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On the domination number of generalized Petersen graphs

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ABSTRACT

Let n and k be integers such that $3 \leq 2k+1 \leq n$. The generalized Petersen graph $GP(n,k) = (V,E)$ is the graph with $V = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and $E = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : 1 \leq i \leq n\}$, where addition is in modulo n. A subset $D \subseteq V$ is a dominating set of $GP(n,k)$ if for each $v \in V \setminus D$ there is a vertex $u \in D$ adjacent to v. The minimum cardinality of a dominating set of $GP(n,k)$ is called the domination number of $GP(n,k)$.
In this paper we give a dynamic programming algorithm for computing the domination number of a given $GP(n,k)$ in $\mathcal{O}(n)$ time and space for every $k = \mathcal{O}(1)$.

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1 Introduction

Let $G = (V,E)$ be a graph with the vertex set V and the edge set E. Here, we study finite, simple and undirected graphs. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of $v \in V$, denoted by $\deg_G(v)$, is the cardinality of $N_G(v)$, that is, $\deg_G(v) = |N_G(v)|$.

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A *dominating set* (DS) of G is a set $D \subseteq V$ with the property that every vertex $v \in V \setminus D$ is adjacent to at least one vertex $u \in D$. The minimum cardinality of a dominating set of G is called the *domination number* of G, denoted by $\gamma(G)$.
Let n and k be integers such that $3 \leq 2k+1 \leq n$. Watkins [5] has introduced the *generalized Petersen* graph $GP(n,k) = (V,E)$ as the graph with the vertex set $V = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : 1 \leq i \leq n\}$, where the subscripts are added modulo n.
Behzad et al. [1] have given an upper bound on, and then Yan et al. [6] and Liu and Zhang [4] have determined the exact value for the domination number of some classes of generalized Petersen graphs. The problem of finding a minimum dominating set of an arbitrary graph is NP-complete [2]. There are polynomial time algorithms to compute the domination number of some of class of graphs such as trees, interval, permutation and series-parallel graphs [3, Chapter 12]. In this paper, we give a linear time and space algorithm based on dynamic programming approach to compute the domination number of $GP(n,k)$, where $k = \mathcal{O}(1)$.

2 Preliminaries

In the rest of the paper we fix integers n and k such that $3 \leq 2k+1 \leq n$. Let $GP(n,3) = (V,E)$ be the generalized Petersen graph with $V = \{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_n\}$ and $E = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : 1 \leq i \leq n\}$. The *semi-generalized Petersen graph* $SGP(n,k) = (V_s, E_s)$ (corresponding to $GP(n,k)$) is a graph with the vertex set

$$V_s = V \cup V_l \cup V_r,$$

where $V_l = \{v_{1-k}, v_{2-k}, \ldots, v_0, u_0\}$ and $V_r = \{u_{n+1}, v_{n+1}, v_{n+2}, \ldots, v_{n+k}\}$ and the edge set

$$E_s = (E \setminus \{u_1u_n, v_{n-k+i}v_i : 1 \leq i \leq k\}) \cup E_l \cup E_r,$$

where $E_l = \{v_{1-k}v_1, v_{2-k}v_2, \ldots, v_0v_k, u_0u_1, u_0v_0\}$ and $E_r = \{u_{n+1}v_{n+1}, u_nu_{n+1}, v_{n-k+1}v_{n+1}, v_{n-k+2}v_{n+2}, \ldots, v_nv_{n+k}\}$. See Fig. 1.
We have $\deg_{SGP(n,k)}(v) = 3$ for every vertex $v \in V$ and $\deg_{SGP(n,k)}(v) < 3$ for every vertex $v \in V_l \cup V_r$.
Let $G' = (V', E')$ be a connected subgraph of $SGP(n,k)$. A subset $D \subseteq V'$ is a *semi dominating set* (SDS) of G' if for each vertex $v \in V' \setminus D$ with $\deg_{G'}(v) = 3$ there is a vertex $u \in D$ adjacent to v. Let G_i^k be the subgraph of $SGP(n,k)$ induced by $V_i = V_l \cup \{u_1, \ldots, u_i\} \cup \{v_1, \ldots, v_i, v_{i+1}, v_{i+k-1}\}$ for each $1 \leq i \leq n+1$. We obtain $G_{n+1}^k = SGP(n,k)$. See Fig. 1(b). Let $b_1, b_2, \ldots, b_{2k+2} \in \{0,1\}$ and let $i \in \{1, 2, \ldots, n+1\}$. In the following we define $\gamma^{b_{2k+2}b_{2k+1}\cdots b_1}(G_i^k)$. Here, $b_{2k+2}, b_{2k+1}, \ldots, b_1$ are corresponding to vertices $v_{i-k}, v_{i-k+1}, \ldots, v_{i-1}, u_{i-1}, u_i, v_i, v_{i+1}, \ldots, v_{i+k-1}$, respectively. Let $j \in \{1, \ldots, 2k+2\}$. The value $\gamma^{b_{2k+2}\cdots b_1}(G_i^k)$ is the minimum cardinality of a SDS D of G_i^k such that if $b_j = 0$, then the corresponding vertex of b_j is not in D and if $b_j = 1$, then the corresponding vertex of b_j is in D. Since there are $2^{2k+2} = 4^{k+1}$ different cases for defining $\gamma^{b_{2k+2}\cdots b_1}(G_j^k)$, in the following we give the complete formal definition of some cases.

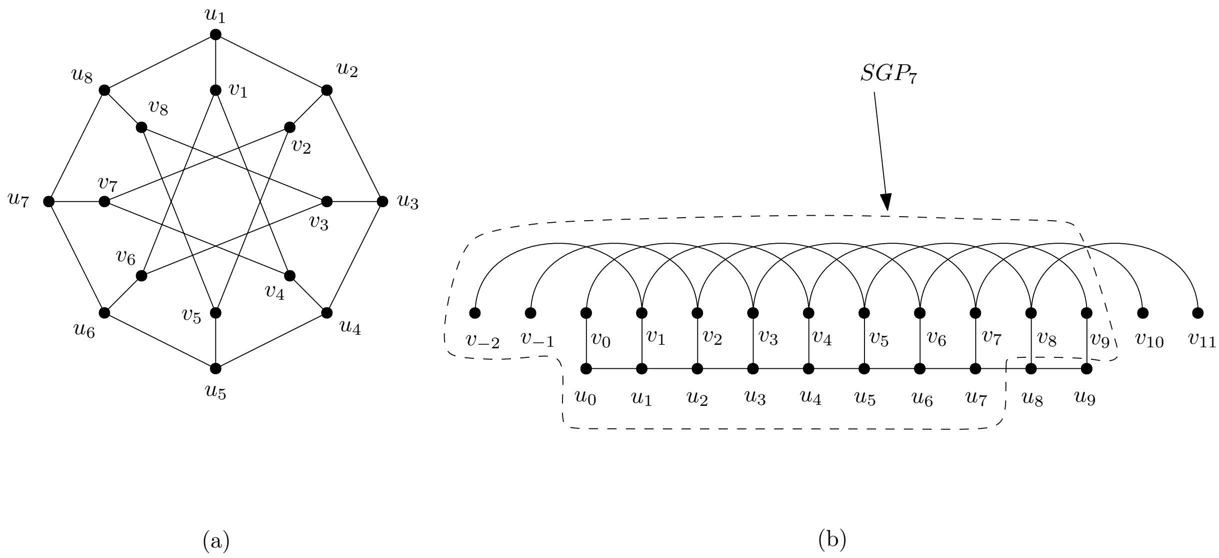

Figure 1: Illustrating (a) $GP(8,3)$ and (b) $SGP(8,3)$ and G_7^3.

- $\gamma^{0\cdots 0}(G_i^k) = \min\{|D| : D$ is a SDS of G_i^k, $v_{i-k} \notin D$, $v_{i-k+1} \notin D, \ldots, v_{i-1} \notin D$, $u_{i-1} \notin D$, $u_i \notin D$, $v_i \notin D$, $v_{i+1} \notin D, \ldots, v_{i+k-1} \notin D\}$,
- $\gamma^{0\cdots 01}(G_i^k) = \min\{|D| : D$ is a SDS of G_i^k, $v_{i-k} \notin D$, $v_{i-k+1} \notin D, \ldots, v_{i-1} \notin D$, $u_{i-1} \notin D$, $u_i \notin D$, $v_i \notin D$, $v_{i+1} \notin D, \ldots, v_{i+k-2} \notin D$, $v_{i+k-1} \in D\}$ and
- $\gamma^{1\cdots 1}(G_i^k) = \min\{|D| : D$ is a SDS of G_i^k, $v_{i-k} \in D$, $v_{i-k+1} \in D, \ldots, v_{i-1} \in D$, $u_{i-1} \in D$, $u_i \in D$, $v_i \in D$, $v_{i+1} \in D, \ldots, v_{i+k-1} \in D\}$.

A $\gamma^{0\cdots 0}(G_i^k)$-set is a minimum SDS D of G_i^k such that $v_{i-k} \notin D$, $v_{i-k+1} \notin D, \ldots, v_{i-1} \notin D$, $u_{i-1} \notin D$, $u_i \notin D$, $v_i \notin D$, $v_{i+1} \notin D, \ldots, v_{i+k-1} \notin D$. Similarly, we define the others. See Fig. 2.

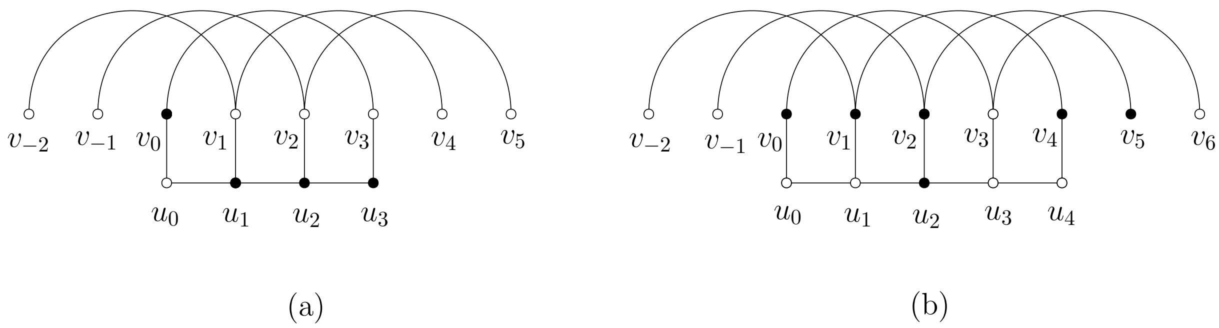

Figure 2: Illustrating (a) a $\gamma^{10011000}(G_3^3)$-set and (b) a $\gamma^{11000110}(G_4^3)$-set; note that the vertices of SDSs are solid.

Let $X_{n,k}$ be the set of all minimum SDS of $SGP(n,k)$ such that

Algorithm 3.1: DT($GP(n,k)$)

Input: The generalized Petersen graph $GP(n,3) = (V,E)$.
Output: The domination number of $GP(n,3)$.

Let $SGP(n,3)$ be the semi generalized Petersen graph corresponding to $GP(n,3)$.
for $b_1,\dots,b_{2k+2} \in \{0,1\}$ **do**
  $\gamma^{b_{2k+2}\cdots b_1}(G_1) = b_1 + \cdots + b_{2k+2}$;
  for $(x_1,\dots,x_{2k+2} \in \{0,1\}) \wedge (x_{2k+2}\cdots x_1 \neq b_{2k+2}\cdots b_1)$ **do**
    $\gamma^{x_{2k+2}\cdots x_1}(G_1) = \infty$;
  for $i = 1$ **to** $n+1$ **do**
    for $x_1,\dots,x_{2k+2} \in \{0,1\}$ **do**
      Compute $\gamma^{x_{2k+2}\cdots x_1}(G_i)$ by Lemma 3.
    $|X_{b_{2k+2}\cdots b_1}| = \gamma^{b_{2k+2}\cdots b_1}(G_{n+1})$;
$\gamma = \min\{|X_{b_{2k+2}\cdots b_1}| - (b_1 + \cdots + b_{2k+2}) : b_1,\dots,b_k \in \{0,1\}\}$;
return γ;

(i) $u_j \in D$ if and only if $u_{n+j} \in D$ for each $j \in \{0,1\}$, and

(ii) $v_j \in D$ if and only if $v_{n+j} \in D$ for each $j \in \{-k+1,-k+2,\dots,k\}$.

The following proposition is clear.
proposition 1 $|X_{n,k}| = 4^{k+1}$.
Let $j \in \{-k+1,-k+2,\dots,k\}$ and $l \in \{0,1\}$ and assume $a_j, d_l \in \{0,1\}$. Let a_j be corresponding to vertices v_j, v_{n+j} and let d_l be corresponding to vertices u_l, u_{n+l}. We define $X_{a_{-k+1}a_{-k+2}\cdots a_0 d_0 d_1 a_1 a_2 \cdots a_k}$ as a minimum SDS of $SGP(n,k)$ such that if $a_j = 0$ (respectively, $d_l = 0$), then their corresponding vertices are not in $X_{a_{-k+1}\cdots a_0 d_0 d_1 a_1 \cdots a_k}$ and if $a_j = 1$ (respectively, $d_k = 1$), then their corresponding vertices are in $X_{a_{-k+1}\cdots a_0 d_0 d_1 a_1 \cdots a_k}$. We obtain $X_{n,k} = \{X_{b_{2k+2}\cdots b_1} : b_1,\dots,b_{2k+2} \in \{0,1\}\}$.

3 Algorithm

In this section we give an algorithm (Algorithm 3.1) to compute the domination number of the generalized Petersen graph $GP(n,k)$. In order to prove that Algorithm 3.1 works correctly we need the following lemmas. The main idea of our algorithm is the following lemma.
Lemma 1. Let $GP(n,k) = (V,E)$ and let D be a set of $X_{n,k}$ such that $|D \cap V| \leq |S \cap V|$ for every set $S \in X_{n,k}$. Then, $D \cap V$ is a minimum DS of $GP(n,k)$. *Proof.* Recall $V_l = \{v_{1-k},\dots,v_0,u_0\}$ and $V_r = \{u_{n+1},v_{n+1},\dots,v_{n+k}\}$. Let $D' = D \cap V$. We first prove that D' is a DS of $GP(n,k)$. By Note **??**, we have $\deg_{SGP(n,k)}(v) = 3$ for every vertex $v \in V$. Assume $v \in V \setminus D'$. Since D is a SDS of $SGP(n,k)$, there is a vertex $u \in D$ adjacent to v. If $N_{SGP(n,k)}(v) \cap D' \neq \emptyset$, then there is nothing to be proven. If $N_{SGP(n,k)}(v) \cap D' = \emptyset$, then $N_{SGP(n,k)}(v) \cap D \subseteq V_l \cup V_r$. Assume without loss of generality

that $u = v_j \in N_{SGP(n,k)}(v) \cap D$ for some $1-k \le j \le 0$. By the definition of $SGP(n,k)$, $N_{SGP(n,3)}(v_j) = \{v_{j+k}\}$ if $j \neq 0$ and $N_{SGP(n,3)}(v_0) = \{v_k, u_0\}$ and so $v = v_{j+k}$. Because $D \in X_{n,k}$ and $v_j \in D$, we deduce $v_{n+j} \in D$. Since $v_{n+j} \in N_{GP(n,k)}(v_{j+k})$, hence D' is a DS of $GP(n,k)$.
Suppose for a contradiction that D' is a not a minimum DS of $GP(n,k)$. Assume that Z' is a DS of $GP(n,k)$ with $|Z'| < |D'|$. We construct the set Z as follows. Initialize Z to be Z'. If $u_1 \in Z'$, then we add u_{n+1} to Z, if $u_n \in Z'$, then we add u_0 to Z, if $v_j \in D$ for some $j \in \{1,2,\dots,k\}$, then we add v_{n+j} to Z and if $v_j \in D$ for some $j \in \{n-k+1, n-k+2, \dots, n\}$, then we add v_{j-n} to Z. So, $Z \in X_{n,k}$ with $|Z \cap V| = |Z'| < |D'| = |D \cap V|$, a contradiction.
In order to compute all sets of $X_{n,k}$ we need the following lemma.
Lemma 2. Let $b_1, b_2, \dots, b_{2k+2} \in \{0,1\}$, let $i \in \{1,2,\dots,n+1\}$ and let either $b_{k+3} + b_{k+2} \ge 1$ or $b_{k+1} = b_1 = 1$. Then,

(i) $\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G^k_{i+1}) = \gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G^k_i)$,

(ii) $\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 1}(G^k_{i+1}) = \min\{\gamma^{0b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G^k_i), \gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G^k_i)\} + 1$,

(iii) $\gamma^{b_{2k+2}\cdots b_{k+4}001b_k\cdots b_2 0}(G^k_{i+1}) = \min\{\gamma^{1b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2}(G^k_i), \gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G^k_i)\} + 1$,

(iv) $\gamma^{b_{2k+2}\cdots b_1}(G^k_{i+1}) = \min\{\gamma^{0b_{2k+2}\cdots b_{k+4}0b_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i), \gamma^{0b_{2k+2}\cdots b_{k+4}1b_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i), \gamma^{1b_{2k+2}\cdots b_{k+4}0b_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i), \gamma^{1b_{2k+2}\cdots b_{k+4}1b_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i), \} + b_{k+1} + b_1$.

Proof Let $j \in \{2, \dots, k-1, k, k+4, \dots, 2k+1, 2k+2\}$. We first prove (i). Let D be a $\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G^k_{i+1})$-set. So, all vertices $v_i, u_i, u_{i+1}, v_{i+k}$ are not in D and the corresponding vertex to b_j is in D if $b_j = 1$ and is not in D if $b_j = 0$. See Fig. 3(a). Since $N_{G^k_{i+1}}(v_i) = \{u_i, v_{i-k}, v_{i+k}\}$, $N_{G^k_{i+1}}(u_i) = \{u_{i-1}, u_{i+1}, v_i\}$ and D is a SDS of G^k_{i+1}, we deduce that both vertices v_{i-k} and u_{i-1} are in D. Hence, D is a SDS of G^k_i such that the corresponding vertex to b_j is in D if $b_j = 1$ and is not in D if $b_j = 0$, $v_{i-k} \in D$, $u_{i-1} \in D$, $u_i \notin D$ and $v_i \notin D$ and so $\gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G^k_i) \le |D|$, that is,

$$\gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G^k_i) \le \gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G^k_{i+1}). \quad (1)$$

Conversely, let S be a $\gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G^k_i)$-set. So, $v_{i-k} \in S$, $u_{i-1} \in S$, $u_i \notin S$, $v_i \notin S$ and the corresponding vertex to b_j is in S if $b_j = 1$ and is not in S if $b_j = 0$. See Fig. 3(b). Since both vertices v_{i-k} and u_{i-1} are in S, we deduce that S is a SDS of G^k_{i+1} such that $v_{i+k} \notin S$, $u_{i+1} \notin S$, $u_i \notin S$, $v_i \notin S$ and the corresponding vertex to b_j is in S if $b_j = 1$ and is not in S if $b_j = 0$ and so $\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G^k_{i+1}) \le |S|$, that is, $\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 0}(G^k_{i+1}) \le \gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G^k_i)$. This, together with Inequality (1), completes the proof of (i).
Now, we prove (ii). Let D be a $\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 1}(G^k_{i+1})$-set. So, all vertices v_i, u_i, u_{i+1} are not in D, $v_{i+k} \in D$ and the corresponding vertex to b_j is in D if $b_j = 1$ and is not in D if $b_j = 0$. Since $N_{G^k_{i+1}}(v_i) = \{u_i, v_{i-k}, v_{i+k}\}$, $N_{G^k_{i+1}}(u_i) = \{u_{i-1}, u_{i+1}, v_i\}$ and D is a SDS

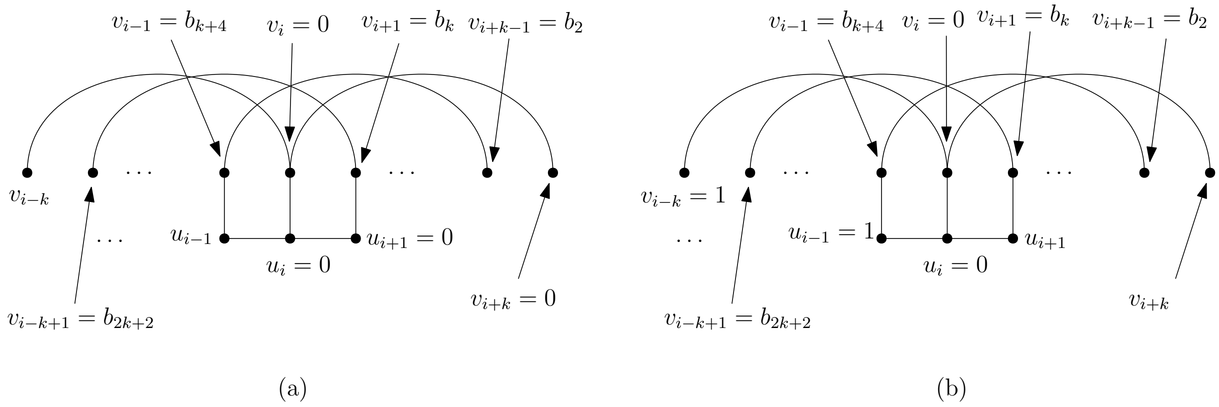

Figure 3: Illustrating the subgraph G_{i+1}^k.

of G_{i+1}^k, we deduce $u_{i-1} \in D$. Because v_i is dominated by $v_{i+k}(\in D)$, either $v_{i-k} \in D$ or $v_{i-k} \notin D$. In the following we consider these cases.

- Assume $v_{i-k} \in D$. Let $X = D \setminus \{v_{i+k}\}$. So, X is a SDS of G_i^k such that the corresponding vertex to b_j is in X if $b_j = 1$ and is not in X if $b_j = 0$, $v_{i-k} \in X$, $u_{i-1} \in X$, $u_i \notin X$ and $v_i \notin X$ and so $\gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G_i^k) \leq |X| = |D| - 1$, that is,
$$\gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G_i^k) + 1 \leq \gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 1}(G_{i+1}^k). \tag{2}$$

- Assume $v_{i-k} \notin D$. Let $X = D \setminus \{v_{i+k}\}$. So, X is a SDS of G_i^k such that the corresponding vertex to b_j is in X if $b_j = 1$ and is not in X if $b_j = 0$, $v_{i-k} \notin X$, $u_{i-1} \in X$, $u_i \notin X$ and $v_i \notin X$ and so $\gamma^{0b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G_i^k) \leq |X| = |D| - 1$, that is,
$$\gamma^{0b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G_i^k) + 1 \leq \gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 1}(G_{i+1}^k). \tag{3}$$

Conversely, let S_0 be a $\gamma^{0b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G_i^k)$-set and let $X_0 = S_0 \cup \{v_{i+k}\}$. So, $v_{i-k} \notin X_0$, $u_{i-1} \in X_0$, $u_i \notin X_0$, $v_i \notin X_0$ and the corresponding vertex to b_j is in X_0 if $b_j = 1$ and is not in X_0 if $b_j = 0$. Because both vertices v_{i+k} and u_{i-1} are in X_0, we deduce that X_0 is a SDS of G_{i+1}^k such that $v_i \notin X_0$, $u_i \notin X_0$, $u_{i+1} \notin X_0$, $v_{i+k} \in X_0$ and the corresponding vertex to b_j is in X_0 if $b_j = 1$ and is not in X_0 if $b_j = 0$ and so $\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 1}(G_{i+1}^k) \leq |X_0| = |S_0| + 1$, that is,

$$\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 1}(G_{i+1}^k) \leq \gamma^{0b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G_i^k) + 1. \tag{4}$$

Let S_1 be a $\gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G_i^k)$-set and let $X_1 = S_1 \cup \{v_{i+k}\}$. So, $v_{i-k} \in X_1$, $u_{i-1} \in X_1$, $u_i \notin X_1$, $v_i \notin X_1$ and the corresponding vertex to b_j is in X_1 if $b_j = 1$ and is not in X_1 if $b_j = 0$. Because both vertices v_{i+k} and u_{i-1} are in X_1, we deduce that X_1 is a SDS of G_{i+1}^k such that $v_i \notin X_1$, $u_i \notin X_1$, $u_{i+1} \notin X_1$, $v_{i+k} \in X_1$ and the corresponding vertex to b_j is in X_1 if $b_j = 1$ and is not in X_1 if $b_j = 0$ and so $\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 1}(G_{i+1}^k) \leq |X_1| =$

$|S_1| + 1$, that is, $\gamma^{b_{2k+2}\cdots b_{k+4}000b_k\cdots b_2 1}(G^k_{i+1}) \leq \gamma^{1b_{2k+2}\cdots b_{k+4}100b_k\cdots b_2}(G^k_i) + 1$. This, together with Inequalities (2)–(4), completes the proof of (ii).
Similarly, we can prove (iii).
Here, we prove (iv). Assume $j \in \{2, \ldots, k, k+2, \ldots, 2k+2\}$ and let D be a $\gamma^{b_{2k+2}\cdots b_1}(G^k_{i+1})$-set such that either $b_{k+3} + b_{k+2} \geq 1$ or $b_{k+1} = b_1 = 1$. We have $N_{G^k_{i+1}}(v_i) = \{u_i, v_{i-k}, v_{i+k}\}$ and $N_{G^k_{i+1}}(u_i) = \{u_{i-1}, u_{i+1}, v_i\}$. We first assume $b_{k+3} + b_{k+2} \geq 1$. Hence, either $b_{k+3} = 1$ or $b_{k+2} = 1$.

- If $b_{k+3} = 0$, then $v_i \notin D$ and $b_{k+2} = 1$ and so $u_i \in D$. Hence, u_i dominates v_i.
- If $b_{k+2} = 0$, then $u_i \notin D$ and $b_{k+3} = 1$ and so $v_i \in D$. Hence, v_i dominates u_i.
- If $b_{k+3} = 1$ and $b_{k+2} = 1$, then both vertices $u_i, v_i \in D$.

We deduce either $v_{i-k} \in D$ or $v_{i-k} \notin D$ and either $u_{i-1} \in D$ or $u_{i-1} \notin D$. If $b_{k+1} = b_1 = 1$, then both vertices v_{i+k} and u_{i+1} are in D and so if $v_i \notin D$ (respectively, $u_i \notin D$), then v_{i+k} (respectively, u_{i+1}) dominates v_i (respectively, u_i). Therefore, either $v_{i-k} \in D$ or $v_{i-k} \notin D$ and either $u_{i-1} \in D$ or $u_{i-1} \notin D$. We obtain that if either $b_{k+3} + b_{k+2} \geq 1$ or $b_{k+1} = b_1 = 1$, then either $v_{i-k} \in D$ or $v_{i-k} \notin D$ and either $u_{i-1} \in D$ or $u_{i-1} \notin D$. In the following we consider these cases. Let $X = D$ if $u_{i+1} \notin D$ (i.e., $b_{k+1} = 0$) and $v_{i+k} \notin D$ (i.e., $b_1 = 0$), let $X = D \setminus \{u_{i+1}\}$ if $u_{i+1} \in D$ (i.e., $b_{k+1} = 1$) and $v_{i+k} \notin D$ (i.e., $b_1 = 0$), let $X = D \setminus \{v_{i+k}\}$ if $u_{i+1} \notin D$ (i.e., $b_{k+1} = 0$) and $v_{i+k} \in D$ (i.e., $b_1 = 1$) and let $X = D \setminus \{v_{i+k}, u_{i+1}\}$ if $u_{i+1} \in D$ (i.e., $b_{k+1} = 1$) and $v_{i+k} \in D$ (i.e., $b_1 = 1$). We deduce $|X| = |D| - (b_{k+1} + b_1)$.

(a) Assume $v_{i-k} \in D$ and $u_{i-1} \in D$. So, X is a SDS of G^k_i such that the corresponding vertex to b_j is in X if $b_j = 1$ and is not in X if $b_j = 0$, $v_{i-k} \in X$ and $u_{i-1} \in X$ and so $\gamma^{1b_{2k+2}\cdots b_{k+4}1b_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i) \leq |X| = |D| - (b_{k+1} + b_1)$, that is,

$$\gamma^{1b_{2k+2}\cdots b_{k+4}1b_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i) + b_{k+1} + b_1 \leq \gamma^{b_{2k+2}\cdots 1}(G^k_{i+1}). \quad (5)$$

(b) Assume $v_{i-k} \notin D$ and $u_{i-1} \in D$. Similar to the previous case, we have

$$\gamma^{0b_{2k+2}\cdots b_{k+4}1b_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i) + b_{k+1} + b_1 \leq \gamma^{b_{2k+2}\cdots 1}(G^k_{i+1}). \quad (6)$$

(c) Assume $v_{i-k} \in D$ and $u_{i-1} \notin D$. Similar to Case **(a)**, we obtain

$$\gamma^{1b_{2k+2}\cdots b_{k+4}0b_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i) + b_{k+1} + b_1 \leq \gamma^{b_{2k+2}\cdots 1}(G^k_{i+1}). \quad (7)$$

(d) Assume $v_{i-k} \notin D$ and $u_{i-1} \notin D$. Similar to Case **(a)**, we deduce

$$\gamma^{0b_{2k+2}\cdots b_{k+4}0b_{k+2}b_{k+3}b_k\cdots b_2}(G^k_i) + b_{k+1} + b_1 \leq \gamma^{b_{2k+2}\cdots 1}(G^k_{i+1}). \quad (8)$$

Conversely, assume that S is a SDS of G_i^k, either $b_{k+3}+b_{k+2}\geq 1$ or $b_{k+1}=b_1=1$ and $j\in\{1,\ldots,2k+2\}$. Let $X=S$ if $b_{k+1}=0$ and $b_1=0$, let $X=S\cup\{u_{i+1}\}$ if $b_{k+1}=1$ and $b_1=0$, let $X=S\cup\{v_{i+k}\}$ if $b_{k+1}=0$ and $b_1=1$ and let $X=S\cup\{v_{i+k},u_{i+1}\}$ if $b_{k+1}=1$ and $b_1=1$. We deduce that X is a SDS of G_{i+1}^k with $|X|=|S|+b_{k+1}+b_1$.
Assume that S is a $\gamma^{0b_{2k+2}\cdots b_{k+4}0b_{k+2}b_{k+3}b_k\cdots b_2}(G_i^k)$-set. So, X is a SDS of G_{i+1}^k such that the corresponding vertex to b_j is in X if $b_j=1$ and is not in X if $b_j=0$ and so $\gamma^{0b_{2k+2}\cdots b_{k+4}0b_{k+2}b_{k+3}b_k\cdots b_2}(G_i^k)\leq |X|=|S|+b_{k+1}+b_1$, that is,

$$\gamma^{b_{2k+2}\cdots b_1}(G_{i+1}^k)\leq \gamma^{0b_{2k+2}\cdots b_{k+4}0b_{k+2}b_{k+3}b_k\cdots b_2}(G_i^k)++b_{k+1}+b_1. \tag{9}$$

Similarly, if S is a $\gamma^{0b_{2k+2}\cdots b_{k+4}1b_{k+2}b_{k+3}b_k\cdots b_2}(G_i^k)$-set, then we obtain

$$\gamma^{b_{2k+2}\cdots b_1}(G_{i+1}^k)\leq \gamma^{0b_{2k+2}\cdots b_{k+4}1b_{k+2}b_{k+3}b_k\cdots b_2}(G_i^k)++b_{k+1}+b_1. \tag{10}$$

Similarly, if S is a $\gamma^{1b_{2k+2}\cdots b_{k+4}0b_{k+2}b_{k+3}b_k\cdots b_2}(G_i^k)$-set, then we have

$$\gamma^{b_{2k+2}\cdots b_1}(G_{i+1}^k)\leq \gamma^{1b_{2k+2}\cdots b_{k+4}0b_{k+2}b_{k+3}b_k\cdots b_2}(G_i^k)++b_{k+1}+b_1. \tag{11}$$

Similarly, if S is a $\gamma^{1b_{2k+2}\cdots b_{k+4}1b_{k+2}b_{k+3}b_k\cdots b_2}(G_i^k)$-set, then we deduce

$$\gamma^{b_{2k+2}\cdots b_1}(G_{i+1}^k)\leq \gamma^{1b_{2k+2}\cdots b_{k+4}1b_{k+2}b_{k+3}b_k\cdots b_2}(G_i^k)++b_{k+1}+b_1. \tag{12}$$

Inequalities (5)–(12) complete the proof of (iv).
Now we are in a position to compute all sets of $X_{n,k}$.
Lemma 3. Let $b_1,\ldots,b_{2k+2}\in\{0,1\}$. We can compute $X_{b_{2k+2}\cdots b_1}$ in $\mathcal{O}(4^kn)$ time and space. *Proof* Let $x_1,\ldots,x_{2k+2}\in\{0,1\}$. Because we would like to compute $X_{b_{2k+2}\cdots b_1}$, initialize $\gamma^{b_{2k+2}\cdots b_1}(G_1)$ to be $b_1+\cdots+b_{2k+2}$ and $\gamma^{x_{2k+2}\cdots x_1}(G_1)$ to be ∞ for every $x_{2k+2}\cdots x_1\neq b_{2k+2}\cdots b_1$. Then, by Lemma 3 we compute $\gamma^{x_{2k+2}\cdots x_1}(G_2)$ for each $x_1,\ldots,x_{2k+2}\in\{0,1\}$ and repeat this process to compute $\gamma^{x_{2k+2}\cdots x_1}(G_{n+1})$ for each $x_1,\ldots,x_{2k+2}\in\{0,1\}$. In the end of this process, we have $|X_{b_{2k+2}\cdots b_1}|=\gamma^{b_{2k+2}\cdots b_1}(G_{n+1})$. During this process we can also compute $X_{b_{2k+2}\cdots b_1}$. By Lemma 3, the time and space complexity of this Algorithm is $\mathcal{O}(4^kn)$.
Theorem 1. Algorithm 3.1 on input the generalized Petersen graph $GP(n,k)$ returns the domination number of $GP(n,k)$ in $\mathcal{O}(n16^k)$ time and space. *Proof* Let $b_1,\ldots,b_{2k+2}\in\{0,1\}$ and $GP(n,k)=(V,E)$. By Lemma 3, Algorithm 3.1 on input $GP(n,k)$ in Line 9 computes $|X_{b_{2k+2}\cdots b_1}|$. By the definition of $X_{b_{2k+2}\cdots b_1}$, we deduce that $|X_{b_{2k+2}\cdots b_1}\cap V|=|X_{b_{2k+2}\cdots b_1}|-(b_1+\cdots+b_{2k+2})$. By Lemma 3, $\gamma(GP(n,k))=\min\{|X_{x_{2k+2}\cdots x_1}\cap V|:x_1,\ldots,x_{2k+2}\in\{0,1\}\}$. So, Algorithm 3.1 on input $GP(n,k)$ in Line 10 computes the domination number of $GP(n,k)$ and returns this value in Line 11. We obtain that the time and space complexity of Algorithm 3.1 on input $GP(n,k)$ is $\mathcal{O}(n16^k)$.
By Theorem 3 we have the following result.
Corollary 1. Algorithm 3.1 on input the generalized Petersen graph $GP(n,k)$ returns the domination number of $GP(n,k)$ in $\mathcal{O}(n)$ time and space, where $k\in\mathcal{O}(1)$.

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