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A Survey on Stability Measure of Networks

Peyman Nasehpour*[1]

[1]Department of Algorithms and Computation, Faculty of Engineering Science, University of Tehran

ABSTRACT

In this paper we discuss about tenacity and its properties in stability calculation. We indicate relationships between tenacity and connectivity, tenacity and binding number, tenacity and toughness. We also give good lower and upper bounds for tenacity.

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DEFINITION:

Since we are primarily interested in the case where disruption of the graph is caused by the removal of a vertex or vertices (and the resulting loss of all edges incident with the removed vertices), we shall restrict our discussion to vertex stability measures. In the interest of completeness, however, we have included several related measures of edge stability.

The first two measures provide information about how easily the graph can be broken-up by the removal of specific sets of vertices.

The vertex connectivity [18-21], $\kappa = \kappa(G)$, of a finite, undirected, connected, simple graph G (without loops or multiple edges) is the minimum number of vertices whose removal

*Email: nasehpour@gmail.com

results in a disconnected graph or results in the trivial graph K_1. Graph G is called n-connected if $\kappa \geq n$. Analogously, the edge-connectivity [18-21], $\lambda = \lambda(G)$, of a finite, undirected, connected simple graph G is the minimum number of edges whose removal results in a disconnected or trivial graph K_1. A graph G is called n-edge- connected if $\lambda(G) \geq n$.

A collection of vertices in V(G) is called a cutset if their removal disconnects G, and a collection of edges in V(G) is called an edge-cutset if their removal disconnects G.

The binding number of a graph G was defined by Woodall in [45] as

$$bind(G) = \min_{A}\{\frac{\mid N(A) \mid}{\mid A \mid}\}$$

where $\phi \neq A \subseteq V(G)$ and $N(A) \neq V(G)$. In [46,47] the binding number was called the melting-point of the graph. the reason for the name "binding number" is that, roughly speaking, if bind(G) is large, then the vertices of G are bound tightly together, in the sense that G has many edges fairly well distributed. We stat some of the results in [45].

(1) $bind(K_n) = n - 1$ for $n \geq 1$.

(2) $bind(K_{a,b}) = min(\frac{a}{b}, \frac{b}{a})$ for $(a \geq 1, b \geq 1)$.

(3) If $G = C_n$, with $n \geq 3$, then $bind(G) = \begin{cases} 1, \text{for } n\ even, \\ \frac{n-1}{n-2}, \text{for } n\ odd. \end{cases}$

(4) If $G = P_n$, with $n \geq 1$, then $bind(G) = \begin{cases} 1, \text{for } n\ even \\ \frac{n-1}{n+1}, \text{for } n\ odd. \end{cases}$

Kane, Mohanty and Hales [28], studied the binding numbers of four types of product graphs : cartesian product, tensor product, strong cartesian product and lexicographic product. Since it is difficult to determine the binding numbers of products of arbitrary graphs, they restricted themselves to products of two graphs which could be any one of the following types of graphs : complete graph (K_n), complete bipartite graph $(K_{m,n})$, cycle (C_n) and path (P_n).

In [45] Woodall proved that, if $bind(G) \geq c$, then G contains at least $\frac{|V(G)|c}{c+1}$ disjoint edges if $0 \leq c \leq \frac{1}{2}$, at least $\frac{|V(G)|(3c-2)}{3c} - \frac{2(c-1)}{c}$ disjoint edges if $1 \leq c \leq \frac{4}{3}$, a Hamiltonian circuit if $c \geq \frac{3}{2}$, and a circuit of length at least $\frac{3(|V(G)|-1)(c-1)}{c}$ if $1 < c \leq \frac{3}{2}$.

The next set of measures also take into consideration the structure of the graph G-A. In particular, they reflect how badly the graph G-A has been disconnected. Since we must ultimately face the reconnection problem - repairing a broken network - these measures could prove to be very useful.

The concept of integrity of a graph G was introduced in [9] as a useful measure of the vulnerability of a graph G. If we think of the graph as modeling a network, vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations. The vertex integrity of a graph G, is defined as $I(G) = min\{\mid A \mid + \tau(G - A)\}$, where the minimum is taken over all $A \subseteq V(G)$ and $\tau(G - A)$ is the maximum order of a component of G-A.

In [9], Barefoot, Entringer and Swart compared integrity, connectivity, binding number and toughness for several classes of graphs. The integrities of the several classes of graphs calculated in [9] were determined using ad hoc methods. Any set A with the property that $| A | +\tau(G-A) = I(G)$ is called an I-set of G. The corresponding edge version called the edge-integrity I'(G) is defined as $I'(G) = min\{| E' | +\tau(G-E')\}$, where the minimum is taken over all $E' \subseteq E(G)$. Thus, for instance, a small edge-integrity is in some sence a measure of how a graph can be split into "small pieces" by the removal of a "few" edges. Bagga, Beineke, Lipman and Pippert in [6], first listed some basic facts about the edge integrity : In [24] Fellows and Stuekle studied the computational complexity of edge - integrity. In [1] a new lower bound on the edge integrity of graphs in general is given, but most of the results concern trees.

The toughness of a graph G was introduced by Chvátal in [13], who observed the relationship between this parameter and the existence of Hamilton cycles in the given graph, and several results regarding this invariant were obtained. The original approach to toughness is as follows. A connected graph G is called t-tough if $t\omega(G-A) \leq| A |$ for any subset A of V(G) with $\omega(G-A) > 1$, [13,23,37,38]. If G is not complete, then there is a largest t such that G is t-tough; this number is the toughness of G and denoted by t(G). Thus $t(G) = min\{\frac{|A|}{\omega(G-A)}\}$, where A is a cutset of G. Since a complete graph has no cutset A, we set $t(K_n) = \infty$ for all $n \geq 1$.

An alternate definition is easier to apply in some cases. Let G be an (n,e) gragh of connectivity κ, $G \neq K_n$, $\omega_p = max\{\omega(G-A)\}$,where $| A |= p$, and $t_p = \frac{p}{\omega_p}$. Then G is t-tough for $0 \leq t \leq \min(t_p)$, where $\kappa \leq p$.

There exist other stability measures such as the edge-connectivity vector [39], the ratio of disruption [32], the complement of disruption, the cut frequency vector, cohesion [41,43], and neighbor-connectivity [21].

Tenacity and its Properties:

The tenacity is a new invariant for graphs. It is another stability measure, incorporating ideas of both toughness and integrity. The tenacity of a graph G, T(G) is defined by $T(G) = min\{\frac{|A|+\tau(G-A)}{\omega(G-A)}\}$, where the minimum is taken over all vertex cutset A of G. We define G-A to be the graph induced by the vertices of V-A, $\tau(G-A)$ is the number of vertices in the largest component of the graph induced by G-A and $\omega(G-A)$is the number of components of G-A. A connected graph G is called T-tenacious if $| A | +\tau(G-A) \geq T\omega(G-A)$ holds for any subset A of vertices of G with $\omega(G-A) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $A \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|A|+\tau(G-A)}{\omega(G-A)}$.

We also consider the edge-tenacity, T'(G), defined by $T'(G) = min\{\frac{|F|+\tau(G-F)}{\omega(G-F)}\}$, where the minimum is taken over all edge cutset F of G. A set $F \subseteq E(G)$ is said to be a T'-set of G if $T'(G) = \frac{|F|+\tau(G-F)}{\omega(G-F)}$.

In this paper we will prove a number of basic results about tenacity. We can prove the following propositions:

Proposition 1: If G is a spanning subgragh of H, then $T(G) \leq T(H)$.

Proposition 2: For any graph G, $T(G) \geq \frac{\kappa(G)+1}{\alpha(G)}$.

Proposition 3: If G is not complete, then $T(G) \leq \frac{n-\alpha(G)+1}{\alpha(G)}$, where n is the number of vertices in G.
Proposition 4: If $m \leq n$ then $T(K_{m,n}) = \frac{m+1}{n}$.

Without attempting to obtain the best possible result, we can prove the following relation between T(G) and t(G). This result gives us a number of corollaries.

Theorem 1: For any graph G, $T(G) \geq t(G) + \frac{1}{\alpha(G)}$.
Proof: Let $A \subseteq V(G)$ be a t-set and $B \subseteq G$ be a T-set. Then $\frac{|B|+\tau(G-B)}{\omega(G-B)} \geq \frac{|B|}{\omega(G-B)} + \frac{1}{\omega(G-B)} \geq \frac{|A|}{\omega(G-A)} + \frac{1}{\alpha(G)}$.
Corollary 1: For any graph G, $T(G^2) > \kappa(G)$.
Corollary 2: Let G be a non-empty graph and let m be the largest integer such that $K_{1,m}$ is an induced subgraph of G. Then $T(G) \geq \frac{\kappa(G)}{m} + \frac{1}{\alpha(G)}$.
Theorem 2: If G is connected and a noncomplete $K_{1,3}$-free graph then $T(G) > \frac{\kappa(G)}{2}$.
Theorem 3: For any nontrivial noncomplete graph G on n vertices and any vertex v, $T(G-v) \geq T(G) - \frac{1}{2}$.
We next obtain some bounds on the tenacity of a graph.

Proposition 5: If G is connected, then $T(G) \geq \frac{1}{\Delta(G)}$.
Proof: K_n is a speacial case, otherwise the removal of any vertex of a connected graph G yields at most $\Delta(G)$ components. Similarly, the removal of any n verices yields at most $n\Delta(G)$ components. Then, from the definition we have $T(G) \geq \frac{n+1}{n\Delta(G)} \geq \frac{1}{\Delta(G)}$.
Lemma 1: If A is a minimal T-set for the graph G then, for each vertex v of A, the induced subgraph $< V(G) - A + v >$ has fewer components than does G-A.
Proof: Let $A' = A - v$. If G-A' has at least as many components as G-A, then $| A' |=| A | -1$ and $\tau(G - A') \leq \tau(G - A) + 1$. Therefore $\frac{|A'|+\tau(G-A')}{\omega(G-A)} = \frac{|A|-1+\tau(G-A')}{\omega(G-A)} \leq \frac{|A|-1+\tau(G-A)+1}{\omega(G-A)} = T(G)$, contrary to our choice of A.
Theorem 4: Let $G = G_1 + G_2$, where $| V(G) |= n$, $| V(G_i) |= p_i$, $T(G) = T$ and $T(G_i) = T_i$ for $i = 1, 2$. Then if $G \neq K_n$ we have

$$min\{\frac{[n+\tau(G_1-A_1)]T_1}{p_1+\tau(G_1-A_1}, \frac{[n+\tau(G_2-A_2)]T_2}{p_2+\tau(G_2-A_2)}\} < T \leq min\{\frac{n-\alpha_1+1}{\alpha_1}, \frac{n-\alpha_2+1}{\alpha 2}\},$$

where α_i is the independence number of G_i, and A_i is a disconnecting set of G_i for $i = 1, 2$.
Theorem 5: Let G be a graph with n vertices and $G \neq K_n$, then $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$.
Proof: We observe that at least one of G or $\overline{G}$ is connected. Suppose $\overline{G}$ is not connected. We proved (Proposition 5) that $T(G) \geq \frac{1}{\Delta(G)} \geq \frac{1}{n-1}$ for any graph G. Thus, $T(G)+T(\overline{G}) \geq \frac{1}{n-1}$. Now suppose G is not connected but $\overline{G}$ is connected. Again by Proposition 5, we have $T(\overline{G}) \geq \frac{1}{n-1}$. Therefore $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$.
Theorem 6: Let G be a graph with $0 < T(G) < \infty$, and let $\lambda(G) = \lambda$, then $T(L(G)) > \frac{\lambda}{2}$.
Theorem 7: For any graph G, $T(G) \geq bind(G) - 1$.

Proof: Let bind(G) = c. If $c < 1$, then $c - 1 < 0$ and the result follows since T(G) is nonnegative. Consider $c \geq 1$. Suppose that A is a subset of V(G) such that $\omega = \omega(G - A) \geq 2$. We want to prove that $\frac{|A|+1}{\omega} > (c - 1)$. If each of the ω components of G-A has at least two vertices, let S consist of the vertices in all the components except the smallest, so that

$$| S |\geq \frac{| V(G) - A | (\omega - 1)}{\omega} \geq \frac{2\omega(\omega - 1)}{\omega} = 2(\omega - 1) \geq \omega.$$

If, on the other hand, V(G)-A contains an isolated vertex, let $S = V(G) - A$. So that $| S |=| V(G) - A |\geq \omega$. In either case $N(S) \neq V(G)$, and since $bind(G) = c \geq 1$,

$$| S | + | A | +1 >| S | + | A |\geq| N(S) |\geq c | S | .$$

It follows that $| A | +1 > (c - 1) | S |\geq (c - 1)\omega$. Therefore $\frac{|A|+1}{\omega} > c - 1$, so $T > c - 1$.

In [36] Moazzami showed the Hamiltoinan Properties of tenacity. The results follows for a graph G:

1) $1 < \frac{\kappa(G)}{\alpha(G)} < \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$

2) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1.$

Graphs satisfying the second inequality are not Hamiltonian-connected. Graphs satisfying the first inequality are Hamiltonian-connected.

3) $1 + \frac{n+1}{\alpha(G)} \leq \frac{\kappa(G)+1}{\alpha(G)} \leq T(G)$

4) $\frac{\kappa(G)+1}{\alpha(G)} \leq T(G) < 1 + \frac{n+1}{\alpha(G)}$

If G satisfies the forth inequality it is not n-Hamiltonian. If G satisfies the third inequality then G is n-Hamiltonian.

In [36], Moazzami also obtained some bounds on the tenacity of products of graphs. Note that the first inequality in the following theorem, is a corollary to Theorem 1

In [35], Moazzami compared integrity, connectivity, binding number, toughness and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability.

References

[1] Bagga, K.S., L. W. Beineke, M.J. Lipman and R.E. Pippert, Bounds and an algorithm for the edge-integrity of trees.

[2] Bagga, K.S., L. W. Beineke, M.J. Lipman and R.E. Pippert, A classification scheme for vulnerability and reliability parameters of graphs, **Congr. Numer.**

[3] Bagga, K.S., L. W. Beineke, M.J. Lipman and R.E. Pippert, The concept of leverage in network vulnerability.

[4] Bagga, K.S., L. W. Beineke, M.J. Lipman and R.E. Pippert, Extension of an algorithm for computing the edge-integrity of a tree.

[5] Bagga, K.S., L. W. Beineke, M.J. Lipman and R.E. Pippert, The seperation sequence and network reliability.

[6] Bagga, K.S., L. W. Beineke, M.J. Lipman and R.E. Pippert, On the edge- integrity of graphs, **Congr. Numer. 60** (1987), 141-144.

[7] Bagga, K.S., L. W. Beineke, M.J. Lipman, R.E. Pippert and R.L. Sedlmeyer, A good algorithm for the computation of the edge-integrity of trees.

[8] Barefoot, C. A., R. Entringer and H. Swart, Integrity of trees and powers of cycles, **Congr. Numer. 58** (1987), 103-114.

[9] Barefoot, C. A., R. Entringer and H. Swart, Vulnerability in graphs-a comparative survey, **J. Combin. Math. Combin. Comput. 1** (1987), 13-22.

[10] Bundy, J.A., U.S.R. Murty, Graph Theory with Applications (The Macmillan press Ltd, 1976).

[11] Chartrand, G., A graph-theoretical approach to a communication problem, **J.Siam App. Math. 14** (1966) 778-781.

[12] Chartrand, G., S.F. Kapoor and D.r. Lick, n-Hamiltonian graphs, **J. Combin. Theory 9** (1970) 305-312.

[13] Chvátal, V., Tough graphs and Hamiltonian circuits, **Discrete Math.5** (1973), 215-228.

[14] Chvátal,V. and P. Erdös, A note on Hamiltonian Circuites, **Discrete Math. 2** (1972),111-113.

[15] Clark, L., R. C. Entringer and M. R. Fellows, Computational complexity of integrity, **J. Combin. Math. Combin. Comput. 2** (1987), 179-191.

[16] Cozzens, M.B, D. Moazzami, and S. Stueckle, The tenacity of the Harary Graphs, **J. Combin. Math. Combin. Comput. 16** (1994), 33-56.

[17] Cozzens, M.B, D. Moazzami, S. Stueckle, The tenacity of a graph, **Graph Theory, Combinatorics, and Algorithms (Yousef Alavi and Allen Schwenk eds.) Wiley**, New York, (1995), 1111 - 1122.

[18] Cozzens, M.B. and S.S. Wu, Graphs that are n-edge connected and k-edge critical, **Disc, Math.**

[19] Cozzens, M.B. and S.S. Wu, Maximum critical n-edge connected graphs, **J. of Graph Theory.**

[20] Cozzens, M.B. and S.S. Wu, On minimum critical n-edge connected graphs, **Siam J. of Alg. and Disc. Methods 8 (4)** (October 1987), 659-669.

[21] Cozzens, M.B. and S.S. Wu, Critical neighborhood connectivity, **Ars Combinatoria.** Vol.29 (1990), 144-160.

[22] Doty, L.L., A large class of maximally tough graphs, **OR Spektrum 13** (1991), 147-151.

[23] Enomoto, H., B. Jackson, P. Katerinis and A. Saito, Toughness and the existence of k-factors, **J. Graph Theory 9** (1985), 87-95.

[24] Fellows, M.R. and S. Stueckle, The immersion order, forbidden subgraphs and the complexity of integrity, **Graphs and Combin.**

[25] Goddard, W.D. and H.C. swart, On the toughness of a graph, **Quaestiones Math. 13** (1990), 217-232.

[26] Guichard, D.R., Binding number of the Cartesian product of two cycles, **Ars Combin. 19** (1985), 175-178.

[27] Kane,V.G. and S.P. Mohanty, Binding number, cycles and complete graphs, **Lect. Notes in Math. (Combinatorics and Graph Theory, Proceedings, Calcutta 1980) 885,**, (S.B. Rao, Ed.), Springer, Berlin, (1981), 290-296.

[28] Kane, V. G., S. P. Mohanty and R. S. Hales, Product graphs and binding number, **Ars Combin. 11** (1981), 201-224.

[29] Kane, V. G., S. P. Mohanty and E. G. Straus, Which rational numbers are binding numbers?, **J. Graph Theory 5** (1981), 379-384.

[30] Katerinis, P. and D. R. Woodall, Binding numbers of graphs and the existence of k-factors, **Quart. J. Math., Oxford, Ser.(2) 38** (1987), 221-228.

[31] Lesniak, L.M. and R.E. Pippert, On the edge-connectivity vector of a graph.

[32] Lipman, M.J. and R.E. Pippert, Toward a measure of vulnerability; the ratio of disruption, **Graph Theory with Applications to Algorithms and Computer Science**, (Y. Alavi, et al., Ed.), Wiley, New York, (1985), 507-517.

[33] Liu, J. and S. Tian, The binding number of Cartesian products of n circuits, **J. Shandong Coll. Ocean. 14** (1984), 97-101.

[34] Moazzami, D., The NSM of a graph, **Combinatorics Advances** (C. J. Colbourn and E.S. Mahmoodian, eds.), Kluwer, 1995, 243-250.

[35] Moazzami, D., "Stability Measure of a graph - a Survey, **J. Utilitas Mathematica 57** (2000), 171-191.

[36] Moazzami, D., "Vulnerability in Graphs - a Comparative Survey", **J. Combin. Math. Combin. Comput. 30** (1999), 23-31.

[37] Molluzzo, J.C., Toughness, Hamiltonian connectedness and n-Hamiltonicity, in **Annals N.Y. Acad. Sci. 319**, Proceeding of Second Int'l Conf. on Combin. Math., New York, (1979) (A. Gewirtz, et al., eds.), 402-404.

[38] Pippert, R.E., On the toughness of a graph, **Lect. Notes in Math. (Graph Theory and its Applications) 303,** (Yousef. Alavi, et al., ed.), Springer, Berlin, (1972), 225-233.

[39] Pippert, R.E. and M.J. Lipman, Toward a measure of vulnerability, the edge connectivity vector, **Graph Theory with Applications to Algorithms and Computer Science**, (Y. Alavi, et al., Ed.) Wiley, New York, (1985), 651-657.

[40] Saito, A. and S.L. Tian, The binding number of Line Graphs and Total Graphs, **Graphs Combin. 1** (1985), 351-356.

[41] Shi, R., The binding number of a graph and its pancyclicism, **Acta Math. Appl. Sin. 3** (1987), 257-269.

[42] Tokushinge, N., Binding number and minimum degree for k-factors, **J. Graph Theory 13** (1989), 607-617.

[43] Wang, J., S. Tian and J. Liu, The binding number of product graphs,**Lect. Notes in Math.(Graph Theory, Singapore 1983) 1073**, (K.M. Koh,et al., Ed.), Springer, Berlin, 1984, 119-128.

[44] Wang, J., S. Tian and J.Liu, The binding number of lexicographic products of graphs, **Graph Theory with Applications to Algorithms and Computer Science**, (Y. Alavi, et al., Ed.), Wiley, New York, (1985), 761-776.

[45] Woodall, D. R., The binding number of a graph and its Anderson number, **J. Combin. Theory B 15** (1973), 225-255.

[46] Woodall,D.R., Problems 1 to 3, in **Combinatorics (Proc. 1972 Oxford Combinatorial Conference)** (D.J.Welsh and D.R. Woodall, eds.), Institute of Mathematics and Applications, Southend-Sea, Essex, England, 1972, 359-360.

[47] Woodall, D.R., Abstract No. 20 (The melting-point of a graph, and its Anderson number), **Graph Theory News letter 1 (NO. 4)** (1972).