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A note on the approximability of the tenacity of graphs

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ABSTRACT

In this paper we show that, if $NP \neq ZPP$, for any $\epsilon > 0$, the tenacity of graph with n vertices is not approximable in polynomial time within a factor of $\frac{1}{2}\left(\frac{n-1}{2}\right)^{1-\epsilon}$.

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1 Introduction

The concept of tenacity of a graph G was introduced in [8,9], as a useful measure of the "vulnerability" of G. In [9] Cozzens et al. calculated tenacity of the first and second case of the Harary Graphs but they didn't show the complete proof of the third case. In [28], Moazzami showed a new and complete proof for case three of the Harary Graphs. In [22], Moazzami compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [3, 8-11, 18-50],

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the authors studied more about this invariant. We consider only graphs without loops or multiple edges. We use $V(G)$, $\beta(G)$, and $\omega(G)$ to denote the vertex set, independence number and number of components in a graph G, respectively. We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. We denote by $V(G)$, $E(G)$ and $\mid V(G) \mid$ the set of vertices, the set of edges and the order of G, respectively.

The tenacity of a graph G, $T(G)$, is defined by $T(G) = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph $G-S$, and $\omega(G-S)$ be the number of components of $G-S$. A connected graph G is called T-tenacious if $\mid S \mid +\tau(G-S) \geq T\omega(G-S)$ holds for any subset S of vertices of G with $\omega(G-S) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $S \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|S|+\tau(G-S)}{\omega(G-S)}$.

The Mix-tenacity $T_m(G)$ of a graph G is defined as

$$T_m(G) = \min_{A \subset E(G)} \{\frac{\mid A \mid +m(G-A)}{\omega(G-A)}\}$$

where $m(G-A)$ denotes the order (the number of vertices) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$. Cozzens et al. in [8], called this parameter Edge-tenacity, but Moazzami changed the name of this parameter to Mix-tenacity in [26]. It seems Mix-tenacity is a better name for this parameter. $T(G)$ and $T_m(G)$ turn out to have interesting properties.

After the pioneering work of Cozzens, Moazzami, and Stueckle in [8,9], several groups of researchers have investigated tenacity, and its related problems. In [43] and [44] Piazza et al. used the $T_m(G)$ as Edge-tenacity. But this parameter is a combination of cutset $A \subset E(G)$ and the number of vertices of a largest component, $\tau(G-A)$. It may be observed that in the definition of $T_m(G)$, the number of edges removed is added to the number of vertices in a largest component of the remaining graph. Also this parameter didn't seem very satisfactory for Edge-tenacity. Thus Moazzami and Salehian introduced a new measure of vulnerability, the Edge-tenacity, $T_e(G)$, in [26]. The Edge-tenacity $T_e(G)$ of a graph G is defined as

$$T_e = \min_{A \subset E(G)} \{\frac{\mid A \mid +p(G-A)}{\omega(G-A)}\}$$

where $p(G-A)$ denotes the order (the number of edges) of a largest component of $G-A$ and $\omega(G-A)$ is the number of components of $G-A$. This new measure of vulnerability involves edges only and thus is called the Edge-tenacity. Since 1992 there were several interesting questions. But the question " How difficult is it to recognize T-tenacious graphs? " has remained an interesting open problem for some time. The question was first raised by Moazzami in [21]. Our purpose in [29] was to show that for any fixed positive rational number T, it is NP-hard to recognize T-tenacious graphs. To prove this we showed that it is NP-hard to recognize T-tenacious graphs by reducing a well-known NP-complete variant of INDEPENDENT SET.

2 Preliminaries

In this paper we consider only finite, non-complete, undirected and connected graphs without loops or multiple edges. Consider a graph $G = (V, E)$ with n vertices, then $\beta(G)$ denotes the maximum size of an independent set of G. For each subset of vertices S of G, $\omega(G - S)$ denotes number of connected components of $G - S$ which is obtained by removing S from G. We define $\tau(G - S)$ to be the number of vertices in the largest component of the graph $G - S$. The tenacity, $T(G)$, of the graph G is defined as $T(G) = min \left\{ \frac{|A|+\tau(G-A)}{\omega(G-A)} : A \subseteq V, \omega(G - A) \geq 2 \right\}$.
We consider the following minimization problem:
Min Tenacity
Input: A graph G = (V; E).
Output: A set of vertices S of G with the property $\omega(G - S) \geq 2$ such that the ratio $\frac{|A|+\tau(G-A)}{\omega(G-A)}$ is minimized.
An algorithm is a *f(n)-approximation* algorithm for maximization (respectively minimization) problem if for any instance x of the problem of size n, it returns a solution y of value $m(x, y)$ such that $m(x, y) = opt(x)/f(n)$ (respectively $m(x, y) = f(n) \times opt(x)$). An algorithm is a constant factor approximation if $f(n)$ is a constant. An optimization problem is *f(n)-approximable* if there exists a polynomial time *f(n)-approximation* algorithm for it .

3 Result

Johan Hastad [17] studied the possible performance of polynomial time approximation algorithm for Max Clique. He demanded that the algorithm, on input a graph G with n vertices, outputs a number that is always at most the size of the largest clique in G. We say that we have an f(n) approximation algorithm if this number is always at least the largest clique divided by f(n). The best polynomial time approximation algorithm for Max Clique achieves an approximation ratio of $O(\frac{n}{(logn)^2})$, [7], and thus it is of the form $n^{1-o(1)}$.
The main question has been whether this is the correct form of the best approximation function. Hastad in [17] proved that this is the case. He used the connection, discovered by Feige et al. in their seminal paper [12], between multiprover interactive proofs and non-approximability results for Max Clique. There has been a sequence of papers [2, 1, 5, 13, 6, 4] giving stronger and stronger non-approximability results based on very plausible intractability assumption. The previously strongest non-approximability result

by Bellare, Goldreich, and Sudan [4] shows that unless NP=co R it is, for any $\epsilon > 0$, infeasible to approximate Max Clique within a factor $O(n^{\frac{1}{2}-\epsilon})$. This is done by, for any $\delta > 0$, constructing a probabilistically checkable proof for NP, with a polynomial time verifier that uses logarithmic randomness, perfect completeness (the verifier always accepts when the input is in the language) and amortized free-bit complexity $2 + \delta$. This amortized free-bit complexity, which was introduced in [13] and named in [6], is central to the Hastad's paper and let us give an informal definition.
In general a verifier is trying to verify a proof of some NP-statement, i.e. that $x \in L$ for some input x and $L \in NP$. For the sake of concreteness we assume that L is satisfiability. In probabilistically checkable proofs we should think of the proof as written down and the verifier as trying to check the proof as cheaply as possible. The verifier looks at the number of bits. Sometimes the verifier has no idea what the value of the bit should be while some other times the verifier is in a checking mode and knows what to expect and when the value is not the expected value the verifier reject the input. The number of questions of the first type is, informally speaking, the number of free bits. The amortized number of free bits is $\frac{f}{\log_2 g}$ where f is the number of free bits and g is the *gap* which in our case, since we are assuming perfect completeness, is $\frac{1}{\delta}$ where δ is the probability that the verifier accepts when $x \notin L$. The important connection is now that if there is a proof-system for NP with a polynomial time verifier which uses logarithmic randomness and has perfect completeness with k amortized free bits, then for any $\epsilon > 0$ we get an in-approximability result of $n^{\frac{1}{k+1}}$, [51]. Bellare, Goldreich and Sudan [4] proved that in fact we essentially have an equivalence, in that if it is NP-hard to approximate Max Clique within a factor $n^{\frac{1}{k+1}}$ then there is also a proof-system with essentially k amortized bits.
Johan Hastad in his paper, for any $\delta > 0$, found a proof-system which uses δ amortized bits. This shows that unless $NP = ZPP$ then for any $\epsilon > 0$ Max Clique cannot be approximated within a factor $n^{1-\epsilon}$ in polynomial time.
Hastad's paper was the final version of the results announced in [15] and [16].
In this section we show that if $NP \neq ZPP$, then for any $\epsilon > 0$, the tenacity of a graph with n vertices is not approximable in polynomial time within a factor of $\frac{1}{2}\left(\frac{n-1}{2}\right)^{1-\epsilon}$.
Theorem 1. [17], If $NP \neq ZPP$, for any $\epsilon > 0$, the maximum independent set of a graph is not approximable within $n^{1-\epsilon}$ approximation factor.

Lemma 1. [8], If G is not complete, then $T(G) \leq \frac{n-\beta(G)+1}{\beta(G)}$.

Theorem 2. If $NP \neq ZPP$, for any $\epsilon > 0$, the **MIN TENACIOUS** problem is not $\frac{1}{2}\left(\frac{n-1}{2}\right)^{1-\epsilon}$ approximable in polynomial time where n is the number of vertices of the graph.

Proof: We construct a reduction between **MAX INDEPENDENT SET** and **MIN TENACIOUS**. Given a graph G instance of **MAX INDEPENDENT SET** on n vertices, we construct a graph H from G by adding a clique C of size $n+1$ vertices, and making each vertex of C adjacent to each vertex in G. By lemma 1, we have:

$$T(H) \le \frac{2n+2-\beta(H)}{\beta(H)} \le \frac{2n+2}{\beta(H)-1} \le \frac{2n+2}{\beta(H)}$$

$$\beta(H) \le \frac{2n+2}{T(H)}$$

And because $\beta(H) = \beta(G)$, then:

$$\beta(G) \le \frac{2n+2}{T(H)}$$

Now suppose that **MIN TENACIOUS** is $\frac{1}{2}\left(\frac{n-1}{2}\right)^{1-\epsilon}$ approximable. Thus there is an algorithm that applied to H finds a set A of vertices such that:

$$val = \frac{|A| + \tau(G-A)}{\omega(G-A)} \le \frac{1}{2}\left(\frac{2n+1-1}{2}\right)^{1-\epsilon} T(H) \le \frac{1}{2}n^{1-\epsilon}T(H)$$

We consider as solution for G an independent set that contains a vertex from each connected component of $\omega(G-A)$. Thus the size of this independent set is at least $n+1$, because A contains at least vertices of C, then we have:

$$val' = \omega(G-A) \ge \frac{\omega(G-A) \times (n+1)}{|A| + \tau(G-A)} \ge \frac{2n+2}{n^{1-\epsilon} \times T(H)}$$

And by using the previous inequalities we obtain:

$$val' \ge \frac{1}{n^{1-\epsilon}}\beta(G)$$

The $\beta(G)$ is the optimal solution for **MAX INDEPENTEDT SET** problem, and because this is a maximization problem, and by using theorem 1, we can't find an approximation algorithm with approximation factor better than $n^{1-\epsilon}$, and the theorem follows.

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