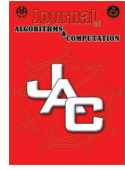




NAKHOD



# JUST CHROMATIC EXCELLENCE IN FUZZY GRAPHS

M. Dharmalingam<sup>\*1</sup> and R. Udaya Suriya<sup>†2</sup>

<sup>1,2</sup>Department of Mathematics, The Madura College, Madurai.

## ABSTRACT

Let  $G$  be a simple fuzzy graph. A family  $\Gamma^f = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$  of fuzzy sets on a set  $V$  is called  $k$ -fuzzy colouring of  $V = (V, \sigma, \mu)$  if i)  $\cup \Gamma^f = \sigma$ , ii)  $\gamma_i \cap \gamma_j = \emptyset$ , iii) for every strong edge  $(x, y)$  (i.e.,  $\mu(xy) > 0$ ) of  $G$   $\min\{\gamma_i(x), \gamma_i(y)\} = 0, (1 \leq i \leq k)$ . The minimum number of  $k$  for which there exists a  $k$ -fuzzy colouring is called the fuzzy chromatic number of  $G$  denoted as  $\chi^f(G)$ . Then  $\Gamma^f$  is the partition of independent sets of vertices of  $G$  in which each sets has the same colour is called the fuzzy chromatic partition. A graph  $G$  is called the just  $\chi^f$ -excellent if every vertex of  $G$  appears as a singleton in exactly one  $\chi^f$ -partition of  $G$ . This paper aims at the study of the new concept namely Just Chromatic excellence in fuzzy graphs. Fuzzy colourful vertex is defined and studied. We explain these new concepts through examples.

*Keyword:* fuzzy chromatic excellent, fuzzy just excellent, fuzzy colorful vertex.

AMS subject Classification: 05C72.

<sup>\*</sup>Corresponding author: Dharmalingam, [kmdharma6902@yahoo.in](mailto:kmdharma6902@yahoo.in)

<sup>†</sup>[udayasuriya20@gmail.com](mailto:udayasuriya20@gmail.com)

## ARTICLE INFO

*Article history:*

Received 22, April 2017

Received in revised form 18, September 2017

Accepted 3 November 2017

Available online 01, December 2017

## 1 Introduction

A fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. The concept of fuzzy sets and fuzzy relations was introduced by L.A.Zadeh in 1965[1] and further studied[2]. It was Rosenfeld[5] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. The concepts of fuzzy trees, blocks, bridges and cut nodes in fuzzy graph has been studied[3]. Computing chromatic sum of an arbitrary graph introduced by Kubica [1989] is known as NP-complete problem. Graph coloring is the most studied problem of combinatorial optimization. As an advancement fuzzy coloring of a fuzzy graph was defined by authors Eslahchi and Onagh in 2004, and later developed by them as Fuzzy vertex coloring[4] in 2006. This fuzzy vertex coloring was extended to fuzzy total coloring in terms of family of fuzzy sets by Lavanya. S and Sattanathan. R[6]. In this paper we are introducing “Just Chromatic excellence in fuzzy graphs”.

## 2 Preliminaries

**Definition 2.1.** A fuzzy graph  $G = (\sigma, \mu)$  is a pair of functions  $\sigma : V \rightarrow [0, 1]$  and  $\mu : V \times V \rightarrow [0, 1]$  where for all  $u, v \in V$ , we have  $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ .

**Definition 2.2.** The order  $p$  and size  $q$  of a fuzzy graph  $G = (\sigma, \mu)$  are defined to be  $p = \sum_{x \in V} \sigma(x)$  and  $q = \sum_{xy \in E} \mu(xy)$ .

**Definition 2.3.** The degree of vertex  $u$  is defined as the sum of the weights of the edges incident at  $u$  and is denoted by  $d(u)$ .

**Definition 2.4.** The union of two fuzzy graphs  $G_1$  and  $G_2$  is defined as a fuzzy graph  $G = G_1 \cup G_2 : ((\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2))$  defined by

$$\begin{aligned} (\sigma_1 \cup \sigma_2)(u) &= \begin{cases} \sigma_1(u), & \text{if } u \in V_1 - V_2 \text{ and} \\ \sigma_2(u) & \text{if } u \in V_2 - V_1 \end{cases} \\ (\mu_1 \cup \mu_2)(uv) &= \begin{cases} \mu_1(uv), & \text{if } uv \in E_1 - E_2 \text{ and} \\ \mu_2(uv) & \text{if } uv \in E_2 - E_1 \end{cases} \end{aligned}$$

**Definition 2.5.** The join of two fuzzy graphs  $G_1$  and  $G_2$  is defined as a fuzzy graph  $G = G_1 + G_2 : ((\sigma_1 + \sigma_2, \mu_1 + \mu_2))$  defined by

$$\begin{aligned} (\sigma_1 + \sigma_2)(u) &= (\sigma_1 \cup \sigma_2)(u) \forall u \in V_1 \cup V_2 \\ (\mu_1 + \mu_2)(uv) &= \begin{cases} (\mu_1 \cup \mu_2)(uv) & \text{if } uv \in E_1 \cup E_2 \text{ and} \\ \sigma_1(u) \wedge \sigma_2(v) & \text{if } uv \in E'. \end{cases} \end{aligned}$$

where  $E'$  is the set of all edges joining the nodes of  $V_1$  and  $V_2$ .

**Definition 2.6.** The cartesian product of two fuzzy graphs  $G_1$  and  $G_2$  is defined as a fuzzy graph  $G = G_1 \times G_2 : (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$  on  $G^* : (V, E)$  where  $V = V_1 \times V_2$  and  $E = \{((\sigma_1, \sigma_2), (\mu_1, \mu_2)) / u_1 = v_1, u_2 v_2 \in E_2 \text{ or } u_2 = v_2, u_1 v_1 \in E_1\}$  with

$$\begin{aligned} (\sigma_1 \times \sigma_2)(u_1, v_1) &= \sigma_1(u_1) \wedge \sigma_2(u_2) \text{ for all } (u_1, u_2) \in V_1 \times V_2 \\ (\mu_1 \times \mu_2)((u_1, u_2)(v_1, v_2)) &= \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2, v_2), & \text{if } u_1 = v_1 \text{ and } u_2 v_2 \in E_2 \\ \sigma_2(u_2) \wedge \mu_1(u_1, v_1), & \text{if } u_2 = v_2 \text{ and } u_1 v_1 \in E_1 \end{cases} \end{aligned}$$

### 3 Main definitions and Results

**Definition 3.1.** Let  $G$  be a fuzzy graph. A family  $\Gamma^f = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$  of fuzzy sets on a set  $V$  is called  $k$ -fuzzy colouring of  $V = (V, \sigma, \mu)$  if

$$(i) \cup \Gamma^f = \sigma,$$

$$(ii) \gamma_i \cap \gamma_j = \emptyset,$$

(iii) for every strong edge  $(x, y)$  (i.e.,  $\mu(xy) > 0$ ) of  $G$   $\min\{\gamma_i(x), \gamma_i(y)\} = 0, (1 \leq i \leq k)$ .

The minimum number of  $k$  for which there exists a  $k$ -fuzzy colouring is called the fuzzy chromatic number of  $G$  denoted as  $\chi^f(G)$ .

**Definition 3.2.**  $\Gamma^f$  is the partition of independent sets of vertices of  $G$  in which each sets has the same colour is called the fuzzy chromatic partition.

**Definition 3.3.** A vertex  $v \in V(G)$  is called  $\Gamma^f$ -good if  $\{v\}$  belongs to a  $\Gamma^f$ -partition. Otherwise  $v$  is said to be  $\Gamma^f$ -bad vertex.

**Definition 3.4.** A graph is called  $\Gamma^f$ -excellent fuzzy graph if every vertex of  $G$  is  $\Gamma^f$ -good.

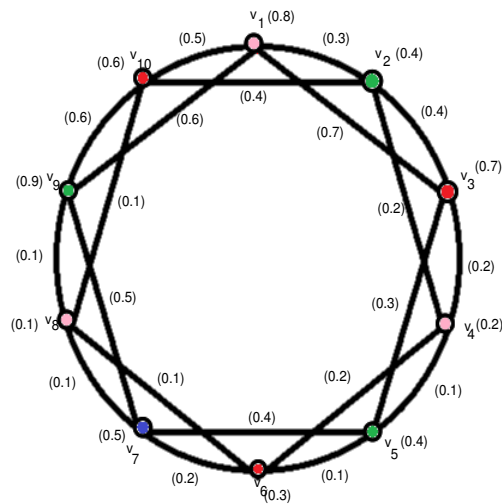
**Definition 3.5.** A graph  $G$  is said to be  $\Gamma^f$ - **commendable** fuzzy graph if the number of  $\Gamma^f$ -good vertices is greater than the number of  $\Gamma^f$ -bad vertices.

A graph  $G$  is said to be  $\Gamma^f$ - **fair** fuzzy graph if the number of  $\Gamma^f$ -good vertices is equal to the number of  $\Gamma^f$ -bad vertices.

A graph  $G$  is said to be  $\Gamma^f$ - **poor** fuzzy graph if the number of  $\Gamma^f$ -good vertices is less than the number of  $\Gamma^f$ -bad vertices.

### 4 Main definitions and Results

**Definition 4.1.** A fuzzy graph  $G$  is just  $\chi^f$ -excellent if every vertex of  $G$  appears as a singleton in exactly one  $\chi^f$ -partition.

**Example 4.2.**

The fuzzy colouring  $\Gamma^f = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$

$$\gamma_1(v_i) = 0.8i = 1$$

$$\gamma_2(v_i) = \begin{cases} 0.4 & i = 2 \\ 0.4 & i = 5 \\ 0.1 & i = 8 \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_3(v_i) = \begin{cases} 0.7 & i = 3 \\ 0.3 & i = 6 \\ 0.9 & i = 9 \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_4(v_i) = \begin{cases} 0.2 & i = 4 \\ 0.5 & i = 7 \\ 0.6 & i = 10 \\ 0 & \text{otherwise} \end{cases}$$

For the above fuzzy graph  $\chi^f(G) = 4$ . Similarly, the  $\chi^f$ -partitions are

$$\begin{aligned}\Gamma_1^f &= \{\{v_1\}, \{v_2, v_5, v_8\}, \{v_3, v_6, v_9\}, \{v_4, v_7, v_{10}\}\} \\ \Gamma_2^f &= \{\{v_2\}, \{v_3, v_6, v_9\}, \{v_4, v_7, v_{10}\}, \{v_5, v_8, v_1\}\} \\ \Gamma_3^f &= \{\{v_3\}, \{v_4, v_7, v_{10}\}, \{v_5, v_8, v_1\}, \{v_6, v_9, v_2\}\} \\ \Gamma_4^f &= \{\{v_4\}, \{v_5, v_8, v_1\}, \{v_6, v_9, v_2\}, \{v_7, v_{10}, v_3\}\} \\ \Gamma_5^f &= \{\{v_5\}, \{v_6, v_9, v_2\}, \{v_7, v_{10}, v_3\}, \{v_8, v_1, v_4\}\} \\ \Gamma_6^f &= \{\{v_6\}, \{v_7, v_{10}, v_3\}, \{v_8, v_1, v_4\}, \{v_9, v_2, v_5\}\} \\ \Gamma_7^f &= \{\{v_7\}, \{v_8, v_1, v_4\}, \{v_9, v_2, v_5\}, \{v_{10}, v_3, v_6\}\} \\ \Gamma_8^f &= \{\{v_8\}, \{v_9, v_2, v_5\}, \{v_{10}, v_3, v_6\}, \{v_1, v_4, v_7\}\} \\ \Gamma_9^f &= \{\{v_9\}, \{v_{10}, v_3, v_6\}, \{v_1, v_4, v_7\}, \{v_2, v_5, v_8\}\} \\ \Gamma_{10}^f &= \{\{v_{10}\}, \{v_1, v_4, v_7\}, \{v_2, v_5, v_8\}, \{v_3, v_6, v_9\}\}\end{aligned}$$

Therefore every vertex in above figure is appears in a singleton in exactly one  $\chi^f$ -partition. Hence above figure is just  $\chi^f$ -excellent.

**Remark 4.3.**

(1) Every just  $\chi^f$ -excellent fuzzy graph is  $\chi^f$ -excellent graph

(2) Let  $G$  be any  $\chi^f$ -excellent graph. Add a vertex  $u$  to every vertex in  $G$  such that  $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$  for every  $v \in V(G)$ . Let the resulting graph be  $H$ . Then  $H$  is  $\chi^f$ -excellent but not just  $\chi^f$ -excellent.

For every  $\chi^f$ -partition of  $H$  contains  $\{u\}$ . Since  $G$  is  $\chi^f$ -excellent, then for any  $v \in V(G)$ , there exists a  $\chi^f$ -partition  $\Gamma^f$  of  $G$  such that  $\{v\} \in \Gamma^f$ . Then  $\Gamma^f \cup \{u\}$  is  $\chi^f$ -partition of  $H$ .

(3) If  $G$  is  $\chi^f$ -excellent, then  $G$  has exactly one  $\chi^f$ -partition (i.e.,  $G$  is uniquely colourable) iff  $G$  is complete.

For: If  $G$  is complete, then  $G$  is  $\chi^f$ -excellent and it has exactly one  $\chi^f$ -partition

Conversely, if  $G$  is  $\chi^f$ -excellent and it has exactly one  $\chi^f$ -partition, then every vertex in  $G$  must appear as a singleton in that  $\chi^f$ -partition. Therefore  $G$  is complete.

**Proposition 4.4.** If  $G$  is not complete fuzzy graph and  $G$  is  $\chi^f$ -excellent, then  $G$  has atleast two  $\chi^f$ -partitions.

**Proof 1.** Let us take  $\Gamma^f$  be a  $\chi^f$ -partition of  $G$ . Since  $G$  is not complete, then there exists atleast non full degree vertex say  $u$ . Let  $\Gamma_1^f = \{\{u\}, V_2, \dots, V_{\chi^f}\}$  be a  $\chi^f$ -partition of  $G$ . Let  $v \in V(G)$  such that  $u$  and  $v$  are not adjacent (i.e.,  $\mu(uv) > \sigma(u) \wedge \sigma(v)$ ). Let  $v \in V_i, 2 \leq i \leq \chi^f$ . Then  $\Gamma_2^f = \{\{V_i - \{v\}, \{u, v\}, V_3, \dots, V_{\chi^f}\}$  is also a  $\chi^f$ -partition of  $G$  not containing  $\{u\}$ .

**Proposition 4.5.** *If  $G$  is not complete fuzzy graph and  $G$  is  $\chi^f$ -excellent, then  $G$  has atleast three  $\chi^f$ -partitions.*

**Proof 2.** *We know that any  $\chi^f$ -excellent non complete fuzzy graph has atleast two  $\chi^f$ -partitions (from the above proposition). Suppose that  $G$  has exactly two  $\chi^f$ -partitions  $\Gamma_1^f$  and  $\Gamma_2^f$ . Let  $\Gamma_1^f = \{V_1, V_2, \dots, V_{\chi^f}\}$  and  $\Gamma_2^f = \{W_1, W_2, \dots, W_{\chi^f}\}$  be the two partitions of  $G$ . Since  $G$  is  $\chi^f$ -excellent and not complete,  $\Gamma_1^f$  has  $r$  singletons and  $\Gamma_2^f$  contains atleast  $n - r$  singletons. Let  $\Gamma_1^f$  contains  $\{u_1\}, \{u_2\}, \dots, \{u_n\}$  and let  $\{u_{r+1}\}, \dots, \{u_n\}$  be the elements of  $\Gamma_2^f$ . Then  $\langle u_1, u_2, \dots, u_r \rangle$  is complete and also  $\langle u_{r+1}, u_{r+2}, \dots, u_n \rangle$  is complete. Therefore in  $\Gamma_1^f$  there will be  $\{u_1\}, \{u_2\}, \dots, \{u_r\}, \{u_{r+1}\}, \dots, \{u_n\}$  elements, a contradiction. Hence there are atleast three  $\chi^f$ -partitions.*

**Remark 4.6.** *A similar arguement as in the above proposition shows that there are atleast four  $\chi^f$ -partitions*

**Remark 4.7.** *There exists fuzzy graphs having not full degree vertex and not just  $\chi^f$ -excellent but  $\chi^f$ -excellent.*

**Proposition 4.8.** *If  $G$  is just  $\chi^f$ -excellent fuzzy graph and  $G \neq K_n$  then  $\chi^f = \lfloor \frac{n+1}{2} \rfloor$ . The converse is not true.*

**Remark 4.9.**  *$P_n$  is not just  $\chi^f$ -excellent fuzzy graph but is an induced subgraph of a just  $\chi^f$ -excellent fuzzy graph. (If  $n$  is odd say  $n = 2k + 1$ , then  $P_n$  is an induced subgraph of  $C_{2k+3}$ . If  $n$  is even say  $n = 2k$ , then  $P_n$  is an induced subgraph of  $C_{2k+1}$ ).*

**Remark 4.10.** *Let  $G \neq K_n$ , be a  $\chi^f$ -excellent fuzzy graph with a full degree vertex. Then  $G$  is not just  $\chi^f$ -excellent.*

**Proof 3.** *Since  $G \neq K_n$ , then  $\chi^f(G) < n$ . Let  $\{u\}$  be a full degree vertex of  $G$ . Then clearly,  $G$  has atleast two  $\chi^f$ -partitions. Then  $\{u\}$  appears in all  $\chi^f$ -partitions of  $G$ . Therefore  $G$  is not just  $\chi^f$ -excellent.*

**Proposition 4.11.** *If  $G$  is a just  $\chi^f$ -excellent fuzzy graph and  $G \neq K_n$ , then any  $\chi^f$ -partition of  $G$  can contain exactly one singleton.*

**Proof 4.** *Let us assume that there exists a  $\chi^f$ -partition  $\Gamma^f$  of  $G$  containing more than one singleton. Let  $\Gamma_1^f = \{\{u_1\}, \{u_2\}, V_3, \dots, V_{\chi^f}\}$  be a partition of  $G$ . Since  $G$  is just  $\chi^f$ -excellent and  $G \neq K_n$ , no vertex of  $V(G)$  is a full degree vertex. Therefore there exists  $v_1 \in V(G)$  such that  $u_1$  and  $v_1$  are not adjacent such that  $\mu(u_1v_1) > \sigma(u_1) \wedge \sigma(v_1)$ . Let  $v_1 \in V_i, 3 \leq i \leq \chi^f$ . Clearly,  $|V_i| \geq 2$ , for if  $V_i = \{v_1\}$ , then  $u_1$  and  $v_1$  are adjacent. Let  $\Gamma_2^f = \{\{u_1, v_1\}, \{u_2\}, V_3, \dots, V_i - \{v_1\}, \dots, V_{\chi^f}\}$ . Then  $\Gamma_2^f$  is a  $\chi^f$ -partition containing  $\{u_2\}$ , which is a contradiction to  $G$  is just  $\chi^f$ -excellent.*

**Corollary 4.12.** *If  $G$  is just  $\chi^f$ -excellent fuzzy graph and  $G \neq K_n$ , then  $\chi^f \leq \lfloor \frac{n+1}{2} \rfloor$ .*

**Proof 5.** *Since  $G$  is just  $\chi^f$ -excellent, then any  $\chi^f$ -partition contains exactly one singleton. Therefore  $n \geq 1 + 2(\chi^f - 1)$ . That is  $n \geq 2(\chi^f - 1)$ . Hence  $\chi^f \leq \lfloor \frac{n+1}{2} \rfloor$ .*

**Remark 4.13.**

(1)  $W_6$  has chromatic number  $4 > \lfloor \frac{n+1}{2} \rfloor$  and  $W_6$  is  $\chi^f$ -excellent. Clearly,  $W_6$  is not just  $\chi^f$ -excellent.

(2) The bound is sharp as seen in  $C_5$  ( $\chi^f(C_5) = 3 = \frac{5+1}{2}$ ) and  $C_5$  is just  $\chi^f$ -excellent.

**Remark 4.14.** The sum of two just  $\chi^f$ -excellent graphs need not be just  $\chi^f$ -excellent. For:  $C_5$  is just  $\chi^f$ -excellent but  $C_5 + C_5$  is not just  $\chi^f$ -excellent.

**Remark 4.15.** If  $G+H$  is just  $\chi^f$ -excellent fuzzy graph then  $G$  and  $H$  are just  $\chi^f$ -excellent graph.

**Proof 6.** Any chromatic partition of  $G+H$  is a union of a chromatic partition of  $G$  and  $H$ . Then  $G+H$  is just  $\chi^f$ -excellent, then  $G$  and  $H$  are just  $\chi^f$ -excellent.

**Proposition 4.16.** If  $G$  and  $H$  are just  $\chi^f$ -excellent fuzzy graph and one of them is not complete if other is  $K_1$  then  $G+H$  is not just  $\chi^f$ -excellent.

**Proof 7.** Let  $G = K_1$ . Then  $H$  is not not complete fuzzy graph. Then  $G+H$  is not complete but it has a full degree vertex. Therefore  $G+H$  is not just  $\chi^f$ -excellent graph. Let  $G \neq K_1$  and  $h \neq K_1$ . Since  $G$  and  $H$  are just  $\chi^f$ -excellent,  $G, H \neq K_n$  for  $n \geq 2$ . Then any  $\chi^f$ -partition of  $G$  and  $H$  contains atleast two elements. Then for any  $\chi^f$ -partition of  $G$  with a singleton element, we can associate several  $\chi^f$ -partitions of  $H$ , giving a  $\chi^f$ -partition of  $G+H$ . Therefore  $G+H$  is not just  $\chi^f$ -excellent.

**Proposition 4.17.** Let  $G$  and  $H$  be two fuzzy graphs.  $G+H$  is just  $\chi^f$ -excellent if and only if both of them are complete graphs.

**Proof 8.** Let us assume that  $G$  and  $H$  are complete fuzzy graph. Then  $G+H$  is complete fuzzy graph and hence just  $\chi^f$ -excellent.

Conversely, assume that  $G+H$  are just  $\chi^f$ -excellent. Therefore both  $G$  and  $H$  are just  $\chi^f$ -excellent. If  $G$  or  $H$  is not complete, then using above remark,  $G+H$  is not just  $\chi^f$ -excellent, a contradiction. Therefore  $G$  and  $H$  are complete. Hence  $G+H$  is complete fuzzy graph.

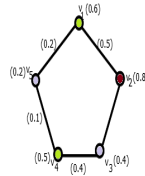
**Proposition 4.18.** Let  $G \neq K_n$  be just  $\chi^f$ -excellent graph. Let  $u \in V(G)$ . Let  $\Gamma^f = \{\{u\}, V_2, \dots, V_{\chi^f}\}$  be a  $\chi^f$ -partition of  $G$ . Then for every vertex in  $V_i, 2 \leq i \leq \chi^f$  is adjacent with atleast one vertex in  $V_j, \text{ for all } j, j \neq i, 2 \leq j \leq \chi^f$ .

**Proof 9.** Since  $G$  is just  $\chi^f$ -excellent,  $|V_i| \geq 2$  for all  $i, 2 \leq i \leq \chi^f$ . Let  $v \in V_i$ . suppose  $v$  is not adjacent to any vertex of some  $V_j$  such that  $\mu(uv) > \sigma(u) \wedge \sigma(v)$ , for  $u \in V_j, j \neq i, 2 \leq j \leq \chi^f$ . Then  $\Gamma_1^f = \{\{u\}, V_2, \dots, V_i - \{v\}, \dots, V_j \cup \{v\}, \dots, V_{\chi^f}\}$  is a  $\chi^f$ -partition of  $G$  (note that  $V_i - \{v\} \neq \emptyset$ ) different from  $\Gamma^f$  a contradiction.

**Definition 4.19.** A vertex of a fuzzy graph  $G$  with respect to a  $\chi^f$ -partition  $\Gamma^f$  of  $G$  is called a fuzzy colourful vertex if it is adjacent to every colour class other than the one to which it belongs.

Let  $\Gamma^f = \{V_1, V_2, \dots, V_{\chi^f}\}$  be a  $\chi^f$ -partition of  $G$ . Let  $u \in V_i$  is said to be fuzzy colourful vertex if  $u$  is adjacent to every colour class in  $\Gamma^f$ -partition but not adjacent to  $V_i$  such that  $\mu(uv_i) \leq \sigma(u) \wedge \sigma(v_i)$  for some vertex  $v_i \in V_1, \dots, V_{i-1}, V_{i+1}, \dots, V_{\chi^f}$  and  $\mu(uv_j) > \sigma(u) \wedge \sigma(v_j)$  for every  $v_j \in V_i$

**Example 4.20.**



For the above figure  $\chi^f(G) = 3$ . Let  $\Gamma^f = \{V_1 = \{v_1\}, V_2 = \{v_2, v_4\}, V_3 = \{v_3, v_5\}\}$  be a  $\chi^f$ -partition. From this partition  $\{v_1\}$  is adjacent to some vertex in  $V_2$  and  $V_3$ ,  $\{v_2\}$  is adjacent to  $V_1$  and  $V_3$ ,  $\{v_4\}$  is adjacent to  $V_3$  but not  $\{v_1\}$ ,  $\{v_3\}$  is adjacent to  $V_2$  but not  $V_1$ ,  $\{v_5\}$  is adjacent to  $V_1$  and  $V_2$ . Hence  $\{v_1, v_2, v_5\}$  are colourful vertices with respect to the  $\Gamma^f$ -partition.

**Corollary 4.21.**

- (1) If  $G$  is just  $\chi^f$ -excellent then every vertex in  $N^f[u], u \in V(G)$  is a fuzzy colourful vertex in the  $\chi^f$ -partition in which  $\{u\}$  is an element. Then the number of colourful vertices is  $\text{deg}^f(u) + 1$ .
- (2) There exists a  $\chi^f$ -partition in which the number of fuzzy colourful vertices is equal to  $\Delta^f(G) + 1$  which is greater than or equal to  $\chi^f(G)$ .

**Theorem 4.22.** Let  $G$  be a just  $\chi^f$ -excellent fuzzy graph which is not complete. Let  $u \in V(G)$  and let  $\Gamma^f = \{\{u\}, V_2, \dots, V_{\chi^f}\}$  be a  $\chi^f$ -partition of  $G$ . If  $|V_i| \geq 3$  for some  $2 \leq i \leq \chi^f$  then there exists a atleast some  $V_j$  with  $|V_j| \geq 3$  containing a vertex not adjacent to  $u$ .

**Proof 10.** Suppose let  $u$  is adjacent to every vertex in  $V_i$  with  $|V_i| \geq 3 (2 \leq i \leq \chi^f)$ .  
**Case(1):**  $|V_i| \geq 3$  for all  $i, 2 \leq i \leq \chi^f$ . Then  $u$  is a full degree vertex and it appears singleton in every  $\chi^f$ -partition of  $G$ , which is a contradiction to  $G$  is just  $\chi^f$ -excellent and  $G \neq K_n$ .  
**Case(2):** Let  $|V_i| \geq 3$  for all  $i, 2 \leq i \leq t$  and  $|V_{t+1}| = 2$ . Let  $|V_{t+1}| = \{v_1, v_2\}$ . Suppose there exists  $V_{t+1}, V_{t+2}, \dots, V_{\chi^f}$  such that  $|V_{t+j}| = 2, 2 \leq j \leq \chi^f - t$  (Note that no  $V_i, (2 \leq i \leq \chi^f)$  is a singleton since  $G$  is just  $\chi^f$ -excellent). Since  $\Gamma^f$  is a  $\chi^f$ -partition,  $u$  is adjacent with atleast one vertex in each of  $V_{t+1}, \dots, V_{\chi^f}$ . Suppose  $u$  is adjacent with  $v_1$  and not adjacent with  $v_2$  in  $V_{t+1}$  such that  $\mu(uv_1) \leq \sigma(u) \wedge \sigma(v_1)$  and  $\mu(uv_2) > \sigma(u) \wedge \sigma(v_2)$  for



$v_1, v_2 \in V_{t+1}$ . Then  $u$  is adjacent with every vertex  $V_{t+j}$ ,  $2 \leq j \leq \chi^f - 1$  such that  $\mu(uv_i) \leq \sigma(u) \wedge \sigma(v_i)$  for every  $v_i \in V_{t+j}$ ,  $2 \leq j \leq \chi^f - 1$ . For: otherwise there exists some vertex  $w \in V_{t+j}$  not adjacent with  $u$ . Therefore  $\Gamma_1^f = \{\{u, v_2, w\}, V_2, \dots, V_t, \{v_1\}, \dots, V_{t+j} - \{w\}, \dots, V_{\chi^f}\}$  which is a contradiction to  $G$  is just  $\chi^f$ -excellent. Hence  $u$  is adjacent with every vertex in  $V - \{v_1\}$ . (Note that if  $V_{t+1} = V_{\chi^f}$  then also  $u$  is adjacent with every vertex in  $V - \{v_2\}$ ). Since  $G$  is just  $\chi^f$ -excellent there exists a  $\chi^f$ -excellent  $\Gamma_2^f = \{\{v_2\}, V'_2, \dots, V'_{\chi^f}\}$ . Therefore  $u \in V'_i$ , a contradiction since  $u$  is adjacent with every vertex in  $V - \{v_2\}$  such that  $\mu(uv_i) \leq \sigma(u) \wedge \sigma(v_i)$  for every vertex  $v_i \in V - \{v_2\}$ . Hence the theorem.

**Remark 4.23.** Let  $G$  be a graph which is just  $\chi^f$ -excellent. If there exists a  $\chi^f$ -partition in which one of the element is a singleton  $\{u\}$  and some other element with cardinality greater than or equal to 3, then there exists a  $\chi^f$ -partition in which none of the elements is singleton.

**Proof 11.** Let  $G$  be a just  $\chi^f$ -excellent fuzzy graph satisfying the hypothesis. Then there exists a  $\chi^f$ -partition  $\Gamma^f = \{\{u\}, V_2, \dots, V_{\chi^f}\}$  in which  $|V_i| \geq 3$  for some  $i$ ,  $2 \leq i \leq \chi^f$  and  $V_i$  contains a non-neighbour, say,  $v$  and  $u$ . Then  $\Gamma_1^f = \{\{u, v\}, V_2, \dots, V_i - \{v\}, \dots, V_{\chi^f}\}$  is a  $\chi^f$ -partition of  $G$  in which each class contains atleast 2 vertices of  $G$ .

**Remark 4.24.** If  $G$  is just  $\chi^f$ -excellent and  $G \neq K_n$  and  $\beta_0^f(G) = 2$ , then the number of  $\chi^f$ -partitions of  $G$  is exactly 'n'. For:

Let  $V(G) = \{u_1, u_2, \dots, u_k\}$ , by the hypothesis there exists a  $\chi^f$ -partitions  $\{\{u_i, V_2, \dots, V_k\}$  and  $|V_i| = 2$  for all  $2 \leq i \leq k$ . Therefore  $|V(G)| = 2k + 1$ . Hence there can not exists a  $\chi^f$ -partitions in which one of the element is a singleton.

**Remark 4.25.** If  $G$  is just  $\chi^f$ -excellent and  $G \neq K_n$ , then  $G$  has exactly 'n'  $\chi^f$ -partitions if and only if in those  $\chi^f$ -partitions in which one element is a singleton, the cardinality of any other element of the partition is 2.

**Remark 4.26.** If  $G$  is just  $\chi^f$ -excellent fuzzy graph, then  $\deg^f(u) \leq n - 3$  for any vertex  $u \in V(G)$

## 5 Application

Fuzzy graph coloring has extensive applications in the following fields and solving different problems as follows: In Human Resource management such as assignment, job allocation, scheduling, In telecommunication process, In Bioinformatics, In traffic light problem.

## 6 Conclusion

In this paper we define new parameter called just chromatic excellence in fuzzy graphs. We can extend this concept to new type of fuzzy chromatic excellence and study the characteristics of this new parameter.

## References

- [1] Zadeh, L.A.,1971, Similarity relations and fuzzy ordering, Information Sciences, 3(2), pp.177-200.
- [2] Kaufmann.A., 1976. Introduction a la theorie des sous-ensembles flous, Io Elements thoriques de base,Paris: Masson et cie.
- [3] M.S Sunitha and Sunil Mathew , Fuzzy Graph Theory: A Survey, Annals of Pure and Applied mathematics, Vol. 4, No.1, 2013, 92-110.
- [4] Eslahchi. C and B.N. Onagh, Vertex Strength of Fuzzy Graphs, International Journal of Mathematics and Mathematical Sciences,Volume 2006.
- [5] Rosenfeld. A, Fuzzy sets and their applications to cognitive and decision Process, Academic press, New york, 1975,77-95.
- [6] Lavanya. S.& Sattanathan. R, Fuzzy total coloring of fuzzy graphs, International journal of information technology and knowledge management, Volume 2, No.1, pp.37-39.
- [7] Sambathkumar. E, Chromatically fixed, free and totally free vertices in a graph, Journal of Combinatorics, Information and system sciences, Vol.17, Nos.1-2,(1992), 130-138.