Skolem Odd Difference Mean Graphs

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ABSTRACT

In this paper we define a new labeling called skolem odd difference mean labeling and investigate skolem odd difference meanness of some standard graphs. Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges. $G$ is said be skolem odd difference mean if there exists a function $f : V(G) \rightarrow \{0, 1, 2, 3, \ldots, p + 3q - 3\}$ satisfying $f$ is $1 - 1$ and the induced map $f^* : E(G) \rightarrow \{1, 3, 5, \ldots, 2q - 1\}$ denoted by $f^*(e) = \left\lceil \frac{|f(u) - f(v)|}{2} \right\rceil$ is a bijection. A graph that admits skolem odd difference mean labeling is called odd difference mean graph. We call skolem odd difference mean labeling as skolem even vertex odd difference mean labeling if all the vertex labels are even.

Keywords: Mean labeling, skolem difference mean labeling, skolem odd difference mean labeling, skolem odd difference mean graph, skolem even vertex odd difference mean labeling.

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1 Introduction

Throughout this paper by a graph we mean a finite, simple and undirected one. The vertex set and the edge set of a graph \( G \) are denoted by \( V(G) \) and \( E(G) \) respectively. Terms and notations not defined here are used in the sense of Harary [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling. An excellent survey of graph labeling is available in [2]. The notion of mean labeling was due to S. Somasundaram and R. Ponraj [7]. A graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges is called a mean graph if there is an injective function \( f \) that maps \( V(G) \) to \( \{0, 1, 2, \ldots, q\} \) such that each edge \( uv \) is labeled with \( \frac{f(u)+f(v)}{2} \) if \( f(u) + f(v) \) is even and \( \frac{f(u)+f(v)+1}{2} \) if \( f(u) + f(v) \) is odd. Then the resulting edge labels are distinct. The concept of odd mean labeling was introduced by K. Manickam and M. Marudai in [3]. Let \( G = (V, E) \) be a graph with \( p \) vertices and \( q \) edges. A graph \( G \) is said to be odd mean if there exists a function \( f : V(G) \to \{0, 1, 2, 3, \ldots, 2q - 1\} \) satisfying \( f \) is \( 1-1 \) and the induced map \( f^* : E(G) \to \{1, 3, 5, \ldots, 2q - 1\} \) defined by

\[
 f^*(uv) = \begin{cases} 
 \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\
 \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd}
\end{cases}
\]

is a bijection.

K. Murugan and A. Subramanian [4] introduced the concept of skolem difference mean labeling and some standard results on skolem difference mean labeling were proved in [5] and [6]. A graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges is said to have skolem difference mean labeling if it is possible to label the vertices \( x \in V \) with distinct elements \( f(x) \) from \( \{1, 2, 3, \ldots, p+q\} \) in such a way that for each edge \( e = uv, f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil \) and the resulting labels of the edges are distinct and are from \( 1, 2, 3, \ldots, q \). A graph that admits skolem difference mean labeling is called skolem difference mean graph. Motivated by the concepts of skolem difference mean labeling [4] and odd mean labeling [3], we introduce a new labeling called skolem odd difference mean labeling. A graph \( G \) is said to be skolem odd difference mean if there exists a function \( f : V(G) \to \{1, 2, 3, \ldots, p+3q-3\} \) satisfying \( f \) is \( 1-1 \) and the induced map \( f^* : E(G) \to \{1, 3, 5, \ldots, 2q - 1\} \) denoted by \( f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil \) is a bijection. A graph that admits skolem odd difference mean labeling is called skolem odd difference mean graph.

We use the following definitions in the subsequent section.

**Definition 1.1.** The corona \( G_1 \odot G_2 \) of the graphs \( G_1 \) and \( G_2 \) is obtained by taking one copy of \( G_1 \) (with \( p \) vertices) and \( p \) copies of \( G_2 \) and then join the \( i^{th} \) vertex of \( G_1 \) to every vertex of the \( i^{th} \) copy of \( G_2 \).
Definition 1.2. The bistar $B_{m,n}$ is a graph obtained from $K_2$ by joining $m$ pendent edges to one end of $K_2$ and $n$ pendent edges to the other end of $K_2$.

Definition 1.3. A caterpillar is a tree with a path $P_m: v_1, v_2, v_3, \ldots, v_m$ called spine with leaves (pendent vertices) known as feet attached to the vertices of the spine by edges known as legs. If every spine $v_i$ is attached with $n_i$ number of leaves, then the caterpillar is denoted by $S(n_1, n_2, \ldots, n_m)$.

Definition 1.4. The graph $P_m \oplus P_n$ is obtained from $P_m$ and $m$ copies of $P_n$ by identifying one pendent vertex of the $i^{th}$ copy of $P_n$ with $i^{th}$ vertex of $P_m$ where $P_m$ is a path of length of $m - 1$.

2 Skolem odd difference mean labeling

Theorem 2.1. Any path $P_n (n \geq 1)$ is a skolem odd difference mean graph.

Proof: Let $V(P_n) = \{u_i : 1 \leq i \leq n\}$. Then $E(P_n) = \{u_iu_{i+1} : 1 \leq i \leq n - 1\}$.

Define $f : V(P_n) \rightarrow \{0, 1, 2, 3, \ldots, p + 3q - 3 = 4n - 6\}$ as follows:

$f(u_{2i-1}) = 4(i - 1)$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$,

$f(u_{2i}) = 4(n - i) - 2$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.

For each vertex label $f$ the induced edge label $f^*$ is defined as follows:

$f^*(u_iu_{i+1}) = 2(n - i) - 1$ for $1 \leq i \leq n - 1$.

It can be verified that $f$ is a skolem odd difference mean labeling of $P_n$. Hence $P_n$ is a skolem odd difference mean graph.

For example, the skolem odd difference mean labeling of $P_6$ is shown in Figure 1.

![Figure 1](image.png)

Theorem 2.2. Any cycle $C_n (n \geq 4)$ is a skolem odd difference mean graph.

Proof: Let $V(C_n) = \{u_i : 1 \leq i \leq n\}$.

Then $E(C_n) = \{u_iu_{i+1}, u_nu_1 : 1 \leq i \leq n - 1\}$.

Define $f : V(C_n) \rightarrow \{0, 1, 2, 3, \ldots, p + 3q - 3\}$ as follows:

Case (i): $n \equiv 0 (mod 4)$

Let $n = 4k$.

$f(v_1) = 0$,

$f(v_{2i+1}) = 4i - 1$ for $1 \leq i \leq k$,

$f(v_{2i}) = 4(n - i) + 1$ for $1 \leq i \leq k$,

$f(v_{2k+2}) = n + 1$,

$f(v_{2k+3}) = n - 4$,

$f(v_{2k+4}) = n + 6$. 
Let $n = 4k + 1$.

For each vertex label, the induced labeling $f^*$ is defined as follows:

Let $e_j = v_jv_{j+1}$ for $1 \leq j \leq n - 1$ and $e_n = v_nv_1$.

$f^*(e_j) = 2n + 1 - 2j$ for $1 \leq j \leq 2k + 1$,

It can be verified that $f$ is a skolem odd difference mean labeling of $C_n$.

Case(ii): $n \equiv 1(\text{mod } 4)$

Let $n = 4k + 2$.

For each vertex label, the induced labeling $f^*$ is defined as follows:

Let $e_j = v_jv_{j+1}$ for $1 \leq j \leq n - 1$ and $e_n = v_nv_1$.

$f^*(e_j) = 2n + 1 - 2j$ for $1 \leq j \leq 2k + 1$,

It can be verified that $f$ is a skolem odd difference mean labeling of $C_n$.

Case(iii): $n \equiv 2(\text{mod } 4)$

Let $n = 4k + 3$.

For each vertex label, the induced labeling $f^*$ is defined as follows:

Let $e_j = v_jv_{j+1}$ for $1 \leq j \leq n - 1$ and $e_n = v_nv_1$.

$f^*(e_j) = 2(i - 2k) - 3$ for $2k + 2 \leq j \leq 4k + 2$.

It can be verified that $f$ is a skolem odd difference mean labeling of $C_n$.

Case(iv): $n \equiv 3(\text{mod } 4)$

Let $n = 4k + 3$. 

\[ f(x) = 4 - (i + 1) \text{ for } 1 \leq i \leq k - 2, \]
\[ f(x) = 4 + (i + 1) + 2 \text{ for } 1 \leq i \leq k - 2. \]
\[ f(v_1) = 0, \]
\[ f(v_{2i+1}) = 4i - 1 \text{ for } 1 \leq i \leq k + 1, \]
\[ f(v_{2i}) = 4(n - i) + 1 \text{ for } 1 \leq i \leq k + 1, \]
\[ f(v_{2k+4}) = n - 2, \]
\[ f(v_{2k+5}) = n + 3, \]
\[ f(v_{2k+6}) = n - 7, \]
\[ f(v_{2k+4+2i}) = n - 7 - 4i \text{ for } 1 \leq i \leq k - 2. \]
\[ f(v_{2k+5+2i}) = n + 3 + 4i \text{ for } 1 \leq i \leq k - 1. \]

For each vertex label, the induced labeling \( f^* \) is defined as follows:

\[ f^*(e_j) = 2i - 1 \text{ for } 1 \leq i \leq n. \]

It can be verified that \( f \) is a skolem odd difference mean labeling of \( C_n \).

The skolem odd difference mean labeling of \( C_7 \) is shown in Figure 2.

\[ \begin{align*}
0 & \quad 10 \\
21 & \quad 25 \\
3 & \quad 7
\end{align*} \]

Figure 2

**Theorem 2.3.** The star graph \( K_{1,n} (n \geq 1) \) admits skolem odd difference mean labeling.

**Proof:** Let \( v_0 \) be the central vertex and \( v_i (1 \leq i \leq n) \) be the pendent vertices of the star \( K_{1,n} \).

Then \( E(K_{1,n}) = \{v_0v_i : 1 \leq i \leq n\} \)

Define \( f : V(K_{1,n}) \to \{0, 1, 2, 3, 4, \ldots, p + 3q - 3 = 4n - 2\} \) as follows:

\[ f(v_0) = 0, \]
\[ f(v_i) = 4i - 2 \text{ for } 1 \leq i \leq n. \]

For each vertex label \( f \) the induced edge label \( f^* \) is defined as follows:

\[ f^*(e_{v_0v_i}) = 2i - 1 \text{ for } 1 \leq i \leq n. \]

It can be verified that \( f \) is a skolem odd difference mean labeling of \( K_{1,n} \). Hence \( K_{1,n} \) admits skolem odd difference mean labeling.

The skolem odd difference mean labeling of \( K_{1,6} \) is shown in Figure 3.
Theorem 2.4. The graph $P_n \odot K_1, n(n \geq 1)$ is a skolem odd difference mean graph.

Proof: Let $u_i(1 \leq i \leq n)$ be the vertices of $P_n$. Let $v_i(1 \leq i \leq n)$ be the pendant vertices joined with $u_i(1 \leq i \leq n)$ by an edge. Define $f : V(P_n \odot K_1) \to \{0, 1, 2, 3, 4, \ldots, p + 3q - 3 = 8n - 6\}$ as follows:

- $f(u_{2i-1}) = 8(n - i) + 2$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil$,
- $f(u_{2i}) = 4 + 8(i - 1)$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,
- $f(v_{2i-1}) = 8(i - 1)$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil$,
- $f(v_{2i}) = 8(n - i) - 2$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

For each vertex label $f$ the induced edge label $f^*$ is defined as follows:

- $f^*(u_iu_{i+1}) = 4(n - i) - 1$ for $1 \leq i \leq n - 1$,
- $f^*(u_iv_j) = 4(n - i) + 1$ for $1 \leq i \leq n$.

It can be verified that $f$ is a skolem odd difference mean labeling of $P_n \odot K_1$. Hence $P_n \odot K_1$ is a skolem odd difference mean graph.

For example, the skolem odd difference mean labeling of $P_5 \odot K_1$ is shown in Figure 4.

Theorem 2.5. The coconut tree $T(n, m)$, obtained by identifying the central vertex of the star $K_1,m$ with a pendant vertex of a path $P_n$ is a skolem odd difference mean graph.

Proof: Let $v_0, v_1, v_2, \ldots, v_n$ be the vertices of a path, having path length $n(n \geq 1)$ and $u_1, u_2, \ldots, u_m$ be the pendant vertices being adjacent with $v_0$. Define $f : V(T(n, m)) \to \{0, 1, 2, 3, 4, \ldots, p + 3q - 3 = 4(m + n) - 2\}$ as follows:

- $f(v_0) = 0$,
- $f(u_i) = 4(m + n - i) + 2$ for $1 \leq i \leq m$,
- $f(v_j) = 4n - 2j$ for $1 \leq j \leq n$ and $j$ is odd,
- $f(v_j) = 2j$ for $1 \leq j \leq n$ and $j$ is even.

For each vertex label $f$ the induced edge label $f^*$ is defined as follows:
\[ f^*(v_0u_i) = 2(m + n - i) + 1 \] for \( 1 \leq i \leq m, \]
\[ f^*(v_0v_1) = 2n - 1, \]
\[ f^*(v_jv_{j+1}) = 2(n - j) - 1 \] for \( 1 \leq j \leq n - 1. \]

It can be verified that \( f \) is a skolem odd difference mean labeling of \( T(n,m) \). Hence \( T(n,m) \) is a skolem odd difference mean graph. \( \Box \)

For example, the skolem odd difference mean labeling of \( T(4,6) \) is shown in Figure 5.

\[ f^*(v_0u_i) = 2(m + n - i) + 1 \] for \( 1 \leq i \leq m, \]
\[ f^*(v_0v_1) = 2n - 1, \]
\[ f^*(v_jv_{j+1}) = 2(n - j) - 1 \] for \( 1 \leq j \leq n - 1. \]

It can be verified that \( f \) is a skolem odd difference mean labeling of \( T(n,m) \). Hence \( T(n,m) \) is a skolem odd difference mean graph. \( \Box \)

For example, the skolem odd difference mean labeling of \( T(4,6) \) is shown in Figure 5.

\[ f^*(v_0u_i) = 2(m + n - i) + 1 \] for \( 1 \leq i \leq m, \]
\[ f^*(v_0v_1) = 2n - 1, \]
\[ f^*(v_jv_{j+1}) = 2(n - j) - 1 \] for \( 1 \leq j \leq n - 1. \]

It can be verified that \( f \) is a skolem odd difference mean labeling of \( T(n,m) \). Hence \( T(n,m) \) is a skolem odd difference mean graph. \( \Box \)

The skolem odd difference mean labeling of \( B_{5,7} \) is shown in Figure 6.
Theorem 2.7. The caterpillar $S(n_1, n_2, \ldots, n_m)$ is a skolem odd difference mean graph.

Proof: Let $V(S(n_1, n_2, \ldots, n_m)) = \{v_j, u_i : 1 \leq i \leq n_j, 1 \leq j \leq m\}$.
Let $E(S(n_1, n_2, \ldots, n_m)) = \{v_jv_{j+1} : 1 \leq j \leq m - 1 \text{ and } v_ju_i^j : 1 \leq i \leq n_j, 1 \leq j \leq m\}$.
Define $f : V(S(n_1, n_2, \ldots, n_m)) \rightarrow \{0, 1, 2, 3, 4, \ldots, p + 3q - 3 = 4(m + n_1 + n_2 + \ldots + n_m) - 6\}$ as follows:

\[ f(v_j) = 4(m + n_1 + n_2 + \ldots + n_m) - 4(n_2 + n_4 + \ldots + n_{j-1}) - 2(j + 2) \text{ for } 1 \leq j \leq m \text{ and } j \text{ is odd}, \]
\[ f(u_i^j) = 4(m + n_1 + n_2 + \ldots + n_{j-1}) + 2(j - 2) \text{ for } 1 \leq j \leq m \text{ and } j \text{ is even}, \]
\[ f(u_i^{j+1}) = 2(m + n_{j+1} + \ldots + n_m) - 2(j - 1) \text{ for } 1 \leq j \leq m - 1, \]
\[ f^*(e_j) = 2(m + n_{j+1} + \ldots + n_m) - 2(i + j) + 1 \text{ for } 1 \leq i \leq n_j, 1 \leq j \leq m. \]

It can be verified that $f$ is a skolem odd difference mean labeling of $S(n_1, n_2, \ldots, n_m)$.
Hence $S(n_1, n_2, \ldots, n_m)$ is a skolem odd difference mean graph. \hfill \blacksquare

An example for skolem odd difference mean labeling of $S(4, 2, 3, 2)$ is shown in Figure 7.

Theorem 2.8. The graph $P_m \circ P_n$ is a skolem odd difference mean graph.

Proof: Let $V(P_m \circ P_n) = \{v_j, u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$.
Let $E(P_m \circ P_n) = \{v_{t+1}v_t, u_iu_{i+1}^t : 1 \leq t \leq m - 1, 1 \leq i \leq n - 1, 1 \leq j \leq m\}$ with $v_j = u_n^j, 1 \leq j \leq m$. 
Define $f : V(P_m @ P_n) = \{0, 1, 2, 3, \ldots, p + 3q - 3 = 4mn - 6\}$

If $n$ is odd, define

$$f(v_j) = \begin{cases} 
2(nj - 1) & \text{for } 1 \leq j \leq m \text{ and } j \text{ is odd} \\
4(mn - 1) - 2n(j - 1) & \text{for } 1 \leq j \leq m \text{ and } j \text{ is even} 
\end{cases}$$

If $n$ is even, define

$$f(v_j) = \begin{cases} 
2n(2m - j) - 2 & \text{for } 1 \leq j \leq m \text{ and } j \text{ is odd} \\
2n(j - 1) & \text{for } 1 \leq j \leq m \text{ and } j \text{ is even} 
\end{cases}$$

Then $E$ is a skolem odd difference mean graph.

Let $e_j = v_j v_{j+1}$ for $1 \leq j \leq m - 1$ and $e^*_j = u^j_1 u^j_{i+1}$ for $1 \leq i \leq n - 1, 1 \leq j \leq m$. For each vertex label $f$ the induced edge label $f^*$ is defined as follows:

$$f^*(e_j) = 2n(m - j) - 1 \text{ for } 1 \leq j \leq m - 1,$$

$$f^*(e^*_j) = 2n(m - j + 1) - 2i - 1 \text{ for } 1 \leq i \leq n - 1, 1 \leq j \leq m \text{ and } j \text{ is odd},$$

$$f^*(e^*_j) = 2n(m - j) + 2i - 1 \text{ for } 1 \leq i \leq n - 1, 1 \leq j \leq m \text{ and } j \text{ is even}.$$ 

It can be verified that $f$ is a skolem odd difference mean labeling of $P_m @ P_n$. Hence $P_m @ P_n$ is a skolem odd difference mean graph.

The skolem odd difference mean labeling of $P_4 @ P_4$ is shown in Figure 8.

![Figure 8](image_url)

**Theorem 2.9.** The graph $P_m @ 2P_n$ is a skolem odd difference mean graph.

**Proof:** Let $V(P_m @ 2P_n) = \{v_j, u^j_{1,i}, u^j_{2,i} : 1 \leq j \leq m, 1 \leq i \leq n\}$ with $u^j_{1,n} = u^j_{2,n} = v_j$ for $1 \leq j \leq m$.

Then $E(P_m @ 2P_n) = \{v_j v_{j+1}, u^j_{1,i} u^j_{1,i+1}, u^j_{2,i} u^j_{2,i+1} : 1 \leq j \leq m - 1, 1 \leq i \leq n - 1\}$.

Define $f : V(P_m @ 2P_n) = \{0, 1, 2, 3, \ldots, p + 3q - 3 = 4m(2n - 1) - 6\}$ as follows:
If \( n \) is odd, define
\[
    f(v_j) = \begin{cases} 
        2j(2n - 1) - 2n & \text{for } 1 \leq j \leq m \text{ and } j \text{ is odd} \\
        4n(2m - j + 1) - 2(2m - j + 7) & \text{for } 1 \leq j \leq m \text{ and } j \text{ is even}
    \end{cases}
\]
and if \( n \) is even, then
\[
    f(v_j) = \begin{cases} 
        2(2n - 1)(2m - j) + 2(n - 2) & \text{for } 1 \leq j \leq m \text{ and } j \text{ is odd} \\
        2j(2n - 1) - 2n & \text{for } 1 \leq j \leq m \text{ and } j \text{ is even}
    \end{cases}
\]
\[
    f(u_{1,i}^j) = 4n(j - 1) - 2(j - i) \quad \text{for } 1 \leq i \leq n, \; 1 \leq j \leq m \text{ and } i \text{ is odd, } j \text{ is odd,}
\]
\[
    f(u_{1,i}^j) = 4m(2n - 1) - 4n(j - 1) + 2(j - i - 2) \quad \text{for } 1 \leq i \leq n, \; 1 \leq j \leq m \text{ and } i \text{ is even, } j \text{ is odd,}
\]
\[
    f(u_{2,i}^j) = 4m(2n - 1) - 2(2n - 1)(j - 2) - 4n - 2i \quad \text{for } 1 \leq i \leq n, \; 1 \leq j \leq m, \; i \text{ is odd, } j \text{ is even,}
\]
\[
    f(u_{2,i}^j) = 4n(j - 1) + 2(i - j) \quad \text{for } 1 \leq i \leq n, \; 1 \leq j \leq m \text{ and } i \text{ is even, } j \text{ is even,}
\]
\[
    f(u_{2,i}^j) = 2j(2n - 1) - 2i \quad \text{for } 1 \leq i \leq n, \; 1 \leq j \leq m \text{ and } i \text{ is odd, } j \text{ is odd,}
\]
\[
    f(u_{2,i}^j) = (2n - 1)(4m - 2j) + 2(n + i - 7) \quad \text{for } 1 \leq i \leq n, \; 1 \leq j \leq m \text{ and } i \text{ is even, } j \text{ is odd,}
\]
\[
    f(u_{2,i}^j) = (2n - 1)(4m - 2j) + 2(i - 2) \quad \text{for } 1 \leq i \leq n, \; 1 \leq j \leq m \text{ and } i \text{ is odd, } j \text{ is even.}
\]

Let \( e_j = v_jv_{j+1} \) for \( 1 \leq j \leq m - 1 \),
\[
    e_1' = u_{1,i}^j, u_{1,i+1}^j \quad \text{for } 1 \leq i \leq n - 1, \; 1 \leq j \leq m, \]
\[
    e_2' = u_{2,i}^j, u_{2,i+1}^j \quad \text{for } 1 \leq i \leq n - 1 \text{ and } 1 \leq j \leq m.
\]

For each vertex label \( f \) the induced edge label \( f^* \) is defined as follows:
\[
    f^*(e_j) = 2(2n - 1)(m - j) - 1 \quad \text{for } 1 \leq j \leq m - 1, \\
    f^*(e_1') = 2(2n - 1)(m - j) + 2(2n - i) - 3 \quad \text{for } 1 \leq i \leq n - 1, \; 1 \leq j \leq m \]
\[
    f^*(e_2') = 2(2n - 1)(m - j) + 2i - 1 \quad \text{for } 1 \leq i \leq n - 1, \; 1 \leq j \leq m.
\]

It can be verified that \( f \) is a skolem odd difference mean labeling of \( P_m @ 2P_n \). Hence \( P_m @ 2P_n \) is a skolem odd difference mean graph.

The skolem odd difference mean labeling of \( P_3 @ 2P_4 \) is shown in Figure 9.

![Figure 9](image)

**Theorem 2.10.** The complete graph \( K_n, n > 3 \) is not a skolem odd difference mean graph.
Proof: In a complete graph $K_n$, the number of edges $q = \frac{n(n-1)}{2}$. Therefore $p + 3q - 3 = \frac{3n^2 - n - 6}{2}$.

To get $2q - 1 = n^2 - n - 1$ as edge label, the minimum vertex label is $2n^2 - 2n - 3$.

But $\frac{3n^2 - n - 6}{2} < 2n^2 - 2n - 3$ for all $n \geq 4$.

Therefore $2q - 1$ cannot occur as an edge label of $K_n$ for $n \geq 4$.

Hence, $K_n$ is not a skolem odd difference mean graph. ■

Theorem 2.11. The graph $K_{2,n}$ is a skolem odd difference mean graph if $n \leq 2$.

Proof: $K_{2,1} = P_3$ and $K_{2,2} = C_4$. Hence $K_{2,n}$ is a skolem odd difference mean graph for $n \leq 2$. ■

Theorem 2.12. The graph $K_{2,n}(n \geq 3)$ is not a skolem odd difference mean graph.

Proof: The graph $K_{2,n}$ has $n + 2$ vertices and $2n$ edges.

For $n \geq 3$, $p + 3q - 3 = 7n - 1$.

That is the maximum possible vertex label of $K_{2,n}$ is $7n - 1$.

Therefore it is not possible to get an edge with label $2q - 1 = 4n - 1$.

Hence $K_{2,n}(n \geq 3)$ is not a skolem odd difference mean labeling. ■

3 Skolem Even Vertex Odd Difference Mean Labeling

A graph $G$ is said to be skolem even vertex odd difference mean if there exists a function $f: V(G) \rightarrow \{0, 2, 4, \ldots, p + 3q - 3\}$ satisfying $f$ is 1-1 and the induced map $f^*: E(G) \rightarrow \{1, 3, 5, \ldots, 2q - 1\}$ defined by $f^*(e) = \left\lceil \frac{|f(u) - f(v)|}{2} \right\rceil$ is a bijection. A graph that admits skolem even vertex odd difference mean labeling is called skolem even vertex odd difference mean graph. That is, we call a skolem odd difference mean labeling $f$ of a graph $G$ as skolem even vertex odd difference mean labeling if all the vertex labels $f(v)$ of $G$ are even.

Theorem 3.1. The following graphs are even vertex odd difference mean.

(i) $P_n(n \geq 1)$

(ii) $K_{1,n}(n \geq 1)$

(iii) $P_n \odot K_1(n \geq 1)$

(iv) The coconut tree $T(n, m)$, obtained by identifying the central vertex of the star $K_{1,m}$ with a pendent vertex of a path $P_n$

(v) $B(m, n)(m, n \geq 1)$

(vi) Caterpillar $S(n_1, n_2, \ldots, n_m)$

(vii) $P_m \oplus P_n(m, n \geq 1)$
(viii) $P_m@2P_n (m,n \geq 1)$

**Proof:** The proof follows from Theorems 2.1, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9. ■

**Theorem 3.2.** The cycle $C_n, (n \geq 3)$ is not a skolem even vertex odd difference mean graph.

**Proof:** Let $f$ be a skolem even vertex odd difference mean labeling of $C_n$. Here $p + 3q - 3 = 4n - 3$ is an odd number. Therefore the maximum possible vertex label in $C_n$ is $4n - 4$. Hence, the edge label $2n - 1$ cannot occur. Thus $C_n (n \geq 3)$ is not a skolem even vertex odd difference mean graph.

**References**

1. F. Harary, Graph theory, Addison Wesley, Massachusetts, (1972).


