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Generalization of DP Curves and Surfaces

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ABSTRACT

In CAGD, the DP curves are known as a normalized totally positive curves that have the linear computational complexity. Because of their geometric properties, these curves will have the shape preserving properties, that is, the form of the curve will maintain the shape of the polygon and optimal stability. In this paper, we first define a new basis functions that are called generalized DP basis functions. Based on these functions, the generalized DP curves and surfaced are defined which have most properties of the classical DP curves and surfaces. These curves and surfaces have geometric properties as the rational DP curves and surfaces. Furthermore, we show that the shape parameters can control the shape of the proposed curve without changing the control points.

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1 Introduction

Parametric representation of curves and surfaces plays a significant role in computer aided geometric design (CAGD) and computer graphic (CG). One of the most important issues that we have to remember when describing curves and surfaces where reference functions are used if we want to maintain curve or surface form [2, 4, 5, 6, 7, 11, 12, 14]. Several forms of work were carried out to design and control curves and surfaces through several types of spline. How we can change the curve shape so that the given data and polygon control is not altered is very important.

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The rational parametric curves such as rational Bézier curves are commonly used for constructing free-form curves and surfaces. In recent years, the rational spline with parameters has received attention in the literature. The rational parametric curves such as rational curves are widely used to create curves and surfaces of free form. In recent years attention has been paid to the rational spline with parameters in the literature [8, 9, 15]. However, due to its rational form and nonpolynomial form the rational parametric curves have many defects. Repeated differentiation for example creates very high degree curves. Delgado and Peña [10, 11] introduced a normalized totally positive curve called *DP curve* that has the linear computational complexity. Because of these geometric properties, this curve will have the shape preserving properties, that is, the form of the curve will maintain the shape of the polygon (see [11, 12, 15, 16, 18] and optimal stability (see [17]). Therefore, the DP curves exhibit potential values in CAGD.
In this paper, we introduce a generalization of DP polynomial basis and DP curves and surfaces which seems to be entirely new. It is based on the novel ideas that the shape of the curves and surfaces is controlled by the shape parameters, not by the control points. These curves and surfaces have geometric properties as the rational DP curves and surfaces.
The organization of this paper is as follows: In Section 2, we construct a new basis functions that is called generalized DP basis functions with shape parameters and also we provide the properties of the basis functions. In Sections 3 and 5, the corresponding generalized DP curves and surfaces are presented and their properties studied respectively. In Section 4, we study the effect of altering values of the shape parameters of the curve on the shape of the curve. Finally, we conclude this paper in Section 6.

2 Generalized DP basis function

In [1], Aphirukmatakun and Dejdumrong defined the DP curve in monomial form as follows.
Definition 1.(See [1]). (DP Monomial Form) An n^{th}-degree DP curve, denoted by $\mathcal{D}^n(t)$, given by a set of $n+1$ control points, denoted by $\{\mathbf{d}_i\}_{i=0}^n$, can be formulated in power basis form by

$$\mathcal{D}^n(t) = \sum_{i=0}^{n}\sum_{j=0}^{n} \mathbf{d}_i \cdot d_{i,j} \cdot t^j,$$

where

$$d_{i,j} = \begin{cases} (-1)^j \binom{n}{j}, & \text{for } i = 0, \\ (-1)^{j-1} \binom{n-i}{j-1}, & \text{for } 0 < i \leq \lfloor \frac{n}{2} \rfloor - 1, \\ (-1)^{j-1}(n-2i)\binom{i+1}{j-1} + (\frac{1}{2})^{n-2i} \left[\binom{0}{j} \right. & \\ \left. -\binom{0}{j-i-1} - (-1)^j \binom{i+1}{j} \right], & \text{for } i = \lfloor \frac{n}{2} \rfloor, \\ (-1)^{j-n+i}(n-2i)\binom{1}{j-n+i-1} + (\frac{1}{2})^{2i-n} \left[\binom{0}{j} \right. & \\ \left. -\binom{0}{j-n+i-1} - (-1)^j \binom{n-i+1}{j} \right], & \text{for } i = \lceil \frac{n}{2} \rceil, \\ (-1)^{j-i} \binom{1}{j-i}, & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n-1, \\ \binom{0}{n-j}, & \text{for } i = n. \end{cases} \tag{1}$$

For more information on the DP polynomials, see [2, 3, 4, 9, 13].

Now, we introduce a generalization of DP basis. **Definition 2.** For every integer $n (n \geq 2)$ and n arbitrarily selected real values of $\lambda_i, i = 1, 2, \dots, n$ the following polynomial functions

$$E_i^n(t) = \sum_{j=0}^{n+1} e_{i,j} t^j \quad 0 \leq i \leq n, \tag{2}$$

are called the generalized DP polynomials of degree n associated with the shape parameters $\{\lambda_i\}_{i=1}^n$, where $\lambda_i \in [-1, 1]$, $e_{i,j} = h_{i,j} + g_{i,j}$ such that

$$h_{i,j} = \begin{cases} d_{i,j} & if j < n+1 \\ 0 & otherwise \end{cases}$$

and

$$g_{i,j} = \begin{cases} (-1)^{j}\binom{n}{j-1}\lambda_1 & if\ i=0\ and\ 0<j<n+1 \\ (-1)^{n+1}\lambda_1 & if\ i=0\ and\ j=n+1 \\ \lambda_i & if\ 0<i<\lceil\frac{n}{2}\rceil\ and\ j=i \\ (-1)^{j-i}\left(\binom{n-i}{j-i}\lambda_i+\binom{n-i}{j-i-1}\lambda_i+\binom{n-i}{j-i-1}\lambda_{i+1}\right) & if\ 0<i<\lceil\frac{n}{2}\rceil\ and\ i<j<n+1 \\ (-1)^{n-i+1}(\lambda_i+\lambda_{i+1}) & if\ 0<i<\lceil\frac{n}{2}\rceil\ and\ j=n+1 \\ \lambda_i & if\ i=\lceil\frac{n}{2}\rceil\ and\ j=i \\ (-1)^{j-i}\left(\binom{n-i}{j-i}\lambda_i+\binom{n-i}{j-i-1}\lambda_i-\binom{n-i}{j-i-1}\lambda_{i+1}\right) & if\ i=\lceil\frac{n}{2}\rceil\ and\ \lceil\frac{n}{2}\rceil<j<n+1 \\ (-1)^{n-i+1}(\lambda_i-\lambda_{i+1}) & if\ i=\lceil\frac{n}{2}\rceil\ and\ j=n+1 \\ -\lambda_i & if\ \lceil\frac{n}{2}\rceil<i<n\ and\ j=i \\ (-1)^{j-i}\left(-\binom{n-i}{j-i}\lambda_i-\binom{n-i}{j-i-1}\lambda_i-\binom{n-i}{j-i-1}\lambda_{i+1}\right) & if\ \lceil\frac{n}{2}\rceil<i<n\ and\ i<j<n+1 \\ (-1)^{n-i}(\lambda_i+\lambda_{i+1}) & if\ \lceil\frac{n}{2}\rceil<i<n\ and\ j=n+1 \\ -\lambda_n & if\ i=n\ and\ j=n \\ \lambda_n & if\ i=n\ and\ j=n+1 \\ 0 & otherwise \end{cases}$$

These basis have the following properties.

- Nonnegativity: $E_i^n(t) \geq 0, i = 0, 1, , \ldots, n$;
- Partition of unity: $\sum_{i=0}^{n} E_i^n(t) = 1$;
- Linear independence: $\sum_{i=0}^{n} a_i E_i^n(t) = 0$ if and only if $a_i = 0 (i = 0, 1, \ldots, n)$;

- Symmetry: for $i = 0, 1, \dots, n$;

$$E_i^n(t; \lambda_1, \dots, \lambda_n) = E_{n-i}^n(1 - t; \lambda_n, \dots, \lambda_1).$$

Figure 1 shows the quadratic generalized DP basis functions for $\lambda_1 = 1, \lambda_2 = 1$ (doted lines), $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}$ (dashed lines) and $\lambda_1 = 0, \lambda_2 = -1$ (solid lines), respectively.

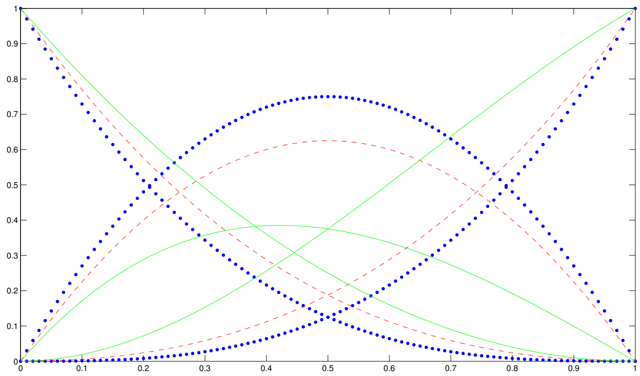


Figure 1: The quadratic generalized DP basis functions

3 Construction of the generalized DP curve

Definition 3. Given control points $\mathbf{p}_i (i = 0, 1, \dots, n) \in \mathbb{R}^2$ or $\mathbb{R}^3$ and $\lambda_i \in [-1, 1], i = 1, 2, \dots, n$, then

$$G(n) = \sum_{i=0}^{n} \mathbf{p}_i E_i^n(t), \quad t \in [0, 1], \tag{3}$$

is called generalized DP curve with shape parameters λ_i, where $E_i^n(t), i = 1, 2, \dots, n$ are generalized DP basis functions. According to (2), we can provide some properties of generalized DP curve in the following theorem. **Theorem 1.** The generalized DP curve (3) have the following properties:

(a) Interpolating endpoints:

$$G(0) = \mathbf{p}_0, \quad G(1) = \mathbf{p}_n. \tag{4}$$

(b) Symmetry: The control points $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n$ and $\mathbf{p}_n, \mathbf{p}_{n-1}, \dots, \mathbf{p}_0$ define the same generalized DP curve in a different parametrization,

$$G(t; \lambda_1, \dots, \lambda_n, \mathbf{p}_0, \dots, \mathbf{p}_n) = G(t; \lambda_n, \dots, \lambda_1, \mathbf{p}_n \dots, \mathbf{p}_0). \tag{5}$$

(c) Geometric invariance: The choice of coordinates for a generalized DP curve is independent of it's shape, i.e., Eq. (3) satisfies the following two equations:

$$\begin{gathered} G(t; \{\lambda_j\}_{j=1}^n; \mathbf{p}_0 + \mathbf{c}, \mathbf{p}_1 + \mathbf{c}, \dots, \mathbf{p}_n + \mathbf{c}) = G(t; \{\lambda_j\}_{j=1}^n; \mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n) + \mathbf{c}, \\ G(t; \{\lambda_j\}_{j=1}^n; \mathbf{p}_0 * \mathbf{M}, \mathbf{p}_1 * \mathbf{M}, \dots, \mathbf{p}_n * \mathbf{M}) = G(t; \{\lambda_j\}_{j=1}^n; \mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n) * \mathbf{M}, \end{gathered} \tag{6}$$

where $\mathbf{c}$ and $\mathbf{M}$ are an arbitrary vector in $\mathbb{R}^2$or $\mathbb{R}^3$ and an arbitrary $d \times d$ matrix, $d = 2$ or 3, respectively.

(d) Convex hull property: The generalized DP curve will never pass outside of the convex hull formed by its control points.

4 Effects of shape parameters on the generalized DP curve

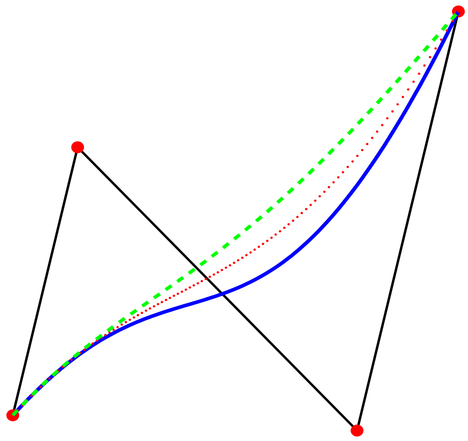
Shape of the generalized DP curve that expressed as (3) can be modified with altering the values of the shape parameters. Clearly, the shape parameters of the generalized DP curves (3) have some effects on the curve. Figures 2(a)-2(e) show the effects on shape of the curve with altering the values of $\lambda_i (i = 1, 2, \dots, n)$ for $n = 3$ and $n = 4$, respectively. At first, we consider cubic case. Figure 2(a), shows the curves that are formed by setting $\lambda_1 = \lambda_2 = 0$ and changing λ_3 to $\lambda_3 = 1$ (solid lines) and $\lambda_3 = 0$ (dotted lines) and $\lambda_3 = -1$ (dashed lines), respectively. Figure 2(b), shows the curves that are formed by setting $\lambda_1 = \lambda_2 = 1$ and changing λ_3 to $\lambda_3 = 0$ (solid lines) and $\lambda_3 = 0.5$ (dashed lines) and $\lambda_3 = 1$ (dotted lines), respectively. Figure 2(c), shows the curves that are formed by setting $\lambda_1 = -1, \lambda_2 = 0$ and changing λ_3 to $\lambda_3 = 0$ (solid lines) and $\lambda_3 = -0.5$ (dashed lines) and $\lambda_1 = 1$ (dotted lines), respectively.
Now, we consider the quartic case ($n = 4$). Figures 2(d) and 2(e) demonstrate the curves with altering shape parameters. Figure 2(d), shows the curves that are formed by setting $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and changing λ_4 to $\lambda_4 = 0$ (solid lines) and $\lambda_4 = 1$ (dotted lines) and $\lambda_4 = -0.5$ (dashed lines) and $\lambda_4 = -1$ (dashdotted lines), respectively. Figure 2(e), shows the curves that are formed by setting $\lambda_1 = -1, \lambda_2 = \lambda_4 = 1$ and changing λ_3 to $\lambda_3 = -1$ (solid lines) and $\lambda_3 = 0$(dotted lines) and $\lambda_3 = 1$ (dashed lines) and $\lambda_3 = 0.5$ (dashdotted lines), respectively.

5 Generalized DP surface

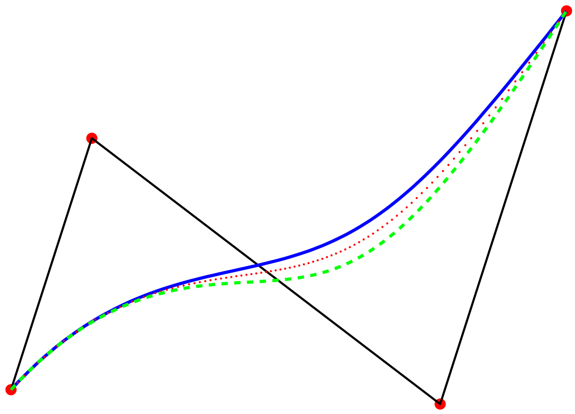
Using tensor product, we can construct generalized DP surface

$$H(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{p}_{i,j} E_i^m(u) E_j^n(v), \quad 0 \leq u, v \leq 1, \tag{7}$$

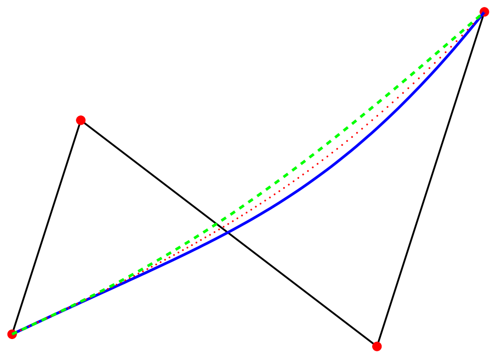
where $E_i^m(u)$ and $E_i^n(v)$ are the generalized DP basis functions (2) and $\mathbf{p}_{i,j} \in \mathbb{R}^3$ are the control points. Tensor product of generalized DP curves has properties similar to those of tensor product of DP curves. Figures 3 and 4 show the generalized DP surfaces with $\lambda_1 = \lambda_2 = -1$ and $\lambda_1 = 0.5, \lambda_2 = 0$ respectively, with $m = n = 2$.



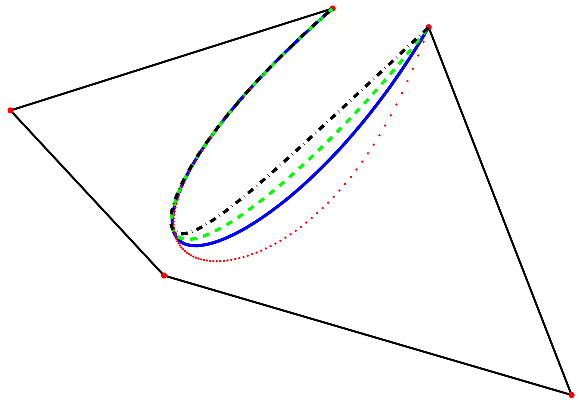
(a) The effect of altering λ_3 on the shape of cubic generalized DP curve with setting $\lambda_1 = \lambda_2 = 0$.



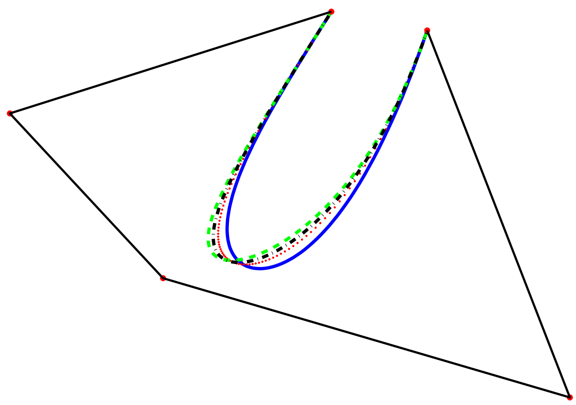
(b) The effect of altering λ_3 on the shape of cubic generalized DP curve with setting $\lambda_1 = \lambda_2 = 1$.



(c) The effect of altering λ_3 on the shape of cubic generalized DP curve with setting $\lambda_1 = -1, \lambda_2 = 0$



(d) The effect of altering λ_4 on the shape of quartic generalized DP curve with setting $\lambda_1 = \lambda_2 = \lambda_3 = 1$.



(e) The effect of altering λ_3 on the shape of quartic generalized DP curve with setting $\lambda_1 = -1, \lambda_2 = \lambda_4 = 1$.

Figure 2: Shape of the curves using the different values of parameters.

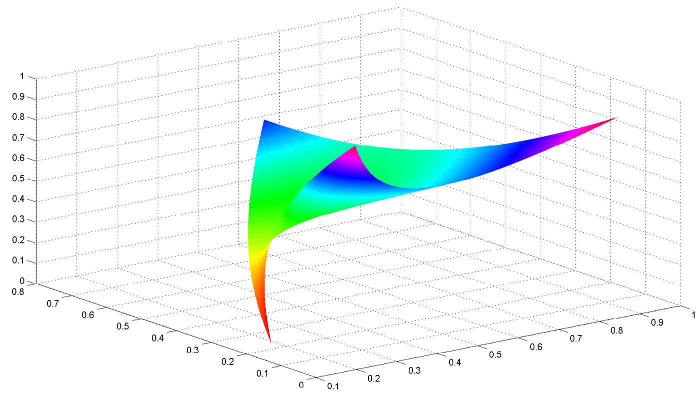


Figure 3: A generalized DP surface with $\lambda_1 = \lambda_2 = -1$ and $m = n = 2$.

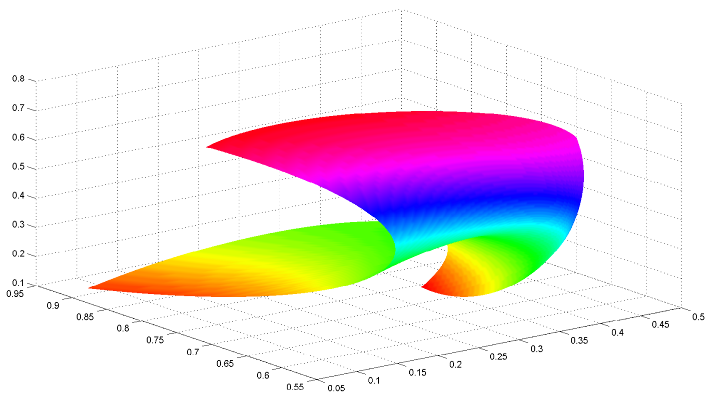


Figure 4: A generalized DP surface with $\lambda_1 = 0.5, \lambda_2 = 0$ and $m = n = 2$.

6 Conclusion

In this paper, we proposed a new polynomial basis functions that is a generalization of DP basis functions. Moreover, we constructed the corresponding curves and surfaces. In addition, we provided the most important properties of these basis and curves and we studied the effects of altering the values of the shape parameters on the shape of the curve.

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