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Three Graceful Operations

Sarah Minion*[1] and Christian Barrientos†[2]

[1, 2]Department of Mathematics, Clayton State University, Morrow, Georgia 30260, USA

ABSTRACT

A graph of size n is said to be *graceful* when is possible to assign distinct integers from $\{0, 1, \ldots, n\}$ to its vertices and $\{|f(u)-f(v)| : uv \in E(G)\}$ consists of n integers. In this paper we present broader families of graceful graphs; these families are obtained via three different operations: the third power of a caterpillar, the symmetric product of G and $\overline{K_2}$, and the disjoint union of G and P_m, where G is a special type of graceful graph named α-graph. Moreover, the majority of the graceful labelings obtained here correspond to the most restrictive kind, they are α-labelings. These labelings are in the core of this research area due to the fact that they can be used to create other types of graph labelings, almost independently of the nature of these labelings.

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*Email: sarah.m.minion@gmail.com
†Email: chr_barrientos@yahoo.com

1 Introduction

A *graceful labeling* of a graph G of size n is an injective function $f : V(G) \to \{0, 1, \ldots, n\}$ such that when each edge uv of G has assigned the *weight* $|f(u) - f(v)|$, all induced weights are distinct. If G admits a graceful labeling is said to be a *graceful graph*. Let f be a graceful labeling of a graph G; suppose that there exists an integer λ such that for each edge uv of G, either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$, then f is said to be an *α-labeling* of G with *boundary value* λ and G is called an *α-graph*. An α-graph is necessarily bipartite and λ is the smaller of the two end-vertices of the edge of weight 1. Suppose G is an α-graph of size n. Let f be an α-labeling of G with boundary value λ. If $\{A, B\}$ is the bipartition of $V(G)$, we may assume, without loss of generality, that $A = \{v \in V(G) : f(v) \leq \lambda\}$ and $B = \{v \in V(G) : f(v) > \lambda\}$. When a positive constant k is added to every vertex in B, the resulting labeling is a *k-graceful labeling*, where the weights are $\{k+1, k+2, \ldots, k+n\}$. This operation was introducted independently by Maheo and Thuiller [8] and Slater [11]. We refer to this operation as an *amplification*.

We study here three operations on graphs that result on graceful graphs. In Section 2 we introduce an operation related to the n^{th} power G^n of a connected graph G, where $n \geq 1$; G^n is that graph with $V(G^n) = V(G)$ for which $uv \in E(G^n)$ if and only if $1 \leq d_G(u, v) \leq n$. In our case, G^n denotes the graph with $V(G^n) = V(G)$ and $E(G^n) = E(G) \cup \{uv : d_G(u, v) = n\}$. In particular, we prove that C^3 is graceful when C is a caterpillar; futhermore, the labeling obtained satisfies the conditions to be an α-labeling. In Section 3 we generalized the result of Seoud and El Sakhawi [10] about the symmetric product $P_n \oplus \overline{K_2}$. We prove that $G \oplus \overline{K_2}$ is an α-graph when G is an α-graph.

Section 4 deals with the union of graphs, specifically we prove that $G \cup P_m$ is graceful provided that G admits an α-labeling that does not assign the integer $\lambda + 2$ as a label, where λ is its boundary value. We finish this section listing several families of α-graphs that admit α-labelings of this kind.

Alpha labelings of graphs are extremely useful in the area of graph decompositions [9]; in addition, they are located in the center of this research area because they can be modified to produce other types of labeling. For more details at this respect, we recommend [4] and [6].

For more information about graph labelings, the interested reader is refered to Gallian's dynamic survey [5]. In this paper we follow the notation and terminology used in [3] and

[5].

2 A Graceful Third Power

In his dynamic survey, Gallian [5] defines P_n^k, the k^{th} *power of* P_n, as the graph obtained from the path of order n by adding edges that join all vertices u and v with $d(u,v) = k$. This definition differs of the one given by Chartrand and Lesniak [3], where $d(u,v) \leq k$. We use here the definition given by Gallian and we extend it to any connected graph. We are aware of only one result related to the gracefulness of a power of a graph. Kang et al [6] proved that P_n^2 is graceful. In our first theorem, we prove that C^3 is graceful for every caterpillar of diameter at least 3; moreover, the resulting labeled graph is an α-graph.
Let C be a caterpillar of diameter at least three. Its cube C^3 is the bipartite graph defined by $V(C^3) = V(C)$ and $E(C^3) = E(C) \cup \{uv : d_C(u,v) = 3\}$.
In his seminal paper, Rosa [9] proved that all caterpillars are α-graphs. For the sake of completeness, we show in Figure 1 the technique used to find this α-labeling.

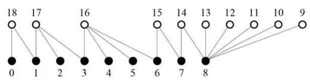

Figure 1: α-labeling of a caterpillar

Suppose C is a caterpillar of size q which has been α-labeled using Rosa's scheme; let g be the α-labeling obtained and λ be its boundary value. If $\{A, B\}$ is the bipartition of $V(C)$ we assume $A = \{u \in V(G) : g(u) \leq \lambda\}$ and $B = \{v \in V(C) : g(v) > \lambda\}$. Thus, $A = \{u_i : 1 \leq i \leq \lambda + 1 \text{ and } g(u_i) = i - 1\}$ and $B = \{v_j : 1 \leq j \leq q - \lambda \text{ and } g(v_j) = q + 1 - j\}$. Let $v \in V(G)$, by $N(v)$ we understand the subset of $V(G)$ consisting of all the vertices adjacent to v, that is the neighborhood of v. In the next theorem we assume that this labeling is known.

Theorem 1. *If C is a caterpillar of diameter at least three, then C^3 is an α-graph.*

Proof. Let n be the size of C^3. Consider the following labeling of the vertices of C^3:

- $f(u_i) = g(u_i)$ for every $u_i \in A$
- $f(v_1) = n$
- $f(v_j) = n + \min\{f(u) : u \in N(v_j)\} - \sum_{i=1}^{j-1} \deg(v_i)$ for every $j \geq 2$.

We claim that f is an α-labeling of C^3.
First, we prove that f is an injective function. Clearly, when f is restricted to A, f is injective. Consider now the labels of v_j and v_{j+1}, $j \geq 2$; we claim that $f(v_{j+1}) > f(v_j)$. Since $d(v_j, v_{j+1}) = 2$, $N(v_j) \cap N(v_{j+1}) \neq \varnothing$, thus $|N(v_j) \cap N(v_{j+1})| > 0$. Then,

$$\begin{gathered}
\min\{f(y) : y \in N(v_{j+1})\} - \min\{f(x) : x \in N(v_j)\} < \deg(v_j) \\
\min\{f(y) : y \in N(v_{j+1})\} - \deg(v_j) < \min\{f(x) : x \in N(v_j)\} \\
n + \min\{f(y) : y \in N(v_{j+1})\} - \sum_{i=1}^{j-1} \deg(v_i) - \deg(v_j) \\
< n + \min\{f(x) : x \in N(v_j)\} - \sum_{i=1}^{j-1} \deg(v_i), \\
n + \min\{f(y) : y \in N(v_{j+1})\} - \sum_{i=1}^{j} \deg(v_i) \\
< n + \min\{f(x) : x \in N(v_j)\} - \sum_{i=1}^{j-1} \deg(v_i), \\
\therefore f(v_{j+1}) < f(v_j).
\end{gathered}$$

Therefore, f is an injective function.
Now, notice that $\max\{f(x) : x \in A\} < \min\{f(y) : y \in B\}$; in fact, $|A| - 1 = \max\{f(x) : x \in A\}$ and $f(v_{|B|}) = |A|$.

Since,

$$\sum_{i=1}^{|A|} \deg(u_i) = \sum_{j=1}^{|B|} \deg(v_j) = n$$
$$\sum_{i=1}^{|A|} \deg(u_i) - n = n - \sum_{j=1}^{|B|} \deg(v_j)$$
$$\sum_{i=1}^{|A|} \deg(u_i) - n + \deg(v_{|B|}) = n - \sum_{j=1}^{|B|-1} \deg(v_j)$$
$$f(v_{|B|}) = \sum_{i=1}^{|A|} \deg(u_i) - n + \deg(v_{|B|}) + \min\{f(y) : y \in N(v_{|B|})\}$$
$$\therefore f(v_{|B|}) = \deg(v_{|B|}) + \min\{f(y) : y \in N(v_{|B|})\} = |A|.$$

In the following step we need to prove that all the induced weights are distinct. Let $v_j \in B$ and $N(v_j) = \{u_{k+1}, u_{k+2}, \ldots, u_{k+\deg(v_j)}\}$. Since $f(u_{k+1}) < f(u_{k+2}) < \cdots < f(u_{k+\deg(v_j)})$, the edge $v_j u$ of biggest weight is $v_j u_{k+1}$, which weight is $f(v_j) - f(u_{k+1})$. Consider now $v_{j-1} \in B$ and $N(v_{j-1}) = \{u_{t+1}, u_{t+2}, \ldots, u_{t+\deg(v_{j-1})}\}$. We know that

$$f(u_{t+\deg(v_{j-1})}) \geq f(u_{k+1}),$$

that is

$$t + \deg(v_{j-1}) \geq k + 1.$$

Since

$$f(v_j) = n + \min\{f(x) : x \in N(v_j)\} - \sum_{i=1}^{j-1} \deg(v_i)$$
$$= n + k + 1 - \sum_{i=1}^{j-2} \deg(v_i) - \deg(v_{j-1})$$
$$\therefore f(v_j) - f(u_{k+1}) = f(v_j) - (k+1) = n - \sum_{i=1}^{j-2} \deg(v_i) - \deg(v_{j-1}).$$

On the other side,

$$f(v_{j-1}) - f(u_{t+\deg(v_{j-1})}) = f(v_{j-1}) - t - \deg(v_{j-1})$$
$$= n + \min\{f(y) : y \in N(v_{j-1})\} - \sum_{i=1}^{j-2} \deg(v_i) - t - \deg(v_{j-1})$$
$$= n - \sum_{i=1}^{j-2} \deg(v_i) - \deg(v_{j-1}) + t + 1 - t = f(v_j) - f(u_{k+1}) + 1.$$

Therefore,

$$(f(v_{j-1}) - f(u_{t+\deg(v_{j-1})})) - (f(v_j) - f(u_{k+1})) = 1$$

Thus, all induced weights are distinct, being n the largest weight. Hence f is a graceful labeling of C^3. Take $\lambda = |A| - 1$; notice that $f(u) \leq \lambda < f(v)$ for every ordered pair $(u, v) \in A \times B$. Thus, f is an α-labeling of C^3 as we claimed. □

We must observe here that the labeling f does not assign the number $\lambda + 2$ as a label of C^3. In Figure 2 we show an example of this construction for a caterpillar of size 18.

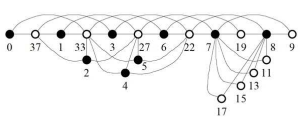

Figure 2: α-labeling of C^3

3 A Graceful Symmetric Product

In [10] Seoud and El Sakhawi introduced the following operation of graphs. The *symmetric product* $G_1 \oplus G_2$, of two graphs G_1 and G_2, is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u, v)(u', v') : uu' \in E(G_1) \text{ or } vv' \in E(G_2) \text{ but not both}\}$. In Figure 3 we show an example of this operation. Seoud and El Sakhawi applied this operation to graphs of the form $P_n \oplus \overline{K_2}$, where P_n is the path of order n and $\overline{K_2}$ is the null graph of order 2, they proved that this graph is arbitrarily graceful. In our next theorem we extend this result proving that when G_1 is an α-graph, the symmetric product $G_1 \oplus \overline{K_2}$ is also an α-graph, which implies that it is arbitrarily graceful.

Theorem 2. *If G is an α-graph, then the symmetric product $G \oplus \overline{K_2}$ is an α-graph.*

Proof. Suppose that g is an α-labeling with boundary value λ of a graph G of size n. Within $G \oplus \overline{K_2}$, we can distinguish two copies of G, named G^1 and G^2. Consider the following labeling of the vertices of $G \oplus \overline{K_2}$:

$$f(v) = \begin{cases} 4g(v) & \text{if } v \in V(G^1), \\ 4g(v)+1 & \text{if } v \in V(G^2) \text{ and } g(v) \leq \lambda, \\ 4g(v)-2 & \text{if } v \in V(G^2) \text{ and } g(v) > \lambda. \end{cases}$$

We claim that f is an α-labeling of $G \oplus \overline{K_2}$.
First of all, notice that f is injective and that the labels assigned by it are in the set $\{0, 1, \ldots, 4n\}$. Let $uv \in E(G)$ such that $g(v) - g(u) = i$ for any $1 \leq i \leq n$. In $G \oplus \overline{K_2}$ there is an edge u^1v^1, with $u^1, v^1 \in V(G^1)$, which weight is $f(v^1) - f(u^1) = 4g(v) - 4g(u) = 4(g(v) - g(u)) = 4i$. Thus, the edges of G^1 have by weights all the integers in $\{1, 2, \ldots, 4n\}$ congruent to 0 modulo 4. In $G \oplus \overline{K_2}$ there is an edge u^2v^2, with $u^2, v^2 \in V(G^2)$, which weight is $f(v^2) - f(u^2) = (4g(v) - 2) - (4g(u) + 1) = 4(g(v) - g(u)) - 3 = 4i - 3$. So the edges of G^2 have as weights all the integers in $\{1, 2, \ldots, 4n\}$ congruent to 1 modulo 4. In $G \oplus \overline{K_2}$ there is an edge u^1v^2 with $u^1 \in V(G^1)$ and $v^2 \in V(G^2)$, which weight is $f(v^2) - f(u^1) = (4g(v) - 2) - 4g(u) = 4(g(v) - g(u)) - 2 = 4i - 2$. Thus the edges of this form have as weights all the integers in $\{1, 2, \ldots, 4n\}$ congruent to 2 modulo 4. Finally, in $G \oplus \overline{K_2}$ there is an edge u^2v^1 with $u^2 \in V(G^2)$ and $v^1 \in V(G^1)$, which weight is $f(v^1) - f(u^2) = 4g(v) - (4g(u) + 1) = 4(g(v) - g(u)) - 1 = 4i - 1$. Hence the edges of this form have as weights all the integers in $\{1, 2, \ldots, 4n\}$ congruent to 3 modulo 4. Therefore, f is a graceful labeling of $G \oplus \overline{K_2}$. Moreover the integer $4\lambda + 1$ is the boundary value for this labeling, which is in fact an α-labeling of $G \oplus \overline{K_2}$. □

In Figure 3 we show an example of this labeling for the disconnected α-graph of order 8 and size 7, $G = C_6 \cup P_2$.
Notice that the α-labeling produced in this theorem does not assign the integer $4\lambda + 2$ as a label of $G \oplus \overline{K_2}$.

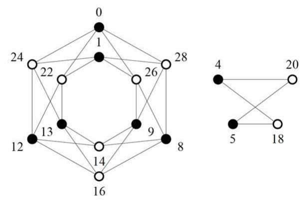

Figure 3: α-labeling of $(C_6 \cup P_2) \oplus \overline{K_2}$

4 A Graceful Union

There are several papers devoted to the gracefulness of disconnected graphs (see [5]). Within this section we analyze the union of a path and a certain type of α-labeled graph. First, we construct an *ad hoc* labeling of the path of size m, P_{m+1}, which is used later together with an amplification of an α-labeling of a graph G to produce an α-labeling of $G \cup P_{m+1}$. We conclude the section with a list of α-graphs that can be used in the place of the graph G just mentioned.

Let G be a graph of size n. Any injective assignment f of nonnegative integers to the vertices of G that induces the weights $1, 2, \dots, n$ is called a *complete labeling* of G. In the next proposition we show that the path P_{m+1} admits a complete labeling other that its well-known α-labeling.

Proposition 3. *The path P_{m+1} admits a complete labeling with labels taken from the set $\{0, 1, \dots, m-1, m+1\}$, $m \geq 3$.*

Proof. Let P_{m+1} be a path of order $m+1$ with $\{A, B\}$ the natural partition of $V(P_{m+1})$; without loss of generality, we are assuming that $|A| \geq |B|$. Let $A = \{u_1, u_2, \dots, u_{|A|}\}$ and $B = \{v_1, v_2, \dots, v_{|B|}\}$, so P_{m+1} is described by the edges u_iv_i and v_iu_{i+1} for every $1 \leq i \leq |B|$. We analyze six cases based on the congruence modulo 3 of $|A|$ and $|B|$, and the fact that $|A| - |B| \leq 1$.

Case I: $|A| \equiv 1 (\text{mod } 3)$ and $|B| \equiv 0 (\text{mod } 3)$.

Let $f : V(P_{m+1}) \to \{0, 1, \dots, m-1, m+1\}$ be the injective function defined below:

For every $i \equiv 2 (\text{mod } 3)$, $f(v_i) = i - 2$, $f(v_{i-1}) = i - 1$, and $f(v_{i+1}) = i$.

For every $i \equiv 1 (\text{mod } 3)$, $f(u_i) = m + 2 - i$, $f(u_{i+1}) = m - i$, and $f(u_{i+2}) = m - 2 - i$.

Case II: $|A| \equiv 0 (\text{mod } 3)$ and $|B| \equiv 0 (\text{mod } 3)$.

The function f is defined as in Case I except for $v_{|B|}$, here $f(v_{|B|}) = |B|$.
Case III: $|A| \equiv 2 (\text{mod } 3)$ and $|B| \equiv 1 (\text{mod } 3)$.
Again, f is defined as in Case I except for $u_{|A|}$, in this case $f(u_{|A|}) = |B| - 1$.
Case IV: $|A| \equiv 1 (\text{mod } 3)$ and $|B| \equiv 1 (\text{mod } 3)$.
In this case the exceptions are $v_{|B|-1}$ and $v_{|B|}$, here $f(v_{|B|-1}) = |B|-1$ and $f(v_{|B|}) = |B|-2$.
Case V: $|A| \equiv 0 (\text{mod } 3)$ and $|B| \equiv 2 (\text{mod } 3)$.
Now we proceed as in Case I except for $u_{|A|-1}$ and $u_{|A|}$, here $f(u_{|A|-1}) = |A| - 1$ and $f(u_{|A|}) = |A|$.
Case VI: $|A| \equiv 2 (\text{mod } 3)$ and $|B| \equiv 2 (\text{mod } 3)$.
In this case we assign the labels as in Case I.
We claim that f, so defined, is a complete labeling of P_{m+1}. Because the proof is similar for all cases, we prove our claim for Case I and leave the remaining cases for the interested reader.
When $i \equiv 2 (\text{mod } 3)$, there is an integer k, $1 \leq k \leq \frac{|B|}{3}$, such that $i = 3k - 1$. It is not difficult to see that the edges $u_{i-1}v_{i-1}$, u_iv_{i-1}, u_iv_i, $u_{i+1}v_i$, $u_{i+1}v_{i+1}$, and $u_{i+2}v_{i+1}$ have weights $m-6k+6$, $m-6k+4$, $m-6k+5$, $m-6k+3$, $m-6k+1$, and $m-6k+2$, respectively. Since $1 \leq k \leq \frac{|B|}{3}$, we have that the induced weights form the set $\{1, 2, \ldots, m\}$. Notice that the larger labels used are $m-1$ and $m+1$, on the vertices u_2 and u_1, respectively. Therefore, f is the complete labeling of P_{m+1} described by the proposition. □

In Figure 4 we show some complete labelings of P_{m+1} corresponding to each case within the proof of the previous proposition.

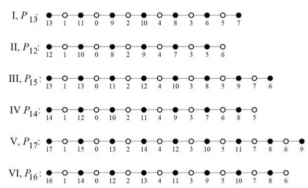

Figure 4: Complete labelings of P_{m+1}

Theorem 4. *Let G be an α-graph of size n and order at most n. If there exists an α-labeling f of G, with boundary value λ, such that $\lambda + 2$ is not a label of G, then $G \cup P_{m+1}$ is a graceful graph for every $m \geq 3$.*

Proof. Suppose that the vertices of G have been labeled using the function f. This labeling is amplified in such a way that the largest induced weight is $m+n$. In this manner, the integer $\lambda+m+2$ is not assigned as a label of G. On the other hand, we apply to P_{m+1} the complete labeling described in Proposition 3 shifted $\lambda+1$ units, so the largest label on P_{m+1} is $\lambda+m+2$. Thus, we have a graceful labeling of $G \cup P_{m+1}$. □

In Figure 5 we show an example for a graph of the form $G \cup P_{m+1}$. The labeling of $G = C_8 \oplus \overline{K_2}$ was obtained using the α-labeling of C_8 given in [6]; the complete labeling of P_8 was obtained using Proposition 3, Case IV, shifted 14 units.

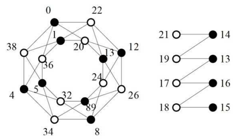

Figure 5: α-labeling of $C_8 \oplus \overline{K_2} \cup P_8$

Notice that when P_{m+1} fits with case I, VI, V, or VI in Proposition 3, the labeling of $G \cup P_{m+1}$ is an α-labeling.
Some α-labeled graphs that satisfy the conditions of Theorem 4 are the complete bipartite graph $K_{m,n}$ for $m, n > 1$, given by Rosa [9] (see also [2]), the kC_4-snake given by Barrientos [1], the cube Q_n for $n \geq 2$ and the book B_{2n} given by Maheo [7]; the symmetric product of the path P_n, $n \geq 2$ with the null graph $\overline{K_2}$, $P_n \oplus \overline{K_2}$, given by Seoud and El Sakhawi [7]; $G_1 \overline{\oplus} G_2$, that is the weak tensor product of two α-graphs, when an α-labeling of G, does not use the label $\lambda+2$, this operation was introduced by Snevily [12]. We must mention that C^3, where C is a caterpillar, labeled as in Section 2 also satisfies the conditions of Theorem 4 as well as those of the graphs $G \oplus \overline{K_2}$ presented in Section 3.

5 Conclusions

The three "graceful" operations considered in this paper provide general results in the area of graph labelings because they work with broader families of graphs. We expect that these results can be extended in forcoming works.

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