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k-Remainder Cordial Graphs

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ABSTRACT

In this paper we generalize the remainder cordial labeling, called k-remainder cordial labeling and investigate the 4-remainder cordial labeling behavior of certain graphs.

Keyword: Path; Cycle; Star; Bistar; Crown; Comb; Complete graph.

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1 Introduction

Graphs considered here are finite and simple. Graph labeling is used in several areas of science and technology like coding theory, astronomy, circuit design etc. For more details refer Gallian [2]. The origin of graph labeling is graceful labeling which was introduced by Rosa (1967). Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. The

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bistar $B_{m,n}$ is the graph obtained by making adjacent the two central vertices of $K_{1,m}$ and $K_{1,n}$. A graph $S(G)$ derived from a graph G by a sequence of edge subdivisions is called a subdivision of a graph G. Cahit[1], introduced the concept of cordial labeling of graphs. Recently Ponraj et al. [4], introduced the remainder labeling of graphs and investigated the remainder cordial labeling behavior of several graphs like path, cycle, complete graph, star, bistar etc. Motivated by these concepts, in this paper we generalize the remainder cordial labeling, called k-remainder cordial labeling and investigate the 4-remainder cordial labeling behavior of certain graphs. Terms are not defined here follows from Harary [3] and Gallian [2].

2 k-Remainder cordial labeling

Definition 1. Let G be a (p, q) graph. Let f be a map from $V(G)$ to the set $\{1, 2, \ldots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \ldots, k\}$ where $v_f(x)$ denote the number of vertices labelled with x and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labeled with even integers and number of edges labelled with odd integers. A graph with a k-remainder cordial labeling is called a k-remainder cordial graph.

Remark 2. *When $k = 2$, number of edges with label 0 is q. So there does not exists a 2-remainder cordial labeling.*

Theorem 3. *Every graph is a subgraph of a connected k-remainder cordial graphs for $k \geq 4$.*

Proof. Let G be a (p, q) graph. Consider the k-copies of the complete graph K_p. Let G_i denotes the i^{th} copy of K_p and $V(G_i) = \{u_j^i : 1 \leq j \leq p\}$. Let $s = \binom{p}{2} - 1$. Next consider the s copies of the path on k vertices and denotes i^{th} copy as $P_k^i : v_1^i v_2^i \ldots v_k^i$ $(1 \leq i \leq s)$. We now construct the super graph G^* of the graph G as given below; Let $V(G^*) = \bigcup\limits_{i=1}^{k} V(G_i) \cup \bigcup\limits_{i=1}^{s} V(P_k^i)$ and $E(G^*) = \bigcup\limits_{i=1}^{k} E(G_i) \cup \bigcup\limits_{i=1}^{s} E(P_k^i) \cup \{u_1^i v_1^{i+1} : 1 \leq i \leq k-1\} \cup \{u_2^2 v_3^1\} \cup \{v_2^i v_3^{i+1} : 1 \leq i \leq s-1\} \cup \{v_3^i v_2^{i+1} : 1 \leq i \leq s-1\} \cup \{v_3^i v_4^{i+1} : 1 \leq i \leq s-1\} \cup \{u_2^2 u_2^3, u_3^2 u_3^3, u_4^2 u_4^3\} \cup \{u_2^3 u_2^4, u_3^3 u_3^4\}$. Clearly G^* has $kp + k\binom{p}{2} - k$ vertices and $2(k+1)\binom{p}{2}$ edges. Let f be this vertex labeling. We now check the vertex and edge condition of the remainder cordiality. $v_f(1) = v_f(2) = \ldots = v_f(k) = p + s$ and $e_f(0) = k\binom{p}{2} + s + 1$, $e_f(1) = k - 2 + 1 + 5 + (k-2)s + s - 1 + s - 1 + s - 1 = k\binom{p}{2} + s + 1$. Hence f is a k-remainder cordial labeling of G^*.

□

We now investigate the 4−remainder cordial labeling behaviors of some graphs.

Theorem 4. *The complete graph K_n is 4−remainder cordial iff $n \leq 3$.*

Proof. Suppose f is a 4−remainder cordial labeling of K_n. The proof is divided into four cases.
Case(i). $n > 3$
Subcase(i). $n \equiv 0 \pmod 4$
Let $n = 4t$. Then $v_f(1) = v_f(2) = v_f(3) = v_f(4) = t$
and we find also $e_f(0) = t^2 + t^2 + t^2 + \binom{t}{2} + t^2 + \binom{t}{2} + \binom{t}{2} + \binom{t}{2}$
$= 4t^2 + 4\binom{t}{2}$.
and $e_f(1) = t^2 + t^2 = 2t^2$.
Then $e_f(0) - e_f(1) = 4t^2 + 4\binom{t}{2} - 2t^2$
$= 2t^2 + 4\binom{t}{2}$
$= 2t^2 + \frac{t(t-1)}{2}$
$= 2t^2 + 2t^2 - 2t$
$= 4t^2 - 2t > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$.
which is a contradiction.

Subcase(ii). $n \equiv 1 \pmod 4$
Let $n = 4t + 1$. Then any one of the following four possibilities are occurs.
Type $A : v_f(1) = t + 1, v_f(2) = t, v_f(3) = t, v_f(4) = t$.
Type $B : v_f(1) = t, v_f(2) = t + 1, v_f(3) = t, v_f(4) = t$.
Type $C : v_f(1) = t, v_f(2) = t, v_f(3) = t + 1, v_f(4) = t$.
Type $D : v_f(1) = t, v_f(2) = t, v_f(3) = t, v_f(4) = t + 1$.

Type $A : v_f(1) = t + 1, v_f(2) = t, v_f(3) = t, v_f(4) = t$.
Then $e_f(0) = t(t + 1) + t(t + 1) + t(t + 1) + t^2 + t^2 + \binom{t+1}{2} + \binom{t}{2} + \binom{t}{2} + \binom{t}{2}$
$= 4t^2 + 4t + t^2 + \frac{(t+1)(t+1)-1}{2} + \frac{t(t-1)}{2} + \frac{t(t-1)}{2} + \frac{t(t-1)}{2}$
$= 5t^2 + 4t + \frac{(t^2+t)}{2} + 3\frac{(t^2-t)}{2}$
$= 7t^2 + 3t$.
and $e_f(1) = t^2 + t^2 = 2t^2$.
Then we find $e_f(0) - e_f(1) = 7t^2 + 3t - 2t^2 = 5t^2 + 3t > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $B : v_f(1) = t, v_f(2) = t + 1, v_f(3) = t, v_f(4) = t$.
Then $e_f(0) = t(t + 1) + t^2 + t^2 + t(t + 1) + \binom{t}{2} + \binom{t}{2} + \binom{t}{2} + \binom{t+1}{2}$
$= 4t^2 + 2t + 3\frac{t(t-1)}{2} + \frac{(t+1)(t+1)-1}{2}$
$= 6t^2 + t$.
and $e_f(1) = t(t + 1) + t^2 = 2t^2 + t$.
Then we find $e_f(0) - e_f(1) = (6t^2 + t) - (2t^2 + t) = 4t^2 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $C : v_f(1) = t, v_f(2) = t, v_f(3) = t+1, v_f(4) = t$.
Then $e_f(0) = t(t+1) + t^2 + t^2 + t^2 + \binom{t}{2} + \binom{t}{2} + \binom{t}{2} + \binom{t+1}{2}$
$= 4t^2 + t + 3\frac{t(t-1)}{2} + \frac{(t+1)(t+1)-1}{2} = 6t^2$.
and $e_f(1) = t(t+1) + t(t+1) = 2t^2 + 2t$.
Then we find $e_f(0) - e_f(1) = 6t^2 - (2t^2 + 2t) = 4t^2 - 2t > 1$ for any positive integer t.
Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $D : v_f(1) = t, v_f(2) = t, v_f(3) = t, v_f(4) = t+1$.
Then $e_f(0) = t^2 + t^2 + t(t+1) + t(t+1) + \binom{t}{2} + \binom{t}{2} + \binom{t}{2} + \binom{t+1}{2}$
$= 4t^2 + 2t + 3\frac{t(t-1)}{2} + \frac{(t+1)(t+1)-1}{2} = 6t^2 + t$.
and $e_f(1) = t^2 + t(t+1) = 2t^2 + t$.
Then we find $e_f(0) - e_f(1) = (6t^2 + t) - (2t^2 + t) = 4t^2 > 1$ for any positive integer t.
Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Subcase(iii). $n \equiv 2 \pmod 4$
Let $n = 4t + 2$. In this case any one of the following arises.
Type $A : v_f(1) = t+1, v_f(2) = t+1, v_f(3) = t, v_f(4) = t$.
Type $B : v_f(1) = t+1, v_f(2) = t, v_f(3) = t+1, v_f(4) = t$.
Type $C : v_f(1) = t+1, v_f(2) = t, v_f(3) = t, v_f(4) = t+1$.
Type $D : v_f(1) = t, v_f(2) = t+1, v_f(3) = t+1, v_f(4) = t$.
Type $E : v_f(1) = t, v_f(2) = t+1, v_f(3) = t, v_f(4) = t+1$.
Type $F : v_f(1) = t, v_f(2) = t, v_f(3) = t+1, v_f(4) = t+1$.

Type $A : v_f(1) = t+1, v_f(2) = t+1, v_f(3) = t, v_f(4) = t$.
Then we find $e_f(0) = (t+1)^2 + t(t+1) + t(t+1) + t(t+1) + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t}{2} + \binom{t}{2}$
$= t^2 + 2t + 1 + 3t(t+1) + 2\frac{(t+1)(t+1)-1}{2} + 2\frac{t(t-1)}{2}$
$= t^2 + 2t + 1 + 3t^2 + 3t + 2\frac{(t^2+t)}{2} + 2\frac{(t^2-t)}{2}$
$= 6t^2 + 5t + 1$.
and also $e_f(1) = t(t+1) + t^2 = 2t^2 + t$.
We get $e_f(0) - e_f(1) = (6t^2 + 5t + 1) - (2t^2 + t) = 4t^2 + 4t + 1 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $B : v_f(1) = t+1, v_f(2) = t, v_f(3) = t+1, v_f(4) = t$.
Now we find $e_f(0) = t(t+1) + (t+1)^2 + t(t+1) + t^2 + \binom{t+1}{2} + \binom{t}{2} + \binom{t+1}{2} + \binom{t}{2}$
$= 2t(t+1) + (t+1)^2 + t^2 + 2\binom{t+1}{2} + 2\binom{t}{2}$
$= 2t^2 + 2t + t^2 + 2t + 1 + t^2 + 2\frac{(t+1)(t+1)-1}{2} + 2\frac{t(t-1)}{2}$
$= 4t^2 + 4t + 1 + 2\frac{(t^2+t)}{2} + 2\frac{(t^2-t)}{2}$
$= 6t^2 + 4t + 1$.
and also $e_f(1) = t(t+1) + t(t+1) = 2t^2 + 2t$.
We get $e_f(0) - e_f(1) = (6t^2 + 4t + 1) - (2t^2 + 2t) = 4t^2 + 2t + 1 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $C: v_f(1) = t+1, v_f(2) = t, v_f(3) = t, v_f(4) = t+1$.
Now we find $e_f(0) = t(t+1) + t(t+1) + (t+1)^2 + t(t+1) + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t}{2} + \binom{t}{2}$
$= 3t(t+1) + (t+1)^2 + 2\binom{t+1}{2} + 2\binom{t}{2}$
$= 3t^2 + 3t + t^2 + 2t + 1 + 2\frac{(t+1)(t+1)-1}{2} + 2\frac{t(t-1)}{2}$
$= 4t^2 + 5t + 1 + 2\frac{(t^2+t)}{2} + 2\frac{t^2-t}{2}$
$= 6t^2 + 5t + 1$.
and also $e_f(1) = t^2 + t(t+1) = 2t^2 + t$.
We get $e_f(0) - e_f(1) = (6t^2 + 5t + 1) - (2t^2 + t) = 4t^2 + 4t + 1 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $D: v_f(1) = t, v_f(2) = t+1, v_f(3) = t+1, v_f(4) = t$.
Now we find $e_f(0) = t(t+1) + t(t+1) + t^2 + t(t+1) + \binom{t}{2} + \binom{t}{2} + \binom{t+1}{2} + \binom{t+1}{2}$
$= 3t(t+1) + t^2 + 2\binom{t}{2} + 2\binom{t+1}{2}$
$= 3t^2 + 3t + t^2 + 2\frac{t(t-1)}{2} + 2\frac{(t+1)(t+1)-1}{2}$
$= 4t^2 + 3t + 2\frac{(t^2-t)}{2} + 2\frac{(t^2+t)}{2}$
$= 6t^2 + 3t$.
and also $e_f(1) = (t+1)^2 + t(t+1) = t^2 + 2t + 1 + t^2 + t = 2t^2 + 3t + 1$.
We get $e_f(0) - e_f(1) = (6t^2 + 3t) - (2t^2 + 3t + 1) = 4t^2 - 1 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $E: v_f(1) = t, v_f(2) = t+1, v_f(3) = t, v_f(4) = t+1$.
Now we find $e_f(0) = t(t+1) + t^2 + t(t+1) + (t+1)^2 + \binom{t}{2} + \binom{t}{2} + \binom{t+1}{2} + \binom{t+1}{2}$
$= 2t(t+1) + t^2 + (t+1)^2 + 2\binom{t}{2} + 2\binom{t+1}{2}$
$= 2t^2 + 2t + t^2 + t^2 + 2t + 1 + 2\frac{t(t-1)}{2} + 2\frac{(t+1)(t+1)-1}{2}$
$= 4t^2 + 4t + 1 + 2\frac{(t^2-t)}{2} + 2\frac{(t^2+t)}{2}$
$= 6t^2 + 4t + 1$.
and also $e_f(1) = t(t+1) + t(t+1) = 2t(t+1) = 2t^2 + 2t$.
We get $e_f(0) - e_f(1) = (6t^2 + 4t + 1) - (2t^2 + 2t) = 4t^2 + 2t + 1 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $F: v_f(1) = t, v_f(2) = t, v_f(3) = t+1, v_f(4) = t+1$.
Now we find $e_f(0) = t^2 + t(t+1) + t(t+1) + t(t+1) + \binom{t}{2} + \binom{t}{2} + \binom{t+1}{2} + \binom{t+1}{2}$
$= t^2 + 3t(t+1) + 2\binom{t}{2} + 2\binom{t+1}{2}$
$= 4t^2 + 3t + 2\frac{t(t-1)}{2} + 2\frac{(t+1)(t+1)-1}{2}$
$= 4t^2 + 3t + 2\frac{(t^2-t)}{2} + 2\frac{(t^2+t)}{2}$
$= 6t^2 + 3t$.
and also $e_f(1) = t(t+1) + (t+1)^2 = t^2 + t + t^2 + 2t + 1 = 2t^2 + 3t + 1$.
We get $e_f(0) - e_f(1) = (6t^2 + 3t) - (2t^2 + 3t + 1) = 4t^2 - 1 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Subcase(iv). $n \equiv 3 \pmod 4$

Let $n = 4t + 3$. In this case any one of the following arises.
Type $A : v_f(1) = t + 1, v_f(2) = t + 1, v_f(3) = t + 1, v_f(4) = t$.
Type $B : v_f(1) = t + 1, v_f(2) = t + 1, v_f(3) = t, v_f(4) = t + 1$.
Type $C : v_f(1) = t + 1, v_f(2) = t, v_f(3) = t + 1, v_f(4) = t + 1$.
Type $D : v_f(1) = t, v_f(2) = t + 1, v_f(3) = t + 1, v_f(4) = t + 1$.

Type $A : v_f(1) = t + 1, v_f(2) = t + 1, v_f(3) = t + 1, v_f(4) = t$.
Now we find $e_f(0) = (t + 1)^2 + (t + 1)^2 + t(t + 1) + t(t + 1) + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t}{2}$
$= 2(t + 1)^2 + 2t(t + 1) + 3\binom{t+1}{2} + \binom{t}{2}$
$= 4t^2 + 6t + 2 + \frac{t(t-1)}{2} + 3\frac{(t+1)(t+1)-1}{2}$
$= 4t^2 + 6t + 2 + \frac{(t^2-t)}{2} + 3\frac{(t^2+t)}{2}$
$= 6t^2 + 7t + 2$.
and also $e_f(1) = (t + 1)^2 + t(t + 1) = t^2 + 2t + 1 + t^2 + t = 2t^2 + 3t + 1$.
We get $e_f(0) - e_f(1) = (6t^2 + 7t + 2) - (2t^2 + 3t + 1) = 4t^2 + 4t + 1 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $B : v_f(1) = t + 1, v_f(2) = t + 1, v_f(3) = t, v_f(4) = t + 1$.
Now we find $e_f(0) = t(t + 1) + (t + 1)^2 + (t + 1)^2 + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t}{2} + t(t + 1)$
$= 2(t + 1)^2 + 2t(t + 1) + 3\binom{t+1}{2} + \binom{t}{2}$
$= 4t^2 + 6t + 2 + \frac{t(t-1)}{2} + 3\frac{(t+1)(t+1)-1}{2}$
$= 4t^2 + 6t + 2 + \frac{(t^2-t)}{2} + 3\frac{(t^2+t)}{2}$
$= 6t^2 + 7t + 2$.
and also $e_f(1) = t(t + 1) + (t + 1)^2 = t^2 + t + t^2 + 2t + 1 = 2t^2 + 3t + 1$.
We get $e_f(0) - e_f(1) = (6t^2 + 7t + 2) - (2t^2 + 3t + 1) = 4t^2 + 4t + 1 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $C : v_f(1) = t + 1, v_f(2) = t, v_f(3) = t + 1, v_f(4) = t + 1$.
Now we find $e_f(0) = (t + 1)^2 + t(t + 1) + (t + 1)^2 + (t + 1)^2 + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t}{2}$
$= 3(t + 1)^2 + t(t + 1) + 3\binom{t+1}{2} + \binom{t}{2}$
$= 4t^2 + 7t + 3 + \frac{t(t-1)}{2} + 3\frac{(t+1)(t+1)-1}{2}$
$= 4t^2 + 7t + 3 + \frac{(t^2-t)}{2} + 3\frac{(t^2+t)}{2}$
$= 6t^2 + 8t + 3$.
and also $e_f(1) = t(t + 1) + t(t + 1) = 2t^2 + 2t$.
We get $e_f(0) - e_f(1) = (6t^2 + 8t + 3) - (2t^2 + 2t) = 4t^2 + 6t + 3 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Type $D : v_f(1) = t, v_f(2) = t + 1, v_f(3) = t + 1, v_f(4) = t + 1$.
Now we find $e_f(0) = t(t + 1) + t(t + 1) + t(t + 1) + (t + 1)^2 + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t+1}{2} + \binom{t}{2}$
$= 3t(t + 1) + (t + 1)^2 + 3\binom{t+1}{2} + \binom{t}{2}$
$= 3t^2 + 3t + t^2 + 2t + 1 + \frac{t(t-1)}{2} + 3\frac{(t+1)(t+1)-1}{2}$
$= 4t^2 + 5t + 1 + \frac{(t^2-t)}{2} + 3\frac{(t^2+t)}{2}$

$= 6t^2 + 6t + 1$.
and also $e_f(1) = (t+1)^2 + (t+1)^2 = 2t^2 + 4t + 2$.
We get $e_f(0) - e_f(1) = (6t^2 + 6t + 1) - (2t^2 + 4t + 2) = 4t^2 + 2t - 1 > 1$ for any positive integer t. Therefore $|e_f(0) - e_f(1)| > 1$. which is a contradiction.

Hence the complete graph K_n is not $4-$remainder cordial for $n > 3$. □

Next is the Path.

Theorem 5. *Any path P_n is $4-$remainder cordial.*

Proof. Let P_n be a path $u_1u_2\dots u_n$. We now divide the proof into the following four cases.
Case(i). $n \equiv 0 \pmod 4$
Assign the labels $1, 2, 3, 4$ respectively to the vertices u_1, u_2, u_3, and u_4. Now we consider the next four vertices u_5, u_6, u_7, and u_8. Assign the labels $1, 2, 3, 4$ to the vertices u_5, u_6, u_7, u_8. The same pattern is continued for the next four vertices. Proceeding like this assign the labels, until we reach the last vertex u_n. Note that in this process the last four vertices namely $u_{n-3}, u_{n-2}, u_{n-1}$, and u_n received the labels $1, 2, 3$, and 4.
Case(ii). $n \equiv 1 \pmod 4$
As in the case(i), assign the labels to the vertices $u_1, u_2, \dots u_{n-1}$. Next assign the label 1 to the vertex u_n.
Case(iii). $n \equiv 2 \pmod 4$
Assign the labels to the vertices $u_i, (1 \le i \le n-1)$,as in the case(ii). Finally assign the label 2 to the vertex u_n.
Case(iv). $n \equiv 3 \pmod 4$
In this case assign the labels to the vertices $u_i, (1 \le i \le n-1)$,as in the case(iii). Finally assign the label 3 to the vertex u_n. The Table 1, establish that this labeling f is a $4-$remainder cordial labeling.

Table 1: Edge condition of 4-remainder cordial labeling of a path

Nature of $n \equiv r \pmod 4$	$e_f(0)$	$e_f(1)$
$n \equiv 0, 2 \pmod 4$	$\frac{n-2}{2}$	$\frac{n}{2}$
$n \equiv 1 \pmod 4$	$\frac{n-1}{2}$	$\frac{n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{n}{2}$	$\frac{n-2}{2}$
$n \equiv 3 \pmod 4$	$\frac{n-1}{2}$	$\frac{n-1}{2}$

□

Next investigation is the cycle graph.

Theorem 6. *All cycles C_n is $4-$remainder cordial.*

Proof. Let $C_n = u_1u_2\dots u_n$ be a cycle.
Case(i). $n \equiv 0 \pmod 4$

Fix the labels $1, 2, 3, 4$ respectively to the four consecutive vertices u_1, u_2, u_3, and u_4. Next assign the labels $4, 3, 2, 1$ respectively to the vertices u_5, u_6, u_7, and u_8. Next assign the labels $4, 3, 2, 1$ to the vertices $u_9, u_{10}, u_{11}, u_{12}$. In this manner assign the labels, until we reach the last vertex u_n. It is easy to verify that the last four vertices $u_{n-3}, u_{n-2}, u_{n-1}$, and u_n received the labels $4, 3, 2, 1$.
Case(ii). $n \equiv 1 \pmod 4$
As in the case(i), assign the labels to the vertices $u_1, u_2, \ldots u_{n-1}$. Next assign the label 4 to the vertex u_n.
Case(iii). $n \equiv 2 \pmod 4$
Assign the labels to the vertices $u_1, u_2, \ldots u_{n-1}$,as in the case(ii). Finally assign the label 3 to the vertex u_n.
Case(iv). $n \equiv 3 \pmod 4$
In this case assign the labels to the vertices $u_1, u_2, \ldots u_{n-1}$,as in the case(iii). Finally assign the label 2 to the vertex u_n. The Table 2, establish that this labeling f is a $4-$ remainder cordial labeling.

Table 2: Edge condition for $4-$ remainder cordial labeling of a cycle

Nature of $n \equiv r \pmod 4$	$e_f(0)$	$e_f(1)$
$n \equiv 0, 2 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}$
$n \equiv 1 \pmod 4$	$\frac{n+1}{2}$	$\frac{n-1}{2}$
$n \equiv 3 \pmod 4$	$\frac{n-1}{2}$	$\frac{n+1}{2}$

□

Next we investigate any comb is $4-$remainder cordial.

Theorem 7. *Any comb $P_n \odot K_1$ is $4-$remainder cordial.*

Proof. Let $P_n = u_1 u_2 \ldots u_n$ be a Path. Let v_i be the pendant vertices attached to $u_i, 1 \leq i \leq n$. Assign the labels to the vertices $u_1, u_2, \ldots u_n$ as in theorem 6.

Case(i). $n \equiv 0 \pmod 4$
We now consider the pendant vertices, fix the labels $4, 3, 2, 1$ respectively to the vertices v_1, v_2, v_3, and v_4. Next assign the labels $1, 2, 3, 4$ to the four vertices v_5, v_6, v_7, and v_8. In similar fashion assign the labels $1, 2, 3, 4$ respectively to the next four consecutive vertices $v_9, v_{10}, v_{11}, v_{12}$. Proceed as above and labels the next four vertices and so on. In this the last four vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n$ received the labels $1, 2, 3, 4$.
Case(ii). $n \equiv 1 \pmod 4$
As in the case(i), assign the labels to the pendant vertices $v_1, v_2, \ldots v_{n-1}$. Next assign the label 1 to the vertex v_n.
Case(iii). $n \equiv 2 \pmod 4$
Assign the labels to the vertices $v_1, v_2, \ldots v_{n-1}$,as in the case(ii). Finally assign the label 2 to the vertex v_n.
Case(iv). $n \equiv 3 \pmod 4$

In this case assign the labels to the vertices $v_1, v_2, \ldots v_{n-1}$,as in the case(iii). Finally assign the label 3 to the vertex v_n. The Table 3, establish that this labeling f is a 4− remainder cordial labeling.

Table 3: Edge condition for 4− remainder cordial labeling of a comb

Nature of $n \equiv r \pmod 4$	$e_f(0)$	$e_f(1)$
$n \equiv 0, 2 \pmod 4$	$\frac{n-2}{2}$	$\frac{n}{2}$
$n \equiv 1, 3 \pmod 4$	$\frac{n+1}{2}$	$\frac{n-3}{2}$

□

4-remainder cordial labeling of $P_5 \odot K_1$ is given in Figure 1.

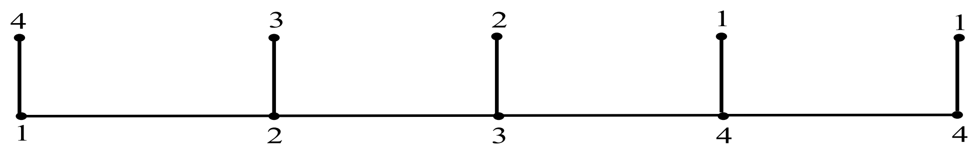

Figure 1:

Next is the Crown $C_n \odot K_1$.

Theorem 8. *All crowns are* $4-$*remainder cordial.*

Proof. The crown $C_n \odot K_1$ is obtained from the comb $P_n \odot K_1$, and by adding the edge $u_n u_1$.

Case(i). $n \equiv 0, 2 \pmod 4$

The vertex labeling as in theorem 7, is also a 4−remainder cordial labeling of crown.

Case(ii). $n \equiv 1, 3 \pmod 4$

Assign the labels 2, 3 to the vertices u_1, u_2 respectively and assign the labels 2, 3 to the next two vertices u_3, u_4 . Continuing in this way until we reach the vertex u_{n-1}. That is assign the labels $2, 3, 2, 3, \ldots 2, 3$ to the vertices $u_1, u_2, \ldots, u_{n-1}$. Now assign the label 2 to the last vertex u_n. Next we consider the pendant vertices, assign the labels to the vertices $v_1, v_2, \ldots v_{n-1}$ in the pattern $1, 4, 1, 4, \ldots 1, 4$. Finally assign the label 4 to the vertex v_n. The following table 4, shows that this labeling f is a 4− remainder cordial labeling.

□

Table 4: Edge condition for 4− remainder cordial labeling of crown

Nature of n	$e_f(0)$	$e_f(1)$
n is even	$\frac{n}{2}$	$\frac{n}{2}$
n is odd	$\frac{n+1}{2}$	$\frac{n-1}{2}$

Theorem 9. *All stars are* $4-$*remainder cordial.*

Proof. Let $K_{1,n}$ be the star with $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. we now give a 4−remainder cordial labeling to the star $K_{1,n}$. Assign the label 3 to the center vertex u.

Case(i). $n \equiv 0 \pmod 4$
let $n = 4t$ Assign the label 1 to the pendant vertices $u_1, u_2, \ldots, u_t$. Next assign the label 2 to the pendant vertices $u_{t+1}, u_{t+2}, \ldots, u_{2t}$. We now assign the label 3 to the next $t-$pendant vertices $u_{2t+1}, u_{2t+2}, \ldots, u_{3t}$. Finally assign the label 4 to the remaining pendant vertices.
Case(ii). $n \equiv 1 \pmod 4$
As in case(i), assign the label to the vertices $u, u_i (1 \leq i \leq n-1)$. Next assign the label 1 to the last vertex u_n.
Case(iii). $n \equiv 2 \pmod 4$
Assign the label to the vertices $u, u_i (1 \leq i \leq n-1)$ as in case(ii). Next assign the label 2 to the vertex u_n.
Case(iv). $n \equiv 3 \pmod 4$
As in the case(iii), assign the label to the vertices $u, u_i (1 \leq i \leq n-1)$. Next assign the label 4 to the vertex u_n. Obviously this vertex labeling f is 4−remainder cordial labeling. □

Theorem 10. *The bistar* $B_{n,n}$ *are* $4-$*remainder cordial for all* n.

Proof. Let $B_{n,n}$ be the bistar with $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \leq i \leq n\}$. Clearly $B_{n,n}$ has $2n + 2$ vertices and $2n + 1$ edges. Assign the label 1, 3 respectively to the central vertices u and v. Consider the pendant vertices u_i.
Case(i). $n \equiv o \pmod 4$
Let $n = 4t$. Assign the label 1 to the pendant vertices $u_1, u_2, \ldots, u_{2t}$ and assign the label 3 to the vertices $u_{2t+1}, u_{2t+2}, \ldots, u_{4t}$. Next we move to the other side pendant vertices v_i. Assign the label 2 to the vertices $v_1, v_2, \ldots, v_{2t}$ and assign the label 4 to the remaining pendant vertices $v_{2t+1}, v_{2t+2}, \ldots, v_{4t}$.
Case(ii). $n \equiv 1 \pmod 4$
Let $n = 4t + 1$. Assign the labels to the vertices $u, v, u_i, v_i (1 \leq i \leq n)$, as in the case(i). Next assign the label 4, 2 respectively to the vertices u_i and v_i.
Case(iii). $n \equiv 2 \pmod 4$
As in the case(ii), assign the label to the vertices $u, v, u_i, v_i (1 \leq i \leq n-1)$. Next assign labels 1, 4 to the vertices u_n and v_n respectively.

Case(iv). $n \equiv 3 \pmod 4$
Assign the labels to the vertices $u, v, u_i, v_i (1 \leq i \leq n-1)$ in case(iii). Finally assign the labels $3, 2$ to the remaining vertices. This vertex labeling is a $4-$remainder cordial labeling follows from table 5.

Table 5: Edge condition of $4-$remainder cordial labeling of bistar

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0, 2 \pmod 4$	$\frac{n}{2}$	$\frac{n-2}{2}$
$n \equiv 1, 3 \pmod 4$	$\frac{n}{2}$	$\frac{n-2}{2}$

For illustration, a $4-$remainder cordial labeling of $B_{5,5}$ is shown in Figure 2.

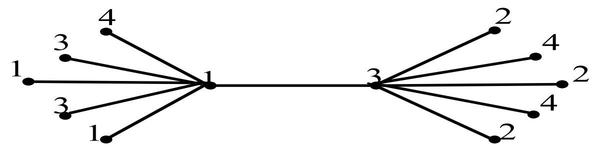

Figure 2:

□

Theorem 11. *The subdivision of the star $S(K_{1,n})$ are $4-$remainder cordial.*

Proof. Let $V(S(K_{1,n})) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{uu_i, u_iv_i : 1 \leq i \leq n\}$. The proof is divided in to four cases given below.
Case(i). $n \equiv 0 \pmod 4$
let $n = 4t$. Assign the label 3 to the vertex u. Next we consider the vertices of degree 2. Assign the label 3 to the vertices $u_1, u_2, \ldots, u_{2t}$ and assign the label 2 to the vertices $u_{2t+1}, u_{2t+2}, \ldots, u_{4t}$. Next we move to the pendant vertices. Assign the label 4 to the vertices $v_1, v_2, \ldots, v_{2t}$ and assign the label 1 to the vertices $v_{2t+1}, v_{2t+2}, \ldots, v_{4t}$.
Case(ii). $n \equiv 1 \pmod 4$
Assign the labels to the vertices $u, u_i, v_i (1 \leq i \leq n-1)$ as in case(i). Next assign the labels $2, 1$ respectively to the vertex u_n and v_n.
Case(iii). $n \equiv 2 \pmod 4$
As in case(ii), assign to labels to the vertices $u, u_i, v_i (1 \leq i \leq n-1)$. Finally assign the labels $4, 3$ to the vertices u_n and v_n respectively.
Case(iv). $n \equiv 3 \pmod 4$
Assign the labels to the vertices $u, u_i, v_i (1 \leq i \leq n-1)$ as in case(iii). Next assign the labels $2, 1$ respectively to the remaining vertices u_n and v_n. The table 6, establish that this vertex labeling f is a $4-$remainder cordial labeling.

□

Table 6: Edge condition of 4−remainder cordial labeling of subdivision of star

Nature of n	$e_f(0)$	$e_f(1)$
$n \equiv 0, 2 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}$
$n \equiv 1, 3 \pmod 4$	$\frac{n-1}{2}$	$\frac{n-1}{2}$

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