

On the tenacity of cycle permutation graph

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ABSTRACT

A special class of cubic graphs are the cycle permutation graphs. A cycle permutation graph $P_n(\alpha)$ is defined by taking two vertex-disjoint cycles on n vertices and adding a matching between the vertices of the two cycles.

In this paper we determine a good upper bound for tenacity of cycle permutation graphs.

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1 Introduction

We consider only finite undirected graphs without loops and multiple edges. Our terminology will be standard except as indicated; a good reference for any undefined terms is [1]. Throughout the paper $G = (V, E)$ will denote a graph with vertex set $V(G)$, edge set $E(G)$. The minimum degree $\delta(G)$, the maximum degree $\Delta(G)$, connectivity $\kappa(G)$, the independence number $\beta(G)$, the number of components $\omega(G)$ and the toughness $\tau(G)$. If no ambiguity is possible, the symbols will be used without reference to G . A cut-set of G

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is a proper subset S of $V(G)$ such that $\omega(G - S) > 1$. The toughness of a graph G was introduced by Chvátal, [3], where he observed the relationships between this parameter and the existence of Hamiltonian cycles in the given graph, and several results regarding this invariant were obtained. The original approach to toughness, $\tau(G)$, as follows. A connected graph G is called τ -tough if $\tau\omega(G - A) \leq |A|$ for any subset A of $V(G)$ with $\omega(G - A) > 1$.

A set $S \subset V(G)$ is said to be a τ -set of G if $\tau(G) = \frac{|S|}{\omega(G-S)}$.

The concept of graph tenacity was introduced by Cozzens, Moazzami and Stueckle in [5, 6], as a measure of network vulnerability and reliability. Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed. The motivation for studying vulnerability measures is derived from design and analysis of networks under hostile environment. Graph tenacity has been an active area of research since the concept was introduced in 1992. Cozzens et al. introduced two measures of network vulnerability termed the tenacity, $T(G)$, and the Mix-tenacity, $T_m(G)$, of a graph.

The tenacity $T(G)$ of a graph G is defined as

$$T(G) = \min_{S \subset V(G)} \left\{ \frac{|S| + m(G - S)}{\omega(G - S)} \right\}$$

where $m(G - S)$ denotes the order (the number of vertices) of a largest component of $G - S$ and $\omega(G - S)$ is the number of components of $G - S$. A set $S \subset V(G)$ is said to be a T -set of G if $T(G) = \frac{|S| + m(G-S)}{\omega(G-S)}$.

The Mix-tenacity T_m of a graph G is defined as

$$T_m = \min_{S \subset E(G)} \left\{ \frac{|S| + m(G - S)}{\omega(G - S)} \right\}$$

where $m(G - S)$ denotes the order (the number of vertices) of a largest component of $G - S$ and $\omega(G - S)$ is the number of components of $G - S$. we called this parameter Mix-tenacity. $T(G)$ and $T_m(G)$ turn out to have interesting properties. After the pioneering work of Cozzens, Moazzami, and Stueckle several groups of researchers have investigated tenacity, and related problems.

Permutation graphs were introduced by Chartrand and Harary [4]. For a labeled graph, G with $V(G) = \{1, 2, \dots, n\}$, and a permutation α in symmetric group S_n , the α -permutation graph, $P_G(\alpha)$, is a graph with two disjoint copies of G , $G_a = (V_a, E_a)$ and $G_b = (V_b, E_b)$, with $V_k = \{k_1, k_2, \dots, k_n\}$ for $k = \{a, b\}$, and with the edges $(a_i, b_{\alpha(i)})$, for

$1 \leq i \leq n$. If G is an n -cycle, labeled consecutively around the cycle, then $P_n(\alpha) = P_{C_n}(\alpha)$ is a *cycle permutation graph*. The copy of C_n labeled a_1, a_2, \dots, a_n will be called the *outer cycle*, the copy of C_n labeled b_1, b_2, \dots, b_n will be called the *inner cycle*, and the edges of the form $(a_i, b_{\alpha(i)})$ will be called *permutation edges* (This is the same definition used in [10]). The permutation cycle graphs are a family of cubic graphs.

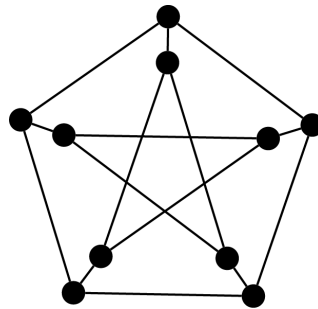


Figure 1: $P_5(1, 3, 5, 2, 4)$, Petersen Graph

For $\alpha = (1, 3, 5, 2, 4)$, Fig.[1] shows $P_5(\alpha)$ that is Petersen graph.

Now, we list some known results on toughness and tenacity.

Proposition 1. [3, 8] For a non-complete graph G

$$\frac{\kappa(G)}{\Delta(G)} \leq \tau(G) \leq \frac{\kappa(G)}{2}.$$

Proposition 2. [5] For a non-complete graph G

$$\tau(G) < T(G).$$

Piazza, Ringeisen and Stueckle in [9] conjectured that a permutation cycle graph has $\tau \leq 4/3$. Goddard in [7] was very close to prove that this conjecture is true.

Proposition 3. [7] Let G be a cycle permutation graph on $2n$ vertices. Then

$$\tau \begin{cases} \leq 4/3 & n \equiv 0, 1 \pmod{4} \\ < 4/3 & n \equiv 2 \pmod{4} \\ \leq 4/3 + 4/(9n - 3) & n \equiv 3 \pmod{4} \end{cases}$$

2 Upper Bound

In this section, we use a special vertex coloring for separate vertices into some independent sets. Consider the outer cycle C_O is $a_1 a_2 \dots a_n a_1$ in clockwise and the inner cycle C_I is

$b_1 b_2 \dots b_n b_1$ in clockwise.

For the outer cycle, assume $\{a_1, a_3, \dots\}$ are colored with black and $\{a_2, a_4, \dots\}$ are colored with white, now if n is an even, a_n will be white otherwise a_n is colored with gray. For the inner cycle, suppose $\alpha(n) = m$, so $\{b_{m+1}, b_{m+3}, b_{m+5}, \dots\}$ are colored with black and $\{b_{m+2}, b_{m+4}, b_{m+6}, \dots\}$ are colored with white and if n is an odd so b_m is colored with gray otherwise b_m is white.

Now we can separate all vertices of $P_n(\alpha)$ into nine subsets that eight of them are independent sets. \mathcal{A}_{bb} is included all **black** vertices of **outer** cycle that they are connected to **black** vertices of **inner** cycle, \mathcal{A}_{bw} is a subset of vertices of **outer** cycle that they are **black** themselves and connected to **white** vertices of **inner** cycle. \mathcal{B}_{bw} is included all **white** vertices of **inner** cycle that they are connected to **black** vertices of **outer** cycle. We can use these definitions same as above to declare \mathcal{A}_{wb} , \mathcal{A}_{ww} , \mathcal{B}_{wb} , \mathcal{B}_{ww} and \mathcal{B}_{bb} and if n is an odd then $\mathcal{D} = \{a_n, b_{\alpha(n)}\}$ otherwise $\mathcal{D} = \emptyset$.

Example 4. For $n = 10$ and $\alpha = (3, 9, 5, 4, 1, 2, 6, 8, 7, 10)$, in $P_{10}(\alpha)$ (Fig.2a) we have:

$$\begin{aligned} \mathcal{A}_{bb} &= \{a_1, a_3, a_5, a_9\}, & \mathcal{A}_{ww} &= \{a_4, a_6, a_8, a_{10}\}, & \mathcal{A}_{bw} &= \{a_7\}, & \mathcal{A}_{wb} &= \{a_2\} \\ \mathcal{B}_{bb} &= \{b_1, b_3, b_5, b_7\}, & \mathcal{B}_{ww} &= \{b_2, b_4, b_8, b_{10}\}, & \mathcal{B}_{bw} &= \{b_6\}, & \mathcal{B}_{wb} &= \{b_9\}. \\ \mathcal{D} &= \emptyset. \end{aligned}$$

For $n = 11$ and $\alpha = (9, 4, 2, 5, 6, 1, 8, 10, 3, 7, 11)$, in $P_{11}(\alpha)$ (Fig. 2b) we have:

$$\begin{aligned} \mathcal{A}_{bb} &= \{a_1, a_9\}, & \mathcal{A}_{ww} &= \{a_2, a_8\}, & \mathcal{A}_{bw} &= \{a_3, a_5, a_7\}, & \mathcal{A}_{wb} &= \{a_4, a_6, a_{10}\} \\ \mathcal{B}_{bb} &= \{b_3, b_9\}, & \mathcal{B}_{ww} &= \{b_4, b_{10}\}, & \mathcal{B}_{bw} &= \{b_2, b_6, b_8\}, & \mathcal{B}_{wb} &= \{b_1, b_5, b_7\} \\ \mathcal{D} &= \{a_{11}, b_{11}\}. \end{aligned}$$

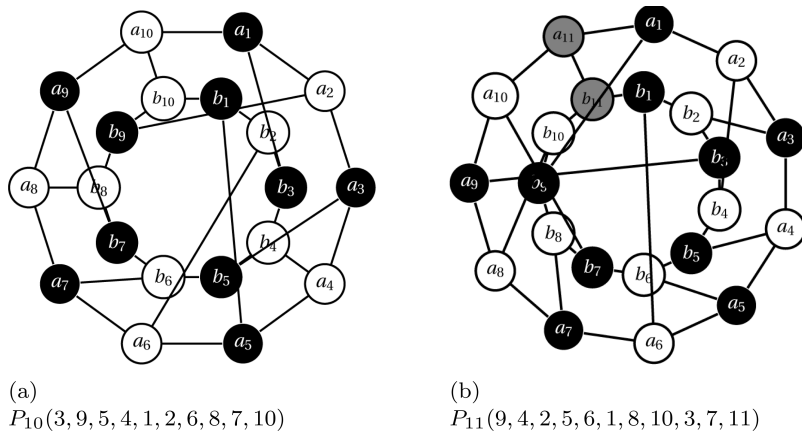


Figure 2: Permutation cycle graphs

Lemma 5. *Let $P_n(\alpha)$ be a cycle permutation graph with $2n$ vertices, then:*

$$|\mathcal{A}_{bb}| = |\mathcal{A}_{ww}| = |\mathcal{B}_{bb}| = |\mathcal{B}_{ww}|$$

$$|\mathcal{A}_{bw}| = |\mathcal{A}_{wb}| = |\mathcal{B}_{bw}| = |\mathcal{B}_{wb}|.$$

Proof. Consider, $|\mathcal{A}_{bb}| = a_{bb}$, $|\mathcal{A}_{bw}| = a_{bw}$, $|\mathcal{A}_{wb}| = a_{wb}$, $|\mathcal{A}_{ww}| = a_{ww}$, $|\mathcal{B}_{bb}| = b_{bb}$, $|\mathcal{B}_{bw}| = b_{bw}$, $|\mathcal{B}_{wb}| = b_{wb}$, $|\mathcal{B}_{ww}| = b_{ww}$ and $|\mathcal{D}| = d$.

It is clear that $a_{bb} = b_{bb}$, $a_{ww} = b_{ww}$, $a_{bw} = b_{bw}$ and $a_{wb} = b_{wb}$.

Assume to the contrary and $a_{bb} \neq a_{ww}$. Without loss of generality, suppose that $a_{bb} \leq a_{ww} - 1$. $k = \lfloor n/2 \rfloor$ so:

$$\begin{aligned} a_{bb} &\leq a_{ww} - 1 \\ \implies a_{bw} &\geq k - a_{ww} + 1 && (\text{ by } a_{bb} + a_{bw} = k) \\ \implies b_{bw} &\geq k - a_{ww} + 1 && (\text{ by } a_{bw} = b_{bw}) \\ \implies b_{ww} &\leq a_{ww} - 1 && (\text{ by } a_{bw} + b_{ww} = k) \\ \implies a_{ww} &\leq a_{ww} - 1 && (\text{ by } b_{ww} = a_{ww}) \end{aligned}$$

that is inconsistency, So the result is desired. □

Consider $S_1 = \mathcal{A}_{bb} \cup \mathcal{A}_{bw} \cup \mathcal{B}_{ww}$, $S_2 = \mathcal{A}_{ww} \cup \mathcal{A}_{wb} \cup \mathcal{B}_{bb}$, $S_3 = \mathcal{B}_{bw}$ and $S_4 = \mathcal{B}_{wb}$. Notice, if $b_{bw} \geq b_{bb}$ we exchange \mathcal{B}_{bb} with \mathcal{B}_{bw} in S_1 and S_3 , \mathcal{B}_{ww} with \mathcal{B}_{wb} in S_2 and S_4 . We know that S_3 and S_4 are independent sets. For Example 4 we have:

$$\begin{aligned} \text{a) } S_1 &= \mathcal{A}_{bb} \cup \mathcal{A}_{bw} \cup \mathcal{B}_{ww} = \{a_1, a_3, a_5, a_9, a_7, b_2, b_4, b_8, b_{10}\} \\ S_2 &= \mathcal{A}_{ww} \cup \mathcal{A}_{wb} \cup \mathcal{B}_{bb} = \{a_4, a_6, a_8, a_{10}, a_2, b_1, b_3, b_5, b_7\} \\ S_3 &= \mathcal{B}_{bw} = \{b_6\}, S_4 = \mathcal{B}_{wb} = \{b_9\}, S_5 = \emptyset. \end{aligned}$$

$$\begin{aligned} \text{b) } S_1 &= \mathcal{A}_{bb} \cup \mathcal{A}_{bw} \cup \mathcal{B}_{wb} = \{a_1, a_9, a_3, a_5, a_7, b_1, b_5, b_7\} \\ S_2 &= \mathcal{A}_{ww} \cup \mathcal{A}_{wb} \cup \mathcal{B}_{bw} = \{a_2, a_8, a_4, a_6, a_{10}, b_2, b_6, b_8\} \\ S_3 &= \mathcal{B}_{bb} = \{b_3, b_9\}, S_4 = \mathcal{B}_{ww} = \{b_4, b_{10}\}, S_5 = \{a_{11}, b_{11}\}. \end{aligned}$$

Lemma 6. *The set S_1 and S_2 are independent sets.*

Proof. It is clear that there are no edges between vertices of \mathcal{A}_{bb} and \mathcal{A}_{bw} . Furthermore vertices of \mathcal{B}_{ww} are connected to white vertices of outer cycle that means \mathcal{A}_{ww} and $\mathcal{A}_{ww} \not\subseteq S_1$, so, we have the desired result. For S_2 , proof is the same. □

Lemma 7. *Every vertex of S_3 has exactly one neighbor in S_1 .*

Proof. At first, every vertex of $S_3 = \mathcal{B}_{bw}$ has at least one neighbor on S_1 , because each vertex of S_3 must be connected to a black vertex of outer cycle and all black vertices of

outer cycle are in S_1 . Now we prove that it is only one and no more. According to the definition, each vertex of S_3 has two black neighbors in inner cycle that they are not in S_1 , so the lemma is proved. \square

This proof can be extended to S_2 and S_4 . By lemma above, for the cut set of $S = S_2 \cup S_4 \cup S_5$, the largest component will be $m(P_n(\alpha)) \leq 2$ and $\omega(P_n(\alpha) - S) = |S_1|$.

Theorem 8. *Let $P_n(\alpha)$ be a cycle permutation with $2n$ vertices and $b = \min(b_{bb}, b_{bw})$, then:*

$$T(P_n(\alpha)) \leq \begin{cases} \frac{n+2}{n-b} & \text{if } n \text{ is even,} \\ \frac{n+3}{n-b-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Now, to prove the theorem we start with even n , let the cut set is $S = S_2 \cup S_4$, then:

$$\begin{aligned} T(P_n(\alpha)) &\leq \frac{|S| + m(P_n(\alpha) - S)}{\omega(P_n(\alpha) - S)} \\ &\leq \frac{a_{bb} + a_{bw} + b_{ww} + b_{wb} + 2}{a_{ww} + a_{wb} + b_{bb}} \quad \text{by } m(P_n(\alpha) - S) \leq 2 \\ &\leq \frac{n+2}{n-b} \end{aligned}$$

For odd n , let the cut set is $S = S_2 \cup S_4 \cup S_5$, so:

$$\begin{aligned} T(P_n(\alpha)) &\leq \frac{|S| + m(P_n(\alpha) - S)}{\omega(P_n(\alpha) - S)} \\ &\leq \frac{a_{bb} + a_{bw} + b_{ww} + b_{wb} + |S_5| + 2}{a_{ww} + a_{wb} + b_{bb}} \quad \text{by } m(P_n(\alpha) - S) \leq 2 \\ &\leq \frac{n+3}{n-b-1} \end{aligned}$$

so the theorem is proved. \square

For example 4, $T(P_{10}(\alpha)) \leq \frac{12}{9}$ and $T(P_{11}(\alpha)) \leq \frac{14}{8}$.

When the number of vertices goes to infinity, if b is small enough, so the tenacity value close to 1^+ . We next investigate the sharpness of the bounds provided above if b is less than or equal 1.

Theorem 9. *The above theorem provide the close answer.*

Proof. The graph $G = P_n(\alpha)$ is defined as follows. Let the cycle of C_1 with n vertices is $a_1 a_2 \cdots a_n a_1$ in clockwise as outer cycle and C_2 with n vertices is $b_1 b_2 \cdots b_n b_1$ in clockwise as inner cycle. At first, suppose n be an even, let the vertices $\{a_1, a_3, \cdots, a_{n-1}, b_1, b_3, \cdots, b_{n-1}\}$

are colored black and the other vertices are colored with white. For permutation edges, suppose each vertex of outer cycle except two optional vertices with opposite color is connected to just one of the vertices of inner cycle of opposite color, and the exceptional outer cycle vertices are matched to the identical color vertices of the inner cycle. By theorem [8], $|S| = n$, $m(G - S) = 2$ and $\omega(G - S) = n - 1$ therefore $T \leq \frac{n+2}{n-1}$.

For odd number n , suppose the vertices $\{a_1, a_3, \dots, a_{n-2}, b_1, b_3, \dots, b_{n-2}\}$ are colored with black and the vertices $\{a_2, a_4, \dots, a_{n-1}, b_2, b_4, \dots, b_{n-1}\}$ are colored with white, a_n and b_n are gray. For permutation edges, let each vertex of outer cycle is connected to just one of the vertices of inner cycle of opposite color, and at last a_n is connected to b_n . By theorem [8], $|S| = n + 1$, $m(G - S) = 2$ and $\omega(G - S) = n - 1$ therefore $T \leq \frac{n+3}{n-1}$.

The second result has obtained is $\beta(G) \geq n - 1$. On the other hand, by preposition [1] for each cut set S , $|S| \geq \omega(G - S)$ so $\beta \leq n$. The independent number $\beta \neq n$, since the graph G is not 3-regular bipartite graph. So let S is an T -set of G , $\omega(G - S) = \omega$ and $m(G - S) = m$:

$$\begin{aligned} T &= \frac{|S| + m}{\omega} \geq \frac{\omega + m}{\omega} \\ &\geq 1 + \frac{2}{\omega} \\ &\geq 1 + \frac{2}{\beta} \\ &= \frac{n + 1}{n - 1} \end{aligned}$$

for n is an odd number, proof is the same, so:

$$\frac{n + 1}{n - 1} \leq T \leq \begin{cases} \frac{n + 2}{n - 1} & n \text{ is even} \\ \frac{n + 3}{n - 1} & n \text{ is odd} \end{cases}$$

□

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