Vertex Equitable Labelings of Transformed Trees

P. Jeyanthi$^1$ and A. Maheswari$^2$

$^1$Govindammal Aditanar College for Women Tiruchendur-628 215, Tamil Nadu, India.
$^2$Department of Mathematics Kamaraj College of Engineering and Technology Virudhunagar- 626 001, Tamil Nadu, India.

ABSTRACT

Let $G$ be a graph with $p$ vertices and $q$ edges and let $A = \{0, 1, 2, \ldots, \left\lceil \frac{q}{2} \right\rceil \}$. A vertex labeling $f : V(G) \rightarrow A$ induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$. For $a \in A$, let $v_f(a)$ be the number of vertices $v$ with $f(v) = a$. A graph $G$ is vertex equitable if there exists a vertex labeling $f$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$. In this paper, we prove that $TÔP_n, TÔ2P_n, TÔC_n, TÔC_n$ are vertex equitable graphs.

1 Introduction

All graphs considered here are simple, finite, connected and undirected. For the basic notations and terminology, we follow [3]. The symbols $V(G)$ and $E(G)$ denote the vertex set and the
edge set of a graph $G$ respectively. Let $G = (p, q)$ be a graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. A labeling $f$ of a graph $G$ is a mapping that assigns elements of a graph to the set of numbers (usually to positive or non-negative integers). If the domain of the mapping is the set of vertices (respectively, the set of edges) then we call the labeling vertex labeling (respectively, edge labeling). The labels of the vertices induce the labels of the edges. There are several types of labeling and a detailed survey on graph labeling can be found in [2]. A vertex labeling $f$ is said to be a difference labeling if it induces the label $|f(x) - f(y)|$ for each edge $xy$ which is called the weight of an edge $xy$. A brief summary of the definitions and known results are given below.

The total graph $T(G)$ of a graph $G$ is a graph such that the vertex set of $T(G)$ corresponds to the vertices and the edges of $G$ and the two vertices are adjacent in $T(G)$ if and only if their corresponding elements are either adjacent or incident in $G$. For each vertex $v$ of a graph $G$, take a new vertex $v'$ and join $v'$ to the vertices of $G$ which are adjacent to $v$. The graph thus obtained is called the splitting graph of $G$ and is denoted by $S'(G)$.

Let $G$ be a graph with $p$ vertices and $q$ edges. A graph $H$ is said to be a super subdivision of $G$ if $H$ is obtained from $G$ by replacing every edge $e_i$ of $G$ by a complete bipartite graph $K_{2,m_i}$ for some $m_i, 1 \leq i \leq q$ in such a way that ends of $e_i$ are merged with the two vertices of the 2-vertices part of $K_{2,m_i}$, after removing the edge $e_i$ from $G$. A super subdivision $H$ of $G$ is said to be an arbitrary super subdivision of $G$ if every edge of $G$ is replaced by $K_{2,m}$ ($m$ vary for each edge arbitrarily). Fusion of two cycles $C_m$ and $C_n$ is a graph $C(m,n)$ obtained by identifying an edge of a cycle $C_m$ with an edge of a cycle $C_n$.

The concept of equitable labeling of graphs was due to Bloom and Ruiz [1]. A function $f : V(G) \rightarrow \{0, 1, \ldots, k-1\}$ is called $k$-equitable labeling if the conditions $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for $i \neq j, i, j = 0, 1, 2, \ldots, k-1$ are satisfied, where $f$ is the induced edge labeling given by $f(uv) = |f(u) - f(v)|$ and $v_f(i)$ and $e_f(i), i \in \{0, 1, \ldots, k-1\}$ are the number of vertices and edges of $G$ respectively with label $i$.

A. Lourdusamy and M. Seenivasan introduced the concept of vertex equitable labeling in [7]. Let $G$ be a graph with $p$ vertices and $q$ edges and let $A = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil \}$. A vertex labeling $f : V(G) \rightarrow A$ induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$. For $a \in A$, let $v_f(a)$ be the number of vertices $v$ with $f(v) = a$. A graph $G$ is said to be vertex equitable if there exists a vertex labeling $f$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$. They proved that the graphs like path, bistar $B(n,n)$, comb graph, cycle $C_n$ if $n \equiv 0$ or $3(mod\ 4)$, $K_{2,n}, C_{3(t)}$ for $t \geq 2$, quadrilateral snake, $K_2 + mK_1, K_{1,n} \cup K_{1,n+k}$ if and only if $1 \leq k \leq 3$, ladder, arbitrary super division of any path and cycle $C_n$ with $n \equiv 0$ or $3(mod\ 4)$ are vertex equitable. Also they proved that the graphs $K_{1,n}$ if $n \geq 4$, any Eulerian graph with $n$ edges where $n \equiv 1$ or $2(mod\ 4)$, the wheel $W_n$, the complete graph $K_n$ if $n > 3$ and triangular cactus with $q \equiv 0$ or $6$ or $9(mod\ 12)$ are not vertex equitable. In addition, they proved that if $G$ is a graph with $p$ vertices and $q$ edges, $q$ is even and $p < \frac{q}{2} + 2$ then $G$ is not vertex equitable.

P. Jeyanthi and A. Maheswari [5, 6] proved that $T_p$-trees, $T \odot K_n$ where $T$ is a $T_p$-tree with an
even number of vertices, the bistar $B(n, n + 1)$, the caterpillar $S(x_1, x_2, \ldots, x_n)$, $C_n \odot \overline{K}_1, P_n^{2}$, tadpoles, $C_m \oplus C_n$, armed crowns, $[Pm; C^n_2]$, $\langle P_n \odot K_{1,n} \rangle$, the graphs obtained by duplicating an arbitrary vertex and an arbitrary edge of a cycle $C_n$, total graph of $P_n$, splitting graph of $P_n$ and $C(m, n)$ are vertex equitable graphs.

In this paper, we prove that $T\hat{O}P_n, T\hat{O}2P_n, T\hat{O}C_n, T\hat{O}C_n$ are vertex equitable graphs. We use the following definitions.

**Definition 1.1.** [4] Let $T$ be a tree and $u_0$ and $v_0$ be two adjacent vertices in $T$. Let $u$ and $v$ be two pendant vertices of $T$ such that the length of the path $u_0-u$ is equal to the length of the path $v_0-v$. If the edge $u_0v_0$ is deleted from $T$ and $u$ and $v$ are joined by an edge $uv$, then such a transformation of $T$ is called an elementary parallel transformation (ept) and the edge $u_0v_0$ is called transformable edge.

If by the sequence of ept’s, $T$ can be reduced to a path, then $T$ is called a $T_p$-tree (transformed tree) and such a sequence regarded as a composition of mappings (ept’s) denoted by $P$, is called a parallel transformation of $T$. The path, the image of $T$ under $P$ is denoted as $P(T)$. A $T_p$-tree and a sequence of two ept’s reducing it to a path are illustrated in Figure-1.

![Figure 1: A $T_p$-tree and a sequence of two ept’s reducing it to a path](image)

**Definition 1.2.** Let $G_1$ be a graph with $p$ vertices and $G_2$ be any graph. A graph $G_1 \hat{o} G_2$ is obtained from $G_1$ and $p$ copies of $G_2$ by identifying one vertex of $i^{th}$ copy of $G_2$ with $i^{th}$ vertex of $G_1$.

**Definition 1.3.** Let $G_1$ be a graph with $p$ vertices and $G_2$ be any graph. A graph $G_1 \hat{o} G_2$ is obtained from $G_1$ and $p$ copies of $G_2$ by joining one vertex of $i^{th}$ copy of $G_2$ with $i^{th}$ vertex of $G_1$ by an edge.

## 2 Main Result

In this paper, we prove that $T\hat{O}P_n, T\hat{O}2P_n, T\hat{O}C_n, T\hat{O}C_n$ are vertex equitable graphs.

**Theorem 2.1.** Let $T$ be a $T_p$-tree on $m$ vertices. Then the graph $T\hat{o}P_n$ is a vertex equitable graph.
Proof. Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T)E_{d})\Box E_{p}$ where $E_{d}$ is the set of edges deleted from $T$ and $E_{p}$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts used to arrive at the path $P(T)$. Clearly, $E_{d}$ and $E_{p}$ have the same number of edges.

Now denote the vertices of $P(T)$ successively by $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other one. Let $u^i_1, u^i_2, \ldots, u^i_m(1 \leq j \leq m)$ be the vertices of $j^{th}$ copy of $P_n$. Then $V(T\hat{o}P_n) = \{u^i_j : 1 \leq i \leq n, 1 \leq j \leq m \}$ with $u^i_m = v_j$. The graph $T\hat{o}P_n$ has $mn$ vertices and $mn - 1$ edges. Let $A = \{0, 1, 2, \ldots, \left\lceil \frac{mn-1}{2} \right\rceil \}$.

We define a vertex labeling $f : V(T\hat{o}P_n) \to A$ as follows:

For $1 \leq i \leq n$, let $f(u^i_j) = \left\{ \begin{array}{ll} \frac{n(j-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor & \text{if } j \text{ is even, } 1 \leq j \leq m \\ \frac{n(j-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor & \text{if } j \text{ is odd, } 1 \leq j \leq m. \end{array} \right.$

Let $v_iv_j$ be a transformed edge in $T$ for some indices $i, j, 1 \leq i \leq j \leq m$ and let $P_1$ be the ept that deletes the edge $v_iv_j$ and adds the edge $v_{i+t}v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ as also the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity, that is, $i$ is odd and $j$ is even or vice-versa.

The induced label of the edge $v_iv_j$ is given by

$$f^*(v_iv_j) = f^*(v_{i+v_{i+2t+1}})$$
$$= f(v_i) + f(v_{i+2t+1})$$
$$= \left\{ \begin{array}{ll} \frac{n(i-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor + \frac{n(i+2t+1)}{2} - \left\lfloor \frac{n}{2} \right\rfloor & \text{if } i \text{ is odd} \\ \frac{n(i-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor + \frac{n(i+2t+1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor & \text{if } i \text{ is even.} \end{array} \right.$$
$$= n(i + t) \text{ and}$$

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+t+1})$$
$$= f(v_{i+t}) + f(v_{i+t+1})$$
$$= n(i + t).$$

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

It can be verified that the induced edge labels of $T\hat{o}P_n$ are $1, 2, 3, \ldots, mn - 1$ and for $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence, $T\hat{o}P_n$ is a vertex equitable graph. \hfill \square

The vertex equitable labeling of $T\hat{o}P_5$ where $T$ is a $T_p$-tree with 13 vertices is given in Figure 2.
Theorem 2.2. Let $T$ be a $T_p$-tree on $m$ vertices. Then the graph $T\tilde{o}2P_n$ is a vertex equitable graph.

Proof. Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T) - E_d) \square E_p$ where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges.

Now denote the vertices of $P(T)$ successively by $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other one. Let $u_{i,1}^j, u_{i,2}^j, \ldots, u_{i,n}^j$ and $u_{2,1}^j, u_{2,2}^j, \ldots, u_{2,n}^j$ $(1 \leq j \leq m)$ be the vertices of the two vertex disjoint paths joined by the $j^{th}$ vertex of $T$ such that $v_j = u_{1,n}^j = u_{2,n}^j$. Then $V(T\tilde{o}2P_n) = \{v_j, u_{i,1}^j, u_{i,n}^j : 1 \leq i \leq n, 1 \leq j \leq m \text{ with } u_{1,n}^j = u_{2,n}^j = v_j\}$.

Let $A = \left\{0, 1, 2, \ldots, \left\lfloor \frac{m(2n-1)-1}{2} \right\rfloor \right\}$. Define a vertex labeling $f : V(T\tilde{o}2P_n) \rightarrow A$ as follows:

For $1 \leq i \leq n$, let $f(u_{1,i}^j) = \left\{ \begin{array}{ll}
\frac{(2n-1)j}{2} - n + \left\lfloor \frac{i}{2} \right\rfloor & \text{if } j \text{ is even, } 1 \leq j \leq m \\
\frac{(2n-1)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor & \text{if } j \text{ is odd, } 1 \leq j \leq m
\end{array} \right.$

and $f(u_{2,i}^j) = \left\{ \begin{array}{ll}
\frac{(2n-1)j}{2} - \left\lfloor \frac{i}{2} \right\rfloor & \text{if } j \text{ is even, } 1 \leq j \leq m \\
\frac{(2n-1)(j-1)}{2} + n - \left\lfloor \frac{i}{2} \right\rfloor & \text{if } j \text{ is odd, } 1 \leq j \leq m
\end{array} \right.$

Let $v_iv_j$ be the transformed edge in $T$ for some indices $i, j, 1 \leq i \leq j \leq m$ and let $P_t$ be the ept that deletes the edge $v_iv_j$ and adds the edge $v_{i+t}v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ as
also the distance of \( v_j \) from \( v_{j-t} \). Let \( P \) be a parallel transformation of \( T \) that contains \( P_1 \) as one of the constituent ept's.

Since \( v_{i+t}v_{j-t} \) is an edge in the path \( P(T) \), it follows that \( i + t + 1 = j - t \) which implies \( j = i + 2t + 1 \). Therefore \( i \) and \( j \) are of opposite parity, that is, \( i \) is odd and \( j \) is even or vice-versa.

The induced label of the edge \( v_iv_j \) is given by

\[
\begin{align*}
f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) \\
&= f(v_i) + f(v_{i+2t+1}) \\
&= (2n - 1)(i + t) \quad \text{and} \\
f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) \\
&= f(v_{i+t}) + f(v_{i+v_{i+t+1}}) \\
&= (2n - 1)(i + t).
\end{align*}
\]

Therefore, \( f^*(v_iv_j) = f^*(v_{i+t}v_{j-t}) \).

It can be verified that the induced edge labels of \( T\delta 2P_n \) are 1, 2, 3, \ldots, \( m(2n - 1) - 1 \) and \( |vf(a) - vf(b)| \leq 1 \) for all \( a, b \in A \). Hence, \( T\delta 2P_n \) is a vertex equitable graph. 

The vertex equitable labeling of \( T\delta 2P_4 \) where \( T \) is a \( T_p \)-tree with 10 vertices is given in Figure 3.

![Figure 3](image)

**Theorem 2.3.** Let \( T \) be a \( T_p \)-tree on \( m \) vertices. Then the graph \( T\delta C_n \) is a vertex equitable graph if \( n \equiv 0, 3(\text{mod } 4) \).
Proof. Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T) - E_d) \square E_p$ where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the EPTs $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges.

Now denote the vertices of $P(T)$ successively by $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other one. Let $u^j_1, u^j_2, \ldots, u^j_n (1 \leq j \leq m)$ be the vertices of $j^{th}$ copy of $P_n$.

Then $V(T \diamond C_n) = \{ u^j_i : 1 \leq i \leq n, 1 \leq j \leq m \}$ with $u^j_1 = v_j$. Let $A = \{0, 1, 2, \ldots, \left\lfloor \frac{m(n+1)-1}{2} \right\rfloor \}$. Define a vertex labeling $f : V(T \diamond C_n) \rightarrow A$ as follows:

Case (i) $n \equiv 3 (mod \ 4)$.

For $1 \leq j \leq m$ and $j$ is odd,

let $f(u^j_1) = \left\lceil \frac{n}{2} \right\rceil j$,

$f(u^j_2) = \left\lceil \frac{n}{2} \right\rceil (j-1) + \left\lfloor \frac{i}{2} \right\rfloor$ if $i$ is odd, $2 \leq i \leq n$,

$f(u^j_3) = \left\lceil \frac{n}{2} \right\rceil (j-1) + \left\lfloor \frac{i}{2} \right\rfloor$ if $i$ is even, $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$.

For $1 \leq j \leq m$ and $j$ is even,

let $f(u^j_1) = \left\lceil \frac{n}{2} \right\rceil (j-1)$,

$f(u^j_2) = \left\lceil \frac{n}{2} \right\rceil (j-1) + \left\lfloor \frac{i}{2} \right\rfloor$ if $i$ is even, $2 \leq i \leq n$,

$f(u^j_3) = \left\lceil \frac{n}{2} \right\rceil (j-1) + \left\lfloor \frac{i}{2} \right\rfloor$ if $i$ is odd, $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$.

Let $v_i, v_j$ be a transformed edge in $T$ for some indices $i, j, 1 \leq i \leq j \leq m$ and let $P_1$ be the ept that deletes the edge $v_i, v_j$ and adds the edge $v_{i+t}v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ as also the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity, that is, $i$ is odd and $j$ is even or vice-versa.

The induced label of the edge $v_i v_j$ is given by

$f^*(v_i v_j) = f^*(v_{i+2t+1}) = f(v_i) + f(v_{i+2t+1})$

$= (n+1)(i + t)$ and

$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t+1}) = f(v_{i+t}) + f(v_{i+t+1})$

$= (n+1)(i + t)$.
Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

**Case (ii)** $n \equiv 0 (mod \ 4)$.

For $1 \leq j \leq m$ and $j$ is odd.

let $f(u'_i) = \left\lfloor \frac{(n+1)j}{2} \right\rfloor$, 

$f(u'_i) = \left\lfloor \frac{(n+1)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor \right\rfloor$ if $i$ is odd, $2 \leq i \leq n$, 

$f(u'_i) = \begin{cases} 
\left\lfloor \frac{(n+1)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor \right\rfloor & \text{if } i \text{ is even, } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\left\lfloor \frac{(n+1)(j-1)}{2} + \left\lceil \frac{i}{2} \right\rceil \right\rfloor & \text{if } i \text{ is even, } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}$

For $1 \leq j \leq m$ and $j$ is even.

let $f(u'_i) = \left\lfloor \frac{(n+1)(j-1)}{2} \right\rfloor$, 

$f(u'_i) = \left\lfloor \frac{(n+1)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor \right\rfloor$ if $i$ is odd, $2 \leq i \leq n$, 

$f(u'_i) = \begin{cases} 
\left\lfloor \frac{(n+1)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor \right\rfloor & \text{if } i \text{ is even, } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\left\lfloor \frac{(n+1)(j-1)}{2} + \left\lceil \frac{i}{2} \right\rceil \right\rfloor & \text{if } i \text{ is even, } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}$

Let $v_iv_j$ be a transformed edge in $T$ for some indices $i,j, 1 \leq i \leq j \leq m$ and let $P_1$ be the ept that deletes the edge $v_iv_j$ and adds the edge $v_{i+t}v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ as also the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity, that is, $i$ is odd and $j$ is even or vice-versa.

The induced label of the edge $v_iv_j$ is given by

$$f^*(v_iv_j) = f^*(v_{i+2t+1}) = f(v_i) + f(v_{i+2t+1}) = (n + 1)(i + t)$$ and 

$$f^*(v_{i+t}v_{j-t}) = f^*(v_{i+t}v_{i+t+1}) = f(v_{i+t}) + f(v_{i+t+1}) = (n + 1)(i + t).$$

Therefore, $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t})$.

It can be verified that the induced edge labels of $T\hat{o}C_n$ are $1, 2, 3, \ldots, m(n+1) - 1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $T\hat{o}C_n$ is a vertex equitable graph.

The vertex equitable labeling pattern of $T\hat{o}C_7$, where $T$ is a $T_p$-tree with 8 vertices, is given in Figure 4.
Theorem 2.4. Let $T$ be a $T_p$-tree on $m$ vertices. Then the graph $T\tilde{o}C_n$ is a vertex equitable graph if $n \equiv 0, 3 \mod 4$.

Proof. Let $T$ be a $T_p$-tree with $m$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$ where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the ept’s $P$ used to arrive at the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges.

Now denote the vertices of $P(T)$ successively by $v_1, v_2, \ldots, v_m$ starting from one pendant vertex of $P(T)$ right up to the other one. Let $u^j_1, u^j_2, \ldots, u^j_m$ ( $1 \leq j \leq m$) be the vertices of $j$th copy of $P_n$ then $V(T\tilde{o}C_n) = \{v_i, u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(T\tilde{o}C_n) = E(T) \cup E(C_n) \cup \{v_i u^j_i : 1 \leq j \leq m\}$. Let $A = \{0, 1, 2, \ldots, \left\lfloor \frac{m(n+2)-1}{2} \right\rfloor \}$. Define a vertex labeling $f : V(T\tilde{o}C_n) \to A$ as follows:

Case 1: $n \equiv 3 \mod 4$.

For $1 \leq j \leq m$, let $f(v_i) = \left\{ \begin{array}{ll}
\frac{(i-1)(n+2)}{2} + \frac{n}{2} & \text{if } i \text{ is odd} \\
\frac{(i-2)(n+2)}{2} + 1 + \frac{n}{2} & \text{if } i \text{ is even}
\end{array} \right.$ if $i$ is even and $f(u^j_i) = f(v_i)$.

For $1 \leq j \leq m$ and $j$ is odd,

\[
\begin{align*}
\text{let } f(u^j_i) &= \left\{ \begin{array}{ll}
\frac{(n+2)(j-1)}{2} + \frac{i-1}{2} & \text{if } i \text{ is even, } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\frac{(n+2)(j-1)}{2} + \frac{i}{2} & \text{if } i \text{ is even, } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n,
\end{array} \right. \\
f(u^j_i) &= \frac{(n+2)(j-1)}{2} + \frac{i}{2} \text{ if } i \text{ is odd, } 2 \leq i \leq n.
\end{align*}
\]
For $1 \leq j \leq m$ and $j$ is even,
\[
\begin{aligned}
&\text{let } f(u'_j) = \begin{cases} 
\frac{(n+2)(j-2)}{2} + 1 + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if } i \text{ is odd, } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\frac{(n+2)(j-2)}{2} + 1 + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{i}{2} \right\rfloor & \text{if } i \text{ is even, } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n,
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
f(u'_j) = \frac{(n+2)(j-2)}{2} + 1 + \left\lfloor \frac{n}{2} \right\rfloor + i & \text{ if } i \text{ is even, } 2 \leq i \leq n.
\end{aligned}
\]

Let $v_iv_j$ be a transformed edge in $T$ for some indices $i$ and $j$, $1 \leq i \leq j \leq m$ and let $P_1$ be the ept that deletes this edge $v_iv_j$ and adds the edge $v_{i+t}v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ as also the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$, it follows that $i + t + 1 = j - t$ which implies $j = i + 2t + 1$. Therefore, $i$ and $j$ are of opposite parity, that is, $i$ is odd and $j$ is even or vice-versa.

The induced label of the edge $v_iv_j$ is given by
\[
\begin{aligned}
f^*(v_iv_j) &= f^*(v_iv_{i+2t+1}) = f(v_i) + f(v_{i+2t+1}) \\
&= (n+2)(i+t) \text{ and }
\end{aligned}
\]
\[
\begin{aligned}
f^*(v_{i+t}v_{j-t}) &= f^*(v_{i+t}v_{i+t+1}) = f(v_{i+t}) + f(v_{i+t+1}) \\
&= (n+2)(i+t).
\end{aligned}
\]

Therefore, $f^*(v_iv_j) = f^*(v_{i+t}v_{j-t})$.

**Case (ii)** $n \equiv 0(\text{mod } 4)$.

For $1 \leq i \leq m$, let $f(v_i) = \begin{cases} 
\frac{i(n+2)}{2} & \text{if } i \text{ is odd} \\
\frac{i(n+2)}{2} & \text{if } i \text{ is even},
\end{cases}$

For $1 \leq j \leq m$ and $j$ is odd,
\[
\begin{aligned}
&\text{let } f(u'_j) = \frac{j(n+2)}{2} - 1 \\
&\text{let } f(u'_j) = \begin{cases} 
\frac{(n+2)(j-1)}{2} + \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if } i \text{ is even, } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\frac{(n+2)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor & \text{if } i \text{ is even, } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n,
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
f(u'_j) = \frac{(n+2)(j-1)}{2} + \left\lfloor \frac{i}{2} \right\rfloor & \text{ if } i \text{ is odd, } 2 \leq i \leq n.
\end{aligned}
\]

For $1 \leq j \leq m$ and $j$ is even,
\[
\begin{aligned}
&\text{let } f(u'_j) = \frac{(j-1)(n+2)}{2} + 1, \\
&\text{let } f(u'_j) = \begin{cases} 
\frac{(n+2)(j-1)}{2} + \left\lceil \frac{i}{2} \right\rceil & \text{if } i \text{ is even, } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\
\frac{(n+2)(j-1)}{2} + \left\lceil \frac{i+1}{2} \right\rceil & \text{if } i \text{ is even, } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n,
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
f(u'_j) = \frac{(n+2)(j-1)}{2} + \left\lceil \frac{i}{2} \right\rceil & \text{ if } i \text{ is odd, } 2 \leq i \leq n.
\end{aligned}
\]
Let \( v_i v_j \) be a transformed edge in \( T \) for some indices \( i, j, 1 \leq i \leq j \leq m \) and let \( P_1 \) be the ept that deletes the edge \( v_i v_j \) and adds the edge \( v_{i+t} v_{j-t} \) where \( t \) is the distance of \( v_i \) from \( v_{i+t} \) as also the distance of \( v_j \) from \( v_{j-t} \). Let \( P \) be a parallel transformation of \( T \) that contains \( P_1 \) as one of the constituent epts.

Since \( v_{i+t} v_{j-t} \) is an edge in the path \( P(T) \), it follows that \( i + t + 1 = j - t \) which implies \( j = i + 2t + 1 \). Therefore \( i \) and \( j \) are of opposite parity, that is, \( i \) is odd and \( j \) is even or vice-versa.

The induced label of the edge \( v_i v_j \) is given by

\[
\begin{align*}
    f^*(v_i v_j) &= f^*(v_i v_{i+2t+1}) = f(v_i) + f(v_{i+2t+1}) \\
        &= (n + 2)(i + t) \quad \text{and} \\
    f^*(v_{i+t} v_{j-t}) &= f^*(v_{i+t} v_{i+t+1}) = f(v_{i+t}) + f(v_{i+t+1}) \\
        &= (n + 2)(i + t).
\end{align*}
\]

Therefore, \( f^*(v_i v_j) = f^*(v_{i+t} v_{j-t}) \).

It can be verified that the induced edge labels of \( T\tilde{o}C_n \) are \( 1, 2, 3, \ldots, m(n + 1) - 1 \) and for all \( a, b \in A \). Hence \( T\tilde{o}C_n \) is a vertex equitable graph.

The vertex equitable labeling pattern of \( T\tilde{o}C_n \), where \( T \) is a \( T_p \)-tree with 8 vertices, is given in Figure 5.

![Figure 5](image-url)
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References


