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Constructions of antimagic labelings for some families of regular graphs

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ABSTRACT

In this paper we construct antimagic labelings of the regular complete multipartite graphs and we also extend the construction to some families of regular graphs.

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1 Introduction

All graphs considered in this paper are finite, simple, undirected and connected, unless stated otherwise. In 1990, Hartsfield and Ringel introduced the concept of an antimagic labeling of a graph. They conjectured that every connected connected graph other than K_2 is antimagic. An antimagic labeling of a graph $G = (V, E)$ is a bijection from E to the a set of positive integers $\{1, 2, \ldots, |E|\}$ such that all vertex weights are pairwise distinct, where the *vertex weight* of a vertex v, $wt(v)$, is defined as the sum of the labels of all the edges incident with the vertex v. A graph G is called *antimagic* if there exists an antimagic labeling of G.

Some families of graphs, for example, path P_m , star S_m, cycle C_m, complete graph K_m, wheel W_m and bipartite graph $K_{2,m}$, $m \geq 3$, have been proved to be antimagic in [6]. For more than two decades there have been many attempts to settle the conjecture but although there are now many supporting results, there still are many gaps that need to be filled, for instance, see [1, 3, 7, 8, 9]. More details about antimagic labeling can be found in the dynamic survey by Gallian in [5], see also [2].

In this paper we construct antimagic labelings for regular complete multipartite graphs and then extend them to other regular graphs.

We conclude this section with a definition, a representation of an edge labeling and some notations that will be used throughout the paper.

The join graph $G + H$ of the graphs G and H is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\ uv : u \in V(G) \quad \text{and} \quad v \in V(H)\}$.

An edge labeling l of a graph G will be described by an array L, not necessary rectangular. Each row of L represents a vertex of G and the entries in the row are the edge labels incident with the vertex.

We denote by T^t the transpose of the array T and by $wt_G(v)$ the sum of the labels of all the edges incident with v in a graph G (resp. by $wt_L(v)$ the sum of all entries in the row v, where L is the array of the edge labels of the graph G), or simply, $wt(v)$.

2 Results

The following lemma will be used as a base step for constructing the antimagic labelings of regular multipartite graphs.

Lemma 1. *For $m \geq 2$, the complete bipartite graph $K_{m,m}$ is antimagic.*

Proof. Let T_l, $1 \leq l \leq m$, be the $(m \times 1)$-array of the edges e_i, $1 \leq i \leq m$, of the regular complete bipartite graph $K_{m,m}$. We label the edges e_i, $1 \leq i \leq m$, in the row i of the array T_l with $i + (l - 1)m$. Then the array $A = T_1T_2 \ldots T_m$ is the array of edge labels of $K_{m,m}$, where each row of A is the set of all the labels of the edges incident with a vertex in one partition of vertices while each column is the set of all the labels of edges incident with a vertex in the other partition of vertices. We next prove that all vertex weights of $K_{m,m}$ are pairwise

distinct. Let r_i and c_j for $1 \leq i, j \leq m$, be a row (vertex) and a column (vertex) of the array A, respectively. By the construction of the array A, we have $wt(r_i) = \frac{m^3-m^2+2m}{2} + (i-1)m$ and $wt(c_j) = \frac{m(m+1)}{2} + (j-1)m^2$. It is clear that $wt(r_i) < wt(r_f)$, for $1 \leq i < f \leq m$ and $wt(c_j) < wt(c_g)$, for $i \leq j < g \leq m$. We finally verify that each vertex weight of one partition of the vertices is distinct from each vertex weight in the other partition of the vertices.
For m even, we have $wt(c_{\frac{m}{2}}) = \frac{m^3-m^2+m}{2} < \frac{m^3-m^2+2m}{2} = wt(r_1)$ and $wt(r_m) = \frac{m^3+m^2}{2} < \frac{m^3+m^2+m}{2} = wt(c_{\frac{m}{2}+1})$. Hence all the vertex weights are pairwise distinct.
For m odd, suppose $wt(r_i) = wt(c_j)$, then we have $m(m^2 - 2jm + 2i - 1) = 0$. Since m is a positive integer, $m^2 - 2jm + 2i - 1 = 0$. Therefore, $j^2 - 2i + 1 > 0$ and must be a square. This leads to $i = j$; substituting back into the equation we obtain $m = 2j - 1$. In this case the labeling of $K_{m,m}$ is not yet antimagic. To obtain an antimagic labeling of $K_{m,m}$ we make a small change by swaping the labels $\lfloor \frac{m}{2} \rfloor$ and $\lfloor \frac{m}{2} \rfloor + 1$. □

Theorem 2. *For $m \geq 1$, the complete 3-partite graph $K_{m,m,m}$ is antimagic.*

Proof. Let L be the array obtained from the array A as the one given in the proof of Lemma 1 by arranging the vertices of $K_{m,m}$, in which $wt_L(v_i) \leq wt_L(v_j)$, for $1 \leq i < j \leq 2m$, that is, each row in L represents a vertex of $K_{m,m}$ and the entries in each row represent the labels of edges incident with that row (vertex). Let T_l, $1 \leq l \leq m$, be the $(2m \times 1)$-array of the edges e_i, $1 \leq i \leq 2m$, where e_i are the edges of $K_{m,m,m}$ that do not belong to $K_{m,m}$. We construct the array B of the edge labels of $K_{m,m,m}$ as follows.

Case 1: m even

(1) Label the edge e_i, $1 \leq i \leq 2m$, in the row i of the array T_l, $1 \leq l \leq m$, with $i + (l-1)2m$, for $1 \leq l \leq \frac{m}{2}$; and $i + 2m^2 + (l - (\frac{m}{2} + 1))2m$, for $\frac{m}{2} + 1 \leq l \leq m$;

(2) Replace the edge labels in the array L with the new labels obtained by adding m^2 to each of the original edge labels;

(3) Form the array B as shown below.

$$\begin{array}{ccccccccc} & & & & & & & & T_1^t \\ & & & & & & & & T_2^t \\ & & & & & & & & \vdots \\ & & & & & & & & T_{\frac{m}{2}}^t \\ T_1 & T_2 & \ldots & T_{\frac{m}{2}} & L & T_{\frac{m}{2}+1} & T_{\frac{m}{2}+2} & \ldots & T_m \\ & & & & & & & & T_{\frac{m}{2}+1}^t \\ & & & & & & & & T_{\frac{m}{2}+2}^t \\ & & & & & & & & \vdots \\ & & & & & & & & T_m^t \end{array}$$

By the construction of the array B, it is clear that the weight of each vertex (row) is less that the weight of the vertex below, except in two cases that need to be verified.
Let $e_{f,g}$ be the edge label in the row f and the column g of the array B and let $wt(r_f)$ be the weight of the vertex r_f (row f).

(i) $wt(T^t_{\frac{m}{2}})$ and $wt(r_{\frac{m}{2}+1})$

Since $\Sigma_{g=1}^{\frac{m}{2}} e_{\frac{m}{2},g} + \Sigma_{g=\frac{3m}{2}+1}^{2m} e_{\frac{m}{2},g} = \frac{4m^3-4m^2+2m}{4} < \frac{6m^3-4m^2+4m}{4} = \Sigma_{g=1}^{\frac{m}{2}} e_{\frac{3m}{2},g} + \Sigma_{g=\frac{3m}{2}+1}^{2m} e_{\frac{m}{2}+1,g}$ and $e_{\frac{m}{2},g} < e_{\frac{m}{2}+1,g}$, for $\frac{m}{2}+1 \le g \le \frac{3m}{2}$, $wt(T^t_{\frac{m}{2}}) < wt(r_{\frac{m}{2}+1})$.

(ii) $wt(r_{\frac{m}{2}+2m})$ and $wt(T^t_{\frac{m}{2}+1})$

Since $\Sigma_{g=1}^{\frac{m}{2}} e_{\frac{m}{2}+2m,g} + \Sigma_{g=\frac{3m}{2}+1}^{2m} e_{\frac{m}{2}+2m,g} = \frac{6m^3+4m^2}{4} < \frac{8m^3+4m^2+2m}{4}$
$= \Sigma_{g=1}^{\frac{m}{2}} e_{\frac{m}{2}+2m+1,g} + \Sigma_{g=\frac{3m}{2}+1}^{2m} e_{\frac{m}{2}+2m+1,g}$ and $e_{\frac{m}{2}+2m,g} < e_{\frac{m}{2}+2m+1,g}$, for $\frac{m}{2}+1 \le g \le \frac{3m}{2}$, $wt(r_{\frac{m}{2}+2m}) < wt(T^t_{\frac{m}{2}+1})$.

Case 2: m odd
This case is similar to Case 1 and so the details are omitted here. We only mention how to construct the array T_l, $1 \le l \le m$, and the array B.

(1) Label the edge e_i, $1 \le i \le 2m$, in the row i of the array T_l, $1 \le l \le m$, with $i+(l-1)2m$, for $1 \le l \le \frac{m+1}{2}$; and $i + 2m^2 + m + (l - (\frac{m+1}{2}+1))2m$, for $\frac{m+1}{2}+1 \le l \le m$;

(2) Form the array B as shown below.

$$\begin{array}{cccccccc}
 & & & & & & & T_1^t \\
 & & & & & & & T_2^t \\
 & & & & & & & \vdots \\
 & & & & & & & T^t_{\frac{m+1}{2}} \\
T_1 & T_2 & \dots & T_{\frac{m+1}{2}} & L & T_{\frac{m+1}{2}+1} & T_{\frac{m+1}{2}+2} \quad \dots & T_m \\
 & & & & & & & T^t_{\frac{m+1}{2}+1} \\
 & & & & & & & T^t_{\frac{m+1}{2}+2} \\
 & & & & & & & \vdots \\
 & & & & & & & T^t_m
\end{array}$$

□

By the induction on n, we have

Corollary 3. *For $m \ge 1, n \ge 2$, the complete n-partite graph $K_{\underbrace{m, m, \dots, m}_{n\ times}}$, except $K_2 = K_{1,1}$, is antimagic.*

1	2	3	4	5	6
1	7	8	9	10	19
2	7	11	12	15	20
3	8	11	13	16	21
4	9	12	14	17	22
5	10	13	14	18	23
6	15	16	17	18	24
19	20	21	22	23	24

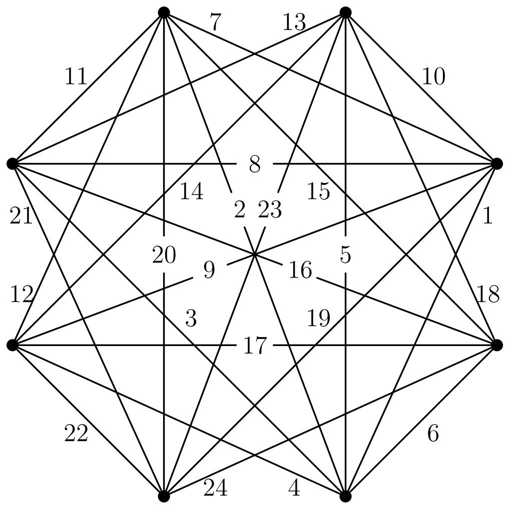

Figure 1: The graph $K_{2,2,2,2}$ and its antimagic labeling.

Figure 1 illustrates $K_{2,2,2,2}$ and its antimagic labeling.
Note that the complete graph K_n, $n \geq 3$, has been proved to be antimagic in [6] and also in [7]. This is a special case of Corollary 3 when $m = 1$, that is $K_n = K_{\underbrace{1,1,\ldots,1}_{n\ times}}$.

We extend our results to more general regular graphs.

Lemma 4. *Let $G = mK_2$, $m \geq 1$. Then $G + (2m-1)K_1$ is antimagic.*

Proof. The array of edge labels of G is $L = (1\ 1\ 2\ 2 \ldots m\ m)^t$. Let T_l, $1 \leq l \leq 2m-1$, be the $(2m \times 1)$-array of the edges e_i, $1 \leq i \leq 2m$, where the e_i's are the edges of $G + (2m-1)K_1$ that do not belong to mK_2. We construct the array B of the edge labels of $G + (2m-1)K_1$ as follows.

(1) Label the edge e_i, $1 \leq i \leq 2m$, in the row i of the array T_l, $1 \leq l \leq 2m-1$, with $i + (l-1)2m$;

(2) Replace the edge labels in the array L with the new labels obtained by adding $2(2m-1)m$ to each of the original edge labels;

(3) Form the array B as shown below.

$$\begin{array}{ccccc} & & & & T_1^t \\ & & & & T_2^t \\ & & & & \vdots \\ & & & & T_m^t \\ T_1 & T_2 & \dots & T_{2m-1} & L \\ & & & & T_{m+1}^t \\ & & & & T_{m+2}^t \\ & & & & \vdots \\ & & & & T_{2m-1}^t \end{array}$$

By the construction of the array B, it is clear that the weight of each vertex is less than the weight of the vertex below with two exceptions that need to be verified.
Let $e_{f,g}$ be the edge label in the row f and the column g of the array B and let $wt(r_f)$ be the weight of the vertex r_f (row f).

(i) $wt(T_m^t)$ and $wt(r_{m+1})$

We have $wt(T_m^t) = 4m^3 - 2m^2 + m < 4m^3 - 2m^2 + 2m = wt(r_{m+1})$.

(ii) $wt(r_{3m})$ and $wt(T_{m+1}^t)$

We have the edge labels of r_{3m} and T_{m+1}^t as shown below.

$$\begin{array}{cccccc} r_{3m}: & 2m & 4m & \dots & (2m-1)2m & (2m-1)2m+m \\ T_{m+1}^t: & 1+2m^2 & 2+2m^2 & \dots & (2m-1)2m^2 & 2m+2m^2 \end{array}$$

We have $wt(r_{3m}) = 4m^3 + 2m^2 - m < 4m^3 + 2m^2 + m = wt(T_{m+1}^t)$.

□

Theorem 5. *Let G be a k-regular (connected or disconnected) graph with p vertices and $k \geq 2$. Then the p-regular graph $G + (p-k)K_1$ is antimagic.*

Proof. We can use the same construction as in the proof of Theorem 2 by replacing $K_{m,m}$ by G. □

By the induction on n, we obtain

Corollary 6. *The $(k+ns)$-regular graph $(((G + \underbrace{sK_1) + sK_1) + \dots) + sK_1}_{n\ times}$ is antimagic, for $s = p - k$.*

Combining Lemma 1, Corollary 3, Lemma 4 and Corollary 6, we have

Theorem 7. *Let G be a k-regular graph, where $k \geq 0$ and $s = p - k$. Then the $(k+ns)$-regular graph $(((G + \underbrace{sK_1) + sK_1) + \dots) + sK_1}_{n\ times}$ is antimagic.*

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References

[1] N. Alon, G. Kaplan, A. Lev, Y. Roditty, R. Yuster, Dense graphs are antimagic, *J. Graph Theory* 47(4) (2004), 297–309.

[2] M. Bača, M. Miller, *Super edge-antimagic graphs: A wealth of problems and some solutions*, BrownWalker Press, Boca Raton, Florida, USA, 2008.

[3] Y. Cheng, A new class of antimagic Cartesian product graphs, *Discrete Math.* 308(24) (2008), 6441–6448.

[4] D. W. Cranston, Regular bipartite graphs are antimagic, *J. Graph Theory* 60(3) (2009), 173–182.

[5] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* 17 (2011), ♯DS6.

[6] N. Hartsfield, G. Ringel, *Pearls in graph theory: A comprehensive introduction*, Academic Press Inc., Boston, MA, 1990.

[7] O. Phanalasy, M. Miller, L. J. Rylands, P. Lieby, On a relationship between completely separating systems and antimagic labeling of regular graphs, *Lecture Notes in Computer Science* 6460 (2011), 238–241.

[8] T. M. Wang, C. C. Hsiao, On antimagic labeling for graph products, *Discrete Math.* 308(16) (2008), 3624–3633.

[9] Y. Zhang, X. Sun, The antimagicness of the Cartesian product of graphs, *Theor. Comput. Sci.* 410 (2009), 727–735.