(d_{ij}, x_j)\} \geq b_i^2, i \in I_2 \qquad (2) \\ & x \in [0,1]^n \end{aligned}$$

Especially, by setting $A = D$ and $b^1 = b^2$, the above problem is converted to max–"Fuzzy Max-Min "fuzzy relational equations.

The theory of fuzzy relational equations (FRE) was firstly proposed by Sanchez and applied in problems of the medical diagnosis [55]. Nowadays, it is well known that many issues associated with a body knowledge can be treated as FRE problems [51]. In addition to the preceding applications, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [41,49].

The solvability determination and the finding of solutions set are the primary (and the most fundamental) subject concerning with FRE problems. Actually, The solution set of FRE is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [5]. This non-convexity property is one of two bottlenecks making major contribution to the increase of complexity in problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. The other bottleneck is concerned with detecting the minimal solutions for FREs [2]. Markovskii showed that solving max-product FRE is closely related to the covering problem which is an NP-hard problem [48]. In fact, the same result holds true for a more general t-norms instead of the minimum and product operators $[2, 3, 12, 13, 22 - 30, 44, 45, 48]$.

Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers $[22-30, 50, 52, 53, 56, 58, 59, 61, 64, 67]$. Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI)$[12, 13, 15-20, 21, 32, 33, 42, 66]$. Li and Yang [42] studied a FRI with addition-min composition and presented an algorithm to search for minimal solutions. Ghodousian et al. [13] focused on the algebraic structure of two fuzzy relational inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$, and studied a mixed fuzzy system formed by the two preceding FRIs, where φ is an operator with (closed) convex solutions.

The problem of optimization subject to FRE and FRI is one of the most interesting and on-going research topic among the problems related to FRE and FRI theory $[1, 8, 11 - 30, 39, 43, 46, 54, 57, 62, 66]$. Fang and Li [9] converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by branch and bound method using jump-tracking technique. In [39] an application of optimizing the linear objective with max-min composition was employed for the streaming media provider. Wu et al. [60] improved the method used by Fang and Li, by decreasing the search domain. The topic of the linear optimization problem was also investigated with max-product operation [11,35,47]. Loetamonphong and Fang defined two sub-problems by separating negative and non-negative coefficients in the objective function and then obtained the optimal solution by combining those of the two sub-problems [47]. Also, in [35] and [11] some necessary conditions of the feasibility and simplification techniques were presented for solving FRE with max-product composition. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [14,38,62], max-Discontinuous t-norms composition [29], max-monotone operators composition [30] and max-t-norm composition $[15 - 20, 22 - 28, 36, 43, 57]$.

Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas $[6, 10, 22 - 28, 33, 40, 46, 63]$. For example, Dempe and Ruziyeva [4] generalized the fuzzy linear optimization problem by considering fuzzy coefficients.

The optimization problem subjected to various versions of FRI could be found in the literature as well $[12, 13, 15-21, 29-33, 65, 66]$. Xiao et al. [66] introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities. Ghodousian et al. [12] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition.
The remainder of the paper is organized as follows. Section 2 takes a brief look at some basic results on the feasible region of Problem (1). These results provide a proper background to design an algorithm for solving the problem. In section 3, Problem (1) is resolved by optimization of the linear objective function considered in section 2. In addition, the existence of an optimal solution is proved if problem (1) is not empty. The preceding results are summarized as an algorithm and, finally in section 4 some numerical examples are described to illustrate.

2 Feasible solutions set of Problem (1)

This section describes the basic definitions and structural properties concerning the intersection of two systems $A \diamond x \le b^1$ and $D \diamond x \ge b^2$. the interesting reader is referred to [31] for the proofs of the lemmas, theorems and corollaries.
Let $S(a_{ij}, b_i^1) = \{x_j \in [0,1]\colon \diamond(a_{ij}, x_j) \le b_i^1\}$, $\forall i \in I_1$ and $\forall i \in J$. Also, define $S(a_i, b_i^1) = \{x \in [0,1]^n\colon \max_{j\in J}\{\diamond(a_{ij}, x_j)\} \le b_i^1\}$, $\forall i \in I_1$. The following lemma determines set $S(a_i, b_i^1)$, $\forall i \in I_1$, where $\underline{W}_{ij}(\lambda) = (b_i^1 - (1-\lambda)a_{ij})/\lambda$.

Lemma 1. For each $i \in I_1$ and each $j \in J$,

$$S(a_{ij}, b_i^1) = \begin{cases} [0, \min\{\underline{W}_{ij}(1-\lambda), 1\}] & , a_{ij} \le b_i^1, 0 \le \lambda < 1 \\ [0,1] & , a_{ij} \le b_i^1, \lambda = 1 \\ [0, \underline{W}_{ij}(\lambda)] & , a_{ij} > b_i^1, (b_i^1 - a_{ij})/(-a_{ij}) \le \lambda \le 1 \\ \varnothing & , a_{ij} > b_i^1, 0 \le \lambda < (b_i^1 - a_{ij})/(-a_{ij}) \end{cases}$$

By the following lemma, the shape of set $S(a_i, b_i^1)$ is attained.

Lemma 2. Suppose that $S(a_i, b_i^1) \ne \varnothing$. Then $S(a_i, b_i^1) = [0, \overline{X}(i)], \forall i \in I_1$, Where, $\overline{X}(i) = [\overline{X}(i)_1, \overline{X}(i)_2, \ldots, \overline{X}(i)_n]$ and

$$\overline{X}(i)_j = \begin{cases} \min\{\underline{W}_{ij}(1-\lambda), 1\} & , a_{ij} \le b_i^1, 0 \le \lambda < 1 \\ 1 & , a_{ij} \le b_i^1, \lambda = 1 \\ \underline{W}_{ij}(\lambda) & , a_{ij} > b_i^1, (b_i^1 - a_{ij})/(-a_{ij}) \le \lambda \le 1 \end{cases}$$

The following theorem shows that set $S(A, b^1)$ is actually a closed convex cell.

Theorem 1. Let $\overline{X} = \min_{i \in I_1} \overline{X}(i)$ and Suppose that $S(a_i, b_i^1) \neq \varnothing, \forall i \in I_1$. Then, $S(A, b^1) = [0, \overline{X}]$.

Remark 1. $S(A, b^1) \neq \varnothing$ iff $0 \in S(A, b^1)$.
Let $S(d_{ij}, b_i^2) = \{x_j \in [0,1] \colon \diamond(d_{ij}, x_j) \geq b_i^2\}$, $\forall i \in I_2$ and $\forall j \in J$.Also, define $S(d_i, b_i^2) = \{x \in [0,1]^n \colon \max_{j \in J}\{\diamond(d_{ij}, x_j)\} \geq b_i^2\}$, $\forall i \in I_2$. The following lemma determines set $S(a_i, b_i^1)$, $\forall i \in I_2$, where $\underline{W}_{ij}(\lambda) = (b_i^2 - (1-\lambda)d_{ij})/\lambda$.

Lemma 3. For each $i \in I_2$ and each $j \in J$,

$$S(d_{ij}, b_i^2) = \begin{cases} [\max\{0, \overline{W}_{ij}(\lambda)\}, 1] & , d_{ij} \geq b_i^2, 0 < \lambda \leq 1 \\ [0,1] & , d_{ij} \geq b_i^2, \lambda = 0 \\ [\overline{W}_{ij}(1-\lambda), 1] & , d_{ij} < b_i^2, 0 \leq \lambda \leq (b_i^2 - 1)/(d_{ij} - 1) \\ \varnothing & , d_{ij} < b_i^2, \lambda > (b_i^2 - 1)/(d_{ij} - 1) \end{cases}$$

By the following lemma, the shape of set $S(d_i, b_i^2)$ is attained.

Lemma 4. Suppose that $S(d_i, b_i^2) \neq \varnothing$. Then, $S(d_i, b_i^2) = \bigcup_{j \in J_1 \cup J_2} [\underline{X}(i,j), 1], \forall i \in I_2$, where $J_1 = \{j \in J \colon d_{ij} \geq b_i^2, \lambda > 0\}$, $J_2 = \{j \in J \colon d_{ij} \geq b_i^2, \lambda = 0\}$, $J_3 = \{j \in J \colon d_{ij} < b_i^2, \lambda \leq (b_i^2 - 1)/(d_{ij} - 1)\}$ and $\underline{X}(i,j) = [\underline{X}(i,j)_1, \underline{X}(i,j)_2, \ldots, \underline{X}(i,j)_n]$ such that

$$\underline{X}(i,j)_k = \begin{cases} \max\{0, max\{0, \overline{W}_{ij}(\lambda)\}\} & , k = j, j \in J_1 \\ 0 & , k = j, j \in J_2 \\ \overline{W}_{ij}(1-\lambda) & , k = j, j \in J_3 \\ 0 & , otherwise \end{cases}$$

The following theorem shows that set $S(d_i, b_i^2)$ is the union of the finite number of closed convex cells.

Theorem 6. Suppose that $S(d_i, b_i^2) \neq \varnothing, \forall i \in I_2$. Then, $S(D, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), 1]$, where $e \colon I_2 \to J_1 \cup J_2 \cup J_3$ and $\underline{X}(e) = [\underline{X}(e)_1, \underline{X}(e)_2, \ldots, \underline{X}(e)_n]$, such that $\underline{X}(e)_j = \max_{i \in I_2}\{\underline{X}(i, e(i))_j\} = \max_{i \in I_2}\{\underline{X}(i, j_i)_j\}, \forall j \in J$.

Remark 2. $S(D, b^2) \neq \varnothing$ iff $1 \in S(D, b^2)$.
The following theorem characterizes the feasible region of Problem (1).

Theorem 3. suppose that $S(A, D, b^1, b^2) \neq \varnothing$.Then $S(A, D, b^1, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), \overline{X}]$

Remark 3. Assume that $S(A, b^1) \neq \varnothing$ and $S(D, b^2) \neq \varnothing$. Then, Since $S(A, D, b^1, b^2) \neq \varnothing$ iff $\overline{X} \in S(D, b^2)$.

3 Optimization of the linear objective function

According to the well-known schemes used for optimization of linear problems such as (1) [9, 13, 15 − 20, 33, 43], problem (1) is converted to the following two sub-problems:

$$\begin{aligned} \min\ Z_1 = &\sum_{j=1}^{n} c_j^{+} x_j \\ &A \diamond x \leq b^1 \qquad (3) \\ &D \diamond x \geq b^2 \\ &x \in [0,1]^n \\ \textit{and}& \\ \min\ Z_2 = &\sum_{j=1}^{n} c_j^{-} x_j \\ &A \diamond x \leq b^1 \qquad (4) \\ &D \diamond x \geq b^2 \\ &x \in [0,1]^n \end{aligned}$$

Where $c_j^{+} = \max\{c_j, 0\}$ and $c_j^{-} = \min\{c_j, 0\}$ for $j = 1, 2, \ldots, n$. It is easy to prove that $\overline{X}$ is the optimal solution of (4), and the optimal solution of (3) is $\underline{X}(e')$ for some $e' \in E_D$.

Theorem 4. Suppose that $S(A, D, b^1, b^2) \neq \varnothing$, and $\overline{X}$ and $\underline{X}(e^*)$ are the optimal solutions of sub-problems (4) and (3), respectively. Then $c^T x$ is the lower bound of the optimal objective function in (1), where $x^* = [x_1^*, x_2^*, \ldots, x_n^*]$ is defined as follows:

$$x_j^* = \begin{cases} \overline{X}_j & , c_j < 0 \\ \underline{X}(e^*)_j & , c_j \geq 0 \end{cases} \qquad (5)$$

for $j = 1, 2, \ldots, n$.
Proof. Let $x \in S(A, D, b^1, b^2)$. Then, from Theorem 3 we have $x \in \bigcup\limits_{e \in E_D} [\underline{X}(e), \overline{X}]$.
Therefore, for each $j \in J$ such that $c_j \geq 0$, inequality $x_j^* \leq x_j$ implies $c_j^{+} x_j^* \leq c_j^{+} x_j$. In addition, for each $j \in J$ such that $c_j < 0$, inequality $x_j^* \geq x_j$ implies $c_j^{-} x_j^* \leq c_j^{-} x_j$.
Hence, $\sum\limits_{j=1}^{n} c_j x_j^* \leq \sum\limits_{j=1}^{n} c_j x_j$.□

Corollary 1. Suppose that $S(A, D, b^1, b^2) \neq \varnothing$. Then, $x^* = [x_1^*, x_2^*, \ldots, x_n^*]$ as defined in (5), is the optimal solution of problem (1).
Proof. According to the definition of vector x^*, we have $\underline{X}(e^*)_j \leq x_j^* \leq \overline{X}_j$, $\forall j \in J$, which implies $x^* \in \bigcup\limits_{e \in E_D} [\underline{X}(e), \overline{X}] = S_{T_F^s}(A, D, b^1, b^2)$.□

We now summarize the preceding discussion as an algorithm. Also, see [31] for the algorithm finding the feasible region of Problem (1).

Algorithm 1 (optimization of problem (1))

Given problem (1):

1. If $0 \notin S(A, b^1)$, then stop; $S(A, b^1)$ is infeasible (Remark 1).
2. If $1 \notin S(D, b^2)$, then stop; $S(D, b^2)$ is infeasible (Remark 2).
3. If $\overline{X} \notin S(D, b^2)$, then stop; $S(A, D, b^1, b^2)$ is infeasible (Remark 3).
4. Find the optimal solution $\underline{X}(e^*)$ for the sub-problem (3) by considering vectors $\underline{X}(e), \forall e \in E_D$.
5. Find the optimal solution $x^* = [x_1^*, x_2^*, \ldots, x_n^*]$ for the problem (1) by (5) (Theorem 4).

4 Numerical example

Example 1. Consider the following linear optimization problem (1):

$$\min Z = 5.8441x_1 + 9.1898x_2 + 3.1148x_3$$
$$\begin{bmatrix} 0.8147 & 0.9134 & 0.2785 \\ 0.9058 & 0.6324 & 0.5469 \\ 0.1270 & 0.0975 & 0.9575 \end{bmatrix} \Delta x \le \begin{bmatrix} 0.9152 \\ 0.9901 \\ 0.9873 \end{bmatrix}$$

$$\begin{bmatrix} 0.9649 & 0.9572 & 0.1419 \\ 0.1576 & 0.4854 & 0.4218 \\ 0.9706 & 0.8003 & 0.9157 \end{bmatrix} \Delta x \ge \begin{bmatrix} 0.1023 \\ 0.0567 \\ 0.6324 \end{bmatrix}$$
$$x \in [0,1]^n$$

Step1: Since $0 \in S(A, b^1)$, set $S(A, b^1)$ is feasible:

$$\begin{bmatrix} 0.8147 & 0.9134 & 0.2785 \\ 0.9058 & 0.6324 & 0.5469 \\ 0.1270 & 0.0975 & 0.9575 \end{bmatrix} \Delta \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4567 \\ 0.4529 \\ 0.4788 \end{bmatrix} \le \begin{bmatrix} 0.9152 \\ 0.9901 \\ 0.9873 \end{bmatrix}$$

Step2: Since $1 \in S(D, b^2)$, set $S(D, b^2)$ is feasible:

$$\begin{bmatrix} 0.9649 & 0.9572 & 0.1419 \\ 0.1576 & 0.4854 & 0.4218 \\ 0.9706 & 0.8003 & 0.9157 \end{bmatrix} \Delta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9825 \\ 0.7427 \\ 0.9853 \end{bmatrix} \ge \begin{bmatrix} 0.1023 \\ 0.0567 \\ 0.6324 \end{bmatrix}$$

Step3: From Lemma 2 and Theorem 1, $\overline{X} = [1.0000 \quad 0.9170 \quad 1.0000]$. Since $\overline{X} \in S(D, b^2)$, set $S(A, D, b^1, b^2)$ is feasible:

$$\begin{bmatrix} 0.9649 & 0.9572 & 0.1419 \\ 0.1576 & 0.4854 & 0.4218 \\ 0.9706 & 0.8003 & 0.9157 \end{bmatrix} \Delta \begin{bmatrix} 1.0000 \\ 0.9170 \\ 1.0000 \end{bmatrix} = \begin{bmatrix} 0.9825 \\ 0.7109 \\ 0.9853 \end{bmatrix} \geq \begin{bmatrix} 0.1023 \\ 0.0567 \\ 0.6324 \end{bmatrix}$$

Step4: For this example, there are 27 feasible vectors $\underline{X}(e)$ (i.e., $\underline{X}(e) \leq \overline{X}$). Minimal solutions are as follows:

$$\begin{aligned} e_1 &= [1 \quad 1 \quad 3] \implies \underline{X}(e_1) = [0 \quad 0 \quad 0.3491] \\ e_2 &= [1 \quad 1 \quad 1] \implies \underline{X}(e_2) = [0.2942 \quad 0 \quad 0] \\ e_3 &= [1 \quad 1 \quad 2] \implies \underline{X}(e_3) = [0 \quad 0.4645 \quad 0] \end{aligned}$$

Vector $\underline{X}(e^*) = [0 \quad 0 \quad 0.3491]$ is the optimal solution of the sub-problem (3) that is obtained by $e^* = e_1$.

Step5: The optimal solution of Problem (1) is resulted as $x^* = [0 \quad 0 \quad 0.3491]$ with optimal objective value $Z^* = 1.0874$.

Example 2. Consider the following linear optimization problem (1):

$$\begin{aligned} &\min Z = -1.2251x_1 - 2.3688x_2 + 5.3103x_3 + 5.9040x_4 \\ &\begin{bmatrix} 0.8147 & 0.6324 & 0.9575 & 0.9572 \\ 0.9058 & 0.0975 & 0.9649 & 0.4854 \\ 0.1270 & 0.2785 & 0.1576 & 0.8003 \\ 0.9134 & 0.5469 & 0.9706 & 0.1419 \end{bmatrix} \Delta x \leq \begin{bmatrix} 0.9701 \\ 0.9901 \\ 0.8800 \\ 0.9888 \end{bmatrix} \\ &\begin{bmatrix} 0.4218 & 0.6557 & 0.6787 & 0.6555 \\ 0.9157 & 0.0357 & 0.7577 & 0.1712 \\ 0.7922 & 0.8491 & 0.7431 & 0.7060 \\ 0.9595 & 0.9340 & 0.3922 & 0.0318 \end{bmatrix} \Delta x \geq \begin{bmatrix} 0.3232 \\ 0.0087 \\ 0.5656 \\ 0.01672 \end{bmatrix} \\ &x \in [0, 1]^n \end{aligned}$$

Step1: Since $0 \in S(A, b^1)$, set $S(A, b^1))$ is feasible:

$$\begin{bmatrix} 0.8147 & 0.6324 & 0.9575 & 0.9572 \\ 0.9058 & 0.0975 & 0.9649 & 0.4854 \\ 0.1270 & 0.2785 & 0.1576 & 0.8003 \\ 0.9134 & 0.5469 & 0.9706 & 0.1419 \end{bmatrix} \Delta \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4788 \\ 0.4824 \\ 0.4002 \\ 0.4853 \end{bmatrix} \leq \begin{bmatrix} 0.9701 \\ 0.9901 \\ 0.8800 \\ 0.9888 \end{bmatrix}$$

Step2: Since $1 \in S(D, b^2)$, set $S(D, b^2)$ is feasible:

$$\begin{bmatrix} 0.4218 & 0.6557 & 0.6787 & 0.6555 \\ 0.9157 & 0.0357 & 0.7577 & 0.1712 \\ 0.7922 & 0.8491 & 0.7431 & 0.7060 \\ 0.9595 & 0.9340 & 0.3922 & 0.0318 \end{bmatrix} \Delta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8394 \\ 0.9578 \\ 0.9245 \\ 0.9798 \end{bmatrix} \geq \begin{bmatrix} 0.3232 \\ 0.0087 \\ 0.5656 \\ 0.01672 \end{bmatrix}$$

Step3: From Lemma 2 and Theorem 1, $\overline{X} = [1.0000 \quad 1.0000 \quad 0.9827 \quad 0.9597]$. Since $\overline{X} \in S(D, b^2)$, set $S(A, D, b^1, b^2)$ is feasible:

$$\begin{bmatrix} 0.4218 & 0.6557 & 0.6787 & 0.6555 \\ 0.9157 & 0.0357 & 0.7577 & 0.1712 \\ 0.7922 & 0.8491 & 0.7431 & 0.7060 \\ 0.9595 & 0.9340 & 0.3922 & 0.0318 \end{bmatrix} \Delta \begin{bmatrix} 1.0000 \\ 1.0000 \\ 0.9827 \\ 0.9597 \end{bmatrix} = \begin{bmatrix} 0.8307 \\ 0.9578 \\ 0.9245 \\ 0.9798 \end{bmatrix} \geq \begin{bmatrix} 0.3232 \\ 0.0087 \\ 0.5656 \\ 0.01672 \end{bmatrix}$$

Step4: For this example, there are 256 feasible vectors $\underline{X}(e)$ (i.e., $\underline{X}(e) \leq \overline{X}$). Minimal solutions are as follows:

$$\begin{aligned} e_1 &= [1 \quad 1 \quad 1 \quad 1] \implies \underline{X}(e_1) = [0.339 \quad 0 \quad 0 \quad 0] \\ e_2 &= [2 \quad 1 \quad 2 \quad 1] \implies \underline{X}(e_2) = [0 \quad 0.2821 \quad 0 \quad 0] \\ e_3 &= [2 \quad 1 \quad 3 \quad 1] \implies \underline{X}(e_3) = [0 \quad 0 \quad 0.3881 \quad 0] \\ e_4 &= [2 \quad 1 \quad 4 \quad 1] \implies \underline{X}(e_4) = [0 \quad 0 \quad 0 \quad 0.4252] \end{aligned}$$

Vector $\underline{X}(e^*) = [0.339 \quad 0 \quad 0 \quad 0]$ is the optimal solution of the sub-problem (3) that is obtained by $e^* = e_1$.
Step5: The optimal solution of Problem (1) is resulted as $x^* = [1 \quad 1 \quad 0 \quad 0]$ with optimal objective value $Z^* = -3.5939$.

Example 3. Consider the following linear optimization problem (1):

$$\min Z = -0.9892x_1 - 8.3236x_2 - 5.4205x_3 + 8.2667x_4 - 6.9524x_5$$
$$\begin{bmatrix} 0.7513 & 0.9593 & 0.8407 & 0.3500 & 0.3517 \\ 0.2551 & 0.5472 & 0.2543 & 0.1966 & 0.8308 \\ 0.5060 & 0.1386 & 0.8143 & 0.2511 & 0.5853 \\ 0.6991 & 0.1493 & 0.2435 & 0.6160 & 0.5497 \\ 0.8909 & 0.2575 & 0.9293 & 0.4733 & 0.9172 \end{bmatrix} \Delta x \leq \begin{bmatrix} 0.9691 \\ 0.9001 \\ 0.82073 \\ 0.7700 \\ 0.9367 \end{bmatrix}$$

$$\begin{bmatrix} 0.2858 & 0.0759 & 0.1299 & 0.1622 & 0.6020 \\ 0.7572 & 0.0540 & 0.5688 & 0.7943 & 0.2630 \\ 0.7537 & 0.5308 & 0.4694 & 0.3112 & 0.6541 \\ 0.3804 & 0.7792 & 0.0119 & 0.5285 & 0.6892 \\ 0.5678 & 0.9340 & 0.3371 & 0.1656 & 0.7482 \end{bmatrix} \Delta x \geq \begin{bmatrix} 0.0198 \\ 0.0423 \\ 0.2222 \\ 0.0033 \\ 0.1221 \end{bmatrix}$$
$$x \in [0,1]^n$$

Step1: Since $0 \in S(A, b^1)$, set $S(A, b^1))$ is feasible:

$$\begin{bmatrix} 0.7513 & 0.9593 & 0.8407 & 0.3500 & 0.3517 \\ 0.2551 & 0.5472 & 0.2543 & 0.1966 & 0.8308 \\ 0.5060 & 0.1386 & 0.8143 & 0.2511 & 0.5853 \\ 0.6991 & 0.1493 & 0.2435 & 0.6160 & 0.5497 \\ 0.8909 & 0.2575 & 0.9293 & 0.4733 & 0.9172 \end{bmatrix} \Delta \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4797 \\ 0.4154 \\ 0.4072 \\ 0.3496 \\ 0.4647 \end{bmatrix} \leq \begin{bmatrix} 0.9691 \\ 0.9001 \\ 0.82073 \\ 0.7700 \\ 0.9367 \end{bmatrix}$$

Step2: Since $1 \in S(D, b^2)$, set $S(D, b^2)$ is feasible:

$$\begin{bmatrix} 0.2858 & 0.0759 & 0.1299 & 0.1622 & 0.6020 \\ 0.7572 & 0.0540 & 0.5688 & 0.7943 & 0.2630 \\ 0.7537 & 0.5308 & 0.4694 & 0.3112 & 0.6541 \\ 0.3804 & 0.7792 & 0.0119 & 0.5285 & 0.6892 \\ 0.5678 & 0.9340 & 0.3371 & 0.1656 & 0.7482 \end{bmatrix} \Delta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8010 \\ 0.9001 \\ 0.81073 \\ 0.7700 \\ 0.9367 \end{bmatrix} \geq \begin{bmatrix} 0.0198 \\ 0.0423 \\ 0.2222 \\ 0.0033 \\ 0.1221 \end{bmatrix}$$

Step3: From Lemma 2 and Theorem 1,
$\overline{X} = [0.8409 \quad 0.9789 \quad 0.8272 \quad 0.9240 \quad 0.9562]$. Since $\overline{X} \in S(D, b^2)$, set $S(A, D, b^1, b^2)$ is feasible:

$$\begin{bmatrix} 0.2858 & 0.0759 & 0.1299 & 0.1622 & 0.6020 \\ 0.7572 & 0.0540 & 0.5688 & 0.7943 & 0.2630 \\ 0.7537 & 0.5308 & 0.4694 & 0.3112 & 0.6541 \\ 0.3804 & 0.7792 & 0.0119 & 0.5285 & 0.6892 \\ 0.5678 & 0.9340 & 0.3371 & 0.1656 & 0.7482 \end{bmatrix} \Delta \begin{bmatrix} 0.8409 \\ 0.9789 \\ 0.8272 \\ 0.9240 \\ 0.9562 \end{bmatrix} = \begin{bmatrix} 0.7791 \\ 0.8592 \\ 0.8052 \\ 0.8790 \\ 0.9565 \end{bmatrix} \geq \begin{bmatrix} 0.0198 \\ 0.0423 \\ 0.2222 \\ 0.0033 \\ 0.1221 \end{bmatrix}$$

Step4: For this example, there are 3125 feasible vectors $\underline{X}(e)$ (i.e., $\underline{X}(e) \leq \overline{X}$).
Minimal solutions are as follows:

$$e_1 = [1 \quad 1 \quad 1 \quad 1 \quad 1] \implies \underline{X}(e_1) = [0 \quad 0 \quad 0 \quad 0 \quad 0]$$

Vector $\underline{X}(e^*) = [0 \quad 0 \quad 0 \quad 0 \quad 0]$ is the optimal solution of the sub-problem (3) that is obtained by $e^* = e_1$.
Step5: The optimal solution of Problem (1) is resulted as
$x^* = [0.8409 \quad 0.9789 \quad 0.82716 \quad 0 \quad 0.9562]$ with optimal objective value
$Z^* = -20.1113$.

Conclusion
In this paper, we proposed an algorithm to find the optimal solution of linear problems subjected to two fuzzy relational inequalities with "Fuzzy Max-Min "averaging operator. Some test problems were then solved by the proposed algorithm. As future works, we aim at testing our algorithm in other type of linear optimization problems whose constraints are defined as FRI with other well-known t-norms.

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