



Tenacity and some related results

D. Moazzami*¹

¹University of Tehran, College of Engineering, Department of Engineering Science.

ABSTRACT

Conceptually graph vulnerability relates to the study of graph intactness when some of its elements are removed. The motivation for studying vulnerability measures is derived from design and analysis of networks under hostile environment. Graph tenacity has been an active area of research since the the concept was introduced in 1992.

The tenacity $T(G)$ of a graph G is defined as

$$T(G) = \min_{ACV(G)} \left\{ \frac{|A| + \tau(G - A)}{\omega(G - A)} \right\}$$

where $\tau(G - A)$ denotes the order (the number of vertices) of a largest component of $G - A$ and $\omega(G - A)$ is the number of components of $G - A$.

In this paper we discuss tenacity and its properties in vulnerability calculation.

Keyword: vertex connectivity, toughness, binding number, independence number, edge-connectivity.

AMS subject Classification: 05C78.

1 Introduction

We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. We denote by $V(G)$, $E(G)$ and $|V(G)|$ the set of vertices, the set of edges and the

*Corresponding author: D. Moazzami Email: dmoazzami@ut.ac.ir

ARTICLE INFO

Article history:

Received 30, June 2016

Received in revised form 18, March 2017

Accepted 3 April 2017

Available online 01, June 2017

order of a graph G , respectively. A set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in any such set is called the vertex independence number of G and is denoted by $\alpha(G)$ or α . The vertex connectivity, $\kappa = \kappa(G)$, of a finite, undirected, simple graph G (without loops or multiple edges) is the minimum number of vertices whose removal results in a disconnected graph or results in the trivial graph K_1 . Graph G is called n -connected if $\kappa \geq n$. Analogously, the edge-connectivity, $\lambda = \lambda(G)$, of a finite, undirected, connected simple graph G is the minimum number of edges whose removal results in a disconnected or trivial graph K_1 . A graph G is called n -edge-connected if $\lambda(G) \geq n$. Given a graph G , the graph G^2 has $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $uv \in E(G)$ or the distance from u to v is 2. The complement \bar{G} of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in \bar{G} if and only if they are not adjacent in G . The line graph of a graph G , denoted $L(G)$, is the graph whose vertices are the edges of G , and two vertices are adjacent whenever the corresponding edges of G are adjacent.

The concept of tenacity of a graph G was introduced as a useful measure of the "vulnerability" of G . We compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [3-25], they studied more about this new invariant. The tenacity of a graph G , $T(G)$, is defined by $T(G) = \min\{\frac{|S| + \tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G . We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph $G-S$, and $\omega(G-S)$ be the number of components of $G-S$. A connected graph G is called T -tenacious if $|S| + \tau(G-S) \geq T\omega(G-S)$ holds for any subset S of vertices of G with $\omega(G-S) > 1$. If G is not complete, then there is a largest T such that G is T -tenacious; this T is the tenacity of G . On the other hand, a complete graph contains no vertex cutset and so it is T -tenacious for every T . Accordingly, we define $T(K_p) = \infty$ for every p ($p \geq 1$). A set $S \subseteq V(G)$ is said to be a T -set of G if $T(G) = \frac{|S| + \tau(G-S)}{\omega(G-S)}$. Any undefined terms can be found in the standard references on graph theory, including Bondy and Murty [1].

Vulnerability Calculation

Let $C_n = (v_1v_2 \cdots v_nv_1)$ be the n -cycle and define the k -th power of the n -cycle, C_n^k by

$$C_n^k = C_n + \{v_iv_j \mid |i-j| \leq k\}.$$

We have the following four propositions.

Proposition 1 : If G is a spanning subgraph of H , then $T(G) \leq T(H)$.

Proposition 2 : For any graph G , $T(G) \geq \frac{\kappa(G)+1}{\alpha(G)}$.

Proposition 3 : If G is not complete, then $T(G) \leq \frac{n-\alpha(G)+1}{\alpha(G)}$.

Proposition 4 : If $k \leq n-k$, then $T(K_{k,n-k}) = \frac{k+1}{n-k}$.

We can prove the following two lemmas:

Lemma 1 : If A is a minimal T -set for C_n^k , then A consists of the union of sets of k

consecutive vertices such that there exists at least one vertex not in A between any two sets of consecutive vertices in A.

Lemma 1 gives us an indication of the size of the cut-set for the tenacity of C_n^k ; the next lemma gives us the size of the largest component.

Lemma 2 : There is a T-set, A, for C_n^k such that all components of C_n^k have order $\tau(C_n^k - A)$ or $\tau(C_n^k - A) - 1$.

These two lemmas allow us to determine precisely the tenacity of the power of cycles.

Theorem 1 : Let C_n^k be a power of cycles and $n = r(k + 1) + s$, for $0 \leq s < k + 1$. Then $T(C_n^k) = k + \frac{1 + \lceil \frac{s}{r} \rceil}{r}$.

Proof : Let A be a minimal T-set of C_n^k . By lemma 1 and lemma 2, $|A| = k\omega$, and $\tau(C_n^k - A) = \lceil \frac{n - k\omega}{\omega} \rceil$. Thus, from the definition of tenacity we have

$$T = \min\left\{\frac{k\omega + \lceil \frac{n - k\omega}{\omega} \rceil}{\omega} \mid 2 \leq \omega \leq r\right\}.$$

Now consider the function $f(\omega) = \frac{k\omega + \lceil \frac{n - k\omega}{\omega} \rceil}{\omega} = k + \frac{\lceil \frac{n}{\omega} - k \rceil}{\omega}$. Let ω_1 and ω_2 be any two integers in $[2, r]$ with $\omega_1 \leq \omega_2$, then $\lceil \frac{n}{\omega_2} \rceil \leq \lceil \frac{n}{\omega_1} \rceil$. Thus $f(\omega_2) = k + \frac{\lceil \frac{n}{\omega_2} - k \rceil}{\omega_2} \leq k + \frac{\lceil \frac{n}{\omega_1} - k \rceil}{\omega_1} = f(\omega_1)$. Hence the function $f(\omega)$ is a non increasing function and the minimum value occurs at the boundary. Thus $\omega = r$ and $\lceil \frac{n - k\omega}{\omega} \rceil = \lceil \frac{r(k+1) + s - kr}{r} \rceil = 1 + \lceil \frac{s}{r} \rceil$. Therefore, $T(C_n^k) = k + \frac{1 + \lceil \frac{s}{r} \rceil}{r}$. □

Now we can discuss about tenacity and its operation on graphs. If the removal of a vertex from a graph results in a complete graph, the tenacity becomes infinite. On the other hand, the removal of a vertex cannot lower by too much. We can easily prove the following two theorems and corollaries:

Theorem 2: For any nontrivial, incomplete graph G with n vertices and any vertex v, $T(G - v) \geq T(G) - \frac{1}{2}$.

The following theorem allow us to find the tenacity of several important classes of graphs.

Theorem 3: If G is a bipartite, r-regular, r-connected graph on n vertices, then $T(G) = \frac{n+2}{n}$.

This result gives several interesting corollaries.

Corollary 1: If G_1 is a bipartite, d-regular, d-connected graph with n_1 vertices and G_2 is a bipartite, q-regular, q-connected graph with n_2 vertices, then $T(G_1 \times G_2) = \frac{n_1 n_2 + 2}{n_1 n_2}$.

Corollary 2: For any integer n, $T(Q_n) = \frac{2^n + 2}{2^n}$.

Corollary 3: For any integers n and m, $T(C_n \times C_m) = \frac{nm + 2}{nm}$.

Corollary 4: For any even integer n, $T(C_n \times K_2) = \frac{n+1}{n}$.

In 1972, Chvátal [2] introduced the concept of the toughness of a graph. It measures in a simple way how tightly various pieces of a graph hold together; therefore he called it toughness. Let G be a graph and t a real number such that the implication $\omega(G - A) > 1 \Rightarrow |A| \geq t \cdot \omega(G - A)$ holds for each set A of vertices of G . Then G will be said to be t -tough.

Proposition 5: $G \subset H \Rightarrow t(G) \leq t(H)$.

Thus toughness is a nondecreasing invariant whose values range from zero to infinity. A graph G is disconnected if and only if $t(G) = 0$; G is complete if and only if $t(G) = +\infty$.

Theorem 4: $t(K_m \times K_n) = \frac{1}{2}(m + n) - 1, (m, n \geq 2)$.

Without attempting to obtain the best possible result, we can prove quite easily the following relation between $T(G)$ and $t(G)$. This result gives us a number of corollaries.

Theorem 5: For any graph G , $T(G) \geq t(G) + \frac{1}{\alpha(G)}$.

Proof: Let $A \subseteq V(G)$ be a t -set and $B \subseteq G$ be a T -set. Then $\frac{|B| + \tau(G-B)}{\omega(G-B)} \geq \frac{|B|}{\omega(G-B)} + \frac{1}{\omega(G-B)} \geq \frac{|A|}{\omega(G-A)} + \frac{1}{\alpha(G)}$. \square

We next obtain some bounds on the tenacity of products of graphs.

Theorem 6: If $n \geq m$, then $\frac{m^2 + mn - 2m + 2}{2m} \leq T(K_m \times K_n) \leq \frac{mn - n + \lceil \frac{n}{m} \rceil}{m}$.

Proof: By Theorem 4, $t(K_m \times K_n) = \frac{m+n-2}{2}$. It is easy to see that $\alpha(K_m \times K_n) = m$. Let $V(K_n) = \{1, 2, 3, \dots, n\}$ and $V(K_m) = \{1, 2, 3, \dots, m\}$. Then $V(K_m \times K_n) = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Also let $n = am + b$, for $0 \leq b < m$, so if $b = 0$ then $a = \lceil \frac{n}{m} \rceil = \frac{n}{m}$ and otherwise $a + 1 = \lceil \frac{n}{m} \rceil$. Now, if $b = 0$, then define the sets W_i as $W_i = \{(i, ia - a + 1), \dots, (i, ia)\}$ for $1 \leq i \leq m$, otherwise define the sets W_i as follows:

$$W_i = \begin{cases} \{(i, ia + i - a), \dots, (i, ia + i)\} & 1 \leq i \leq b \\ \{(i, ia + b - a + 1), \dots, (i, ia + b)\}, & b + 1 \leq i \leq m, \end{cases}$$

and let $W = \bigcup_{i=1}^m W_i$. Define $A = V(G) - W$ and so $|A| = mn - n$. It is easy to see that the

$W_i, 1 \leq i \leq m$, are the components of $G-A$ and so $\tau(G - A) = \lceil \frac{n}{m} \rceil$ and $\omega(G - A) = m$. The result follows. \square

The following our conjecture proved recently:

Conjecture : If $n \geq m \geq 2$ then $T(K_m \times K_n) = \frac{mn - n + \lceil \frac{n}{m} \rceil}{m}$.

Corollary 5: For any integer n , $T(K_n \times K_n) = n - 1 + \frac{1}{n}$.

Corollary 6: For any graph G , $T(G^2) > \kappa(G)$.

Corollary 7: Let G be a non-empty graph and let m be the largest integer such that $K_{1,m}$ is an induced subgraph of G . Then $T(G) \geq \frac{\kappa(G)}{m} + \frac{1}{\alpha(G)}$.

Theorem 7: If G is connected and a non-complete $K_{1,3}$ -free graph then $T(G) > \frac{\kappa(G)}{2}$.

Proof: Suppose G is a non-complete, $K_{1,3}$ -free graph with connectivity $\kappa(G)$. Let A be a T -set, and let W_1, W_2, \dots, W_m be the components of $G-A$.

Since G has a connectivity $\kappa(G)$, it is $\kappa(G)$ -connected and so there exist $\kappa(G)$ internally disjoint paths from $u_i \in W_i$ to $u_j \in W_j$ for all $1 \leq i, j \leq m$ with $i \neq j$. Each of these

paths must contain a vertex of A. Then for each i there are at least $\kappa(G)$ edges coming from W_i to distinct vertices of A. Thus in all there are at least $m\kappa(G)$ edges from G-A to A counting at most one from any component W_i to a particular vertex of A.

Suppose $v \in A$ is adjacent to vertices u_1, u_2 , and u_3 in distinct components of G-A. Then, since $\{u_1, u_2, u_3\}$ is an independent set the graph induced by $\{v, u_1, u_2, u_3\}$ is a $K_{1,3}$, a contradiction. Hence we can conclude any vertex of A is adjacent to at most two components of G-A. Thus, there are at most $2 | A |$ edges coming from G-A to vertices of A, counting at most one edge from any component to a particular vertex of A. Hence $m\kappa(G) \leq 2 | A |$, or $\frac{m\kappa(G)}{2} \leq | A |$. Therefore $\frac{m\kappa(G)}{2} < | A | + 1$, or $\frac{|A|+1}{m} > \frac{\kappa(G)}{2}$. Thus $T(G) = \frac{|A|+\tau(G-A)}{\omega(G-A)} \geq \frac{|A|+1}{m} > \frac{\kappa(G)}{2}$. \square

The effect of removing a vertex is considered first. If the removal of a vertex from a graph results in a complete graph, the tenacity becomes infinite. On the other hand, the removal of a vertex cannot lower T by too much.

Theorem 8: For any nontrivial non-complete graph G on n vertices and any vertex v, $T(G - v) \geq T(G) - \frac{1}{2}$.

Proof: Let $G' = G - v$. If $G' = K_{n-1}$, then $T(G') = \infty$, and the theorem holds. Hence, assume $G' \neq K_{n-1}$. Let A' be a T-set for G', and let $| A' | = m$, then $T(G') = \frac{m+\tau(G'-A')}{\omega(G'-A')}$. Now define $A = A' \cup \{v\}$. Clearly A is a disconnecting set for G and so $T(G) \leq \frac{|A|+\tau(G-A)}{\omega(G-A)}$. But $| A | = m + 1$ and $G - A = G' - A'$, so $T(G) \leq \frac{m+1+\tau(G'-A')}{\omega(G'-A')} = \frac{m+\tau(G'-A')}{\omega(G'-A')} + \frac{1}{\omega(G'-A')} = T(G') + \frac{1}{\omega(G'-A')} \leq T(G') + \frac{1}{2}$, since $\omega(G' - A') \geq 2$. Hence $T(G) \leq T(G') + \frac{1}{2}$. \square

We next obtain some bounds on the tenacity of a graph.

Proposition 6: If G is connected, then $T(G) \geq \frac{1}{\Delta(G)}$.

Proof: K_n is a special case, otherwise the removal of any vertex of a connected graph G yields at most $\Delta(G)$ components. Similarly, the removal of any n vertices yields at most $n\Delta(G)$ components. Then, from the definition we have $T(G) \geq \frac{n+1}{n\Delta(G)} \geq \frac{1}{\Delta(G)}$. \square

Lemma 3: If A is a minimal T-set for the graph G then, for each vertex v of A, the induced subgraph $\langle V(G) - A + v \rangle$ has fewer components than does G-A.

Proof: Let $A' = A - v$. If G-A' has at least as many components as G-A, then $| A' | = | A | - 1$ and $\tau(G - A') \leq \tau(G - A) + 1$. Therefore $\frac{|A'|+\tau(G-A')}{\omega(G-A')} = \frac{|A|-1+\tau(G-A')}{\omega(G-A')} \leq \frac{|A|-1+\tau(G-A)+1}{\omega(G-A)} = T(G)$, contrary to our choice of A. \square

Theorem 9: Let $G = G_1 + G_2$, where $| V(G) | = n$, $| V(G_i) | = p_i$, $T(G) = T$ and $T(G_i) = T_i$ for $i = 1, 2$. Then if $G \neq K_n$ we have

$$\min\left\{\frac{[n + \tau(G_1 - A_1)]T_1}{p_1 + \tau(G_1 - A_1)}, \frac{[n + \tau(G_2 - A_2)]T_2}{p_2 + \tau(G_2 - A_2)}\right\} < T \leq \min\left\{\frac{n - \alpha_1 + 1}{\alpha_1}, \frac{n - \alpha_2 + 1}{\alpha_2}\right\},$$

where α_i is the independence number of G_i , and A_i is a disconnecting set of G_i for $i = 1, 2$.

Proof: Because of the structure of G, the graph cannot be disconnected without removing one of $V(G_1)$ or $V(G_2)$. Having removed the appropriate set, we can then disconnect the graph by disconnecting the remaining graph, either G_1 or G_2 . Candidates for T

are of the form $\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)}$ or $\frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)}$ where $n_i = |A_i|$ for $i = 1, 2$. Then $T = \min\{\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)}, \frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)}\}$, where the minimum is taken over all A_1 and A_2 as defined. Also $T_1 \leq \frac{n_1+\tau(G_1-A_1)}{\omega(G_1-A_1)}$ which implies $\omega(G_1 - A_1) \leq \frac{n_1+\tau(G_1-A_1)}{T_1}$. Thus $\frac{n_1+p_2+\tau(G_1-A_1)}{\omega(G_1-A_1)} \geq \frac{[n_1+p_2+\tau(G_1-A_1)]T_1}{n_1+\tau(G_1-A_1)}$. Similarly, $\frac{n_2+p_1+\tau(G_2-A_2)}{\omega(G_2-A_2)} \geq \frac{[n_2+p_1+\tau(G_2-A_2)]T_2}{n_2+\tau(G_2-A_2)}$. Thus $T \geq \min\{[1 + \frac{p_2}{n_1+\tau(G_1-A_1)}]T_1, [1 + \frac{p_1}{n_2+\tau(G_2-A_2)}]T_2\}$. Also we know that $n_1 < p_1$ and $n_2 < p_2$, therefore $T > \min\{\frac{[n_1+\tau(G_1-A_1)]T_1}{p_1+\tau(G_1-A_1)}, \frac{[n_2+\tau(G_2-A_2)]T_2}{p_2+\tau(G_2-A_2)}\}$. From Proposition 3, we observe that two candidates for T are $\frac{(p_1-\alpha_1)+1+p_2}{\alpha_1}$ and $\frac{(p_2-\alpha_2)+p_1}{\alpha_2}$, which yield $T \leq \min\{\frac{n-\alpha_1+1}{\alpha_1}, \frac{n-\alpha_2+1}{\alpha_2}\}$. \square

Theorem 10: Let G be a graph with n vertices and $G \neq K_n$, then $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$.

Proof: We observe that at least one of G or \overline{G} is connected. Suppose \overline{G} is not connected. We proved (Proposition 6) that $T(G) \geq \frac{1}{\Delta(G)} \geq \frac{1}{n-1}$ for any graph G. Thus, $T(G)+T(\overline{G}) \geq \frac{1}{n-1}$. Now suppose G is not connected but \overline{G} is connected. Again by Proposition 6, we have $T(\overline{G}) \geq \frac{1}{n-1}$. Therefore $T(G) + T(\overline{G}) \geq \frac{1}{n-1}$. \square

Theorem 11: Let G be a graph with $0 < T(G) < \infty$, and let $\lambda(G) = \lambda$, then $T(L(G)) > \frac{\lambda}{2}$.

Proof: Assume there exist vertex cutsets A for L(G) such that A is a t-set. By Theorem 5, $T(L(G)) > t(L(G))$. Let E be those edges of G which are incident to vertices of A. Then E is an edge-cutset of G. Thus we have $t(L(G)) = \min\{\frac{|A|}{\omega(L(G)-A)}\} \geq \min\{\frac{|E|}{\omega(G-E)}\} = t'(G)$, where A is a cutset and E is an edge cutset of G.

Using the result of Chvátal [2] we have $t'(G) = \min\{\frac{|E|}{\omega(G-E)}\} = \frac{\lambda}{2}$. Therefore $T(L(G)) > \frac{\lambda}{2}$. \square

The binding number of a graph G was defined by Woodall in [26] as

$$bind(G) = \min_A \left\{ \frac{|N(A)|}{|A|} \right\}$$

where $\phi \neq A \subseteq V(G)$ and $N(A) \neq V(G)$. The binding number was called the melting-point of the graph. the reason for the name "binding number" is that, roughly speaking, if $bind(G)$ is large, then the vertices of G are bound tightly together, in the sense that G has many edges fairly well distributed.

Theorem 12: For any graph G, $T(G) \geq bind(G) - 1$.

Proof: Let $bind(G) = c$. If $c < 1$, then $c - 1 < 0$ and the result follows since T(G) is nonnegative. Consider $c \geq 1$. Suppose that A is a subset of V(G) such that $\omega = \omega(G - A) \geq 2$. We want to prove that $\frac{|A|+1}{\omega} > (c - 1)$. If each of the ω components of G-A has at least two vertices, let S consist of the vertices in all the components except the smallest, so that

$$|S| \geq \frac{|V(G) - A| (\omega - 1)}{\omega} \geq \frac{2\omega(\omega - 1)}{\omega} = 2(\omega - 1) \geq \omega.$$

If, on the other hand, V(G)-A contains an isolated vertex, let $S = V(G) - A$. So that

$|S| = |V(G) - A| \geq \omega$. In either case $N(S) \neq V(G)$, and since $bind(G) = c \geq 1$,

$$|S| + |A| + 1 > |S| + |A| \geq |N(S)| \geq c |S|.$$

It follows that $|A| + 1 > (c - 1) |S| \geq (c - 1)\omega$. Therefore $\frac{|A|+1}{\omega} > c - 1$, so $T > c - 1$. \square

Acknowledgement

This work was supported by Tehran University. My special thanks go to the University of Tehran, College of Engineering and Department of Engineering Science for providing all the necessary facilities available to me for successfully conducting this research.

References

- [1] Bundy, J.A. and U.S.R. Murty, Graph Theory with Applications (The Macmillan Press Ltd, 1976).
- [2] Chvátal, V., Tough graphs and Hamiltonian circuits, **Discrete Math.**5 (1973), 215-228.
- [3] Aytá, A., On the edge-tenacity of the middle graph of a graph. *Int. J. Comput. Math.* 82 (2005), no. 5, 551-558.
- [4] Cozzens, M.B., D. Moazzami, S. Stueckle, The tenacity of a graph, *Graph Theory, Combinatorics, and Algorithms* (Yousef Alavi and Allen Schwenk eds.) Wiley, New York, (1995), 1111-1112.
- [5] Cozzens, M.B., D. Moazzami, S. Stueckle, The tenacity of the Harary Graphs, *J. Combin. Math. Combin. Comput.* 16 (1994), 33-56.
- [6] Choudum S. A., and N. Priya, Tenacity of complete graph products and grids, *Networks* 34 (1999), no. 3, 192-196.
- [7] Choudum S. A., and N. Priya, Tenacity-maximum graphs, *J. Combin. Math. Combin. Comput.* 37 (2001), 101-114.
- [8] Harary, F., The maximum connectivity of a graph, *Proc. Nat. Acad. Sci, USA*, Vol. 48, (1962), 1142-1146.
- [9] Li, Y.K., Q.N. Wang, Tenacity and the maximum network. *Gongcheng Shuxue Xuebao* 25 (2008), no. 1, 138-142.
- [10] Li, Y.K. , S.G. Zhang, X.L. Li, Y. Wu, Relationships between tenacity and some other vulnerability parameters. *Basic Sci. J. Text. Univ.* 17 (2004), no. 1, 1-4. Li, Yin-kui; Zhang, Sheng-gui; Li, Xue-liang; Wu, Yun

- [11] Ma, J.L., Y.J. Wang, X.L. Li, Tenacity of the torus $P_n \times C_m$. (Chinese) Xibei Shifan Daxue Xuebao Ziran Kexue Ban 43 (2007), no. 3, 15-18.
- [12] Moazzami, D., Vulnerability in Graphs - a Comparative Survey, J. Combin. Math. Combin. Comput. 30 (1999), 23-31.
- [13] Moazzami, D., Stability Measure of a Graph - a Survey, Utilitas Mathematica, 57 (2000), 171-191.
- [14] Moazzami, D., On Networks with Maximum Graphical Structure, Tenacity T and number of vertices p, J. Combin.Math. Combin. Comput. 39 (2001).
- [15] Moazzami, D., A note on Hamiltonian properties of tenacity, Poceedings of the International conference," Paul Erdős and his Mathematics" Budapest, July 4 - 11, (1999), 174-178.
- [16] Moazzami, D., S. Salehian, On the edge-tenacity of graphs. Int. Math. Forum 3 (2008), no. 17-20, 929-936.
- [17] Moazzami, D., S. Salehian, Some results related to Tenacity and Existence of k-tree, Discrete Applied Mathematics, 157 (2009), no. 8, PP.1794-1798.
- [18] Moazzami, D., Tenacity of a Graph with Maximum Connectivity, Discrete Applied Mathematics, 159 (2011) PP. 367-380.
- [19] Moazzami, D., M. Dadvand, A. Moeini, "Recognizing Tenacious Graphs is NP-hard, ARS Combinatoria 115, (2014) PP. 163-174.
- [20] Moazzami, D. B. Bafandeh, On the higher-order edge-tenacity of a graph, accepted in ARS Combinatoria.
- [21] Moazzami, D., Jelodar, D., Tenacity of Cycle Permutation Graph, Journal of Algorithms and Computation, 48 (2016) PP. 37 - 44.
- [22] Moazzami, D., Bafandeh, B., On the first-order edge tenacity of a graph, Discrete Applied Mathematics, 205 (2016), 8-15.
- [23] Moazzami, D., Toughness of the Networks with Maximum Connectivity, Journal of Algorithms and Computation, Vol 46 (2015), PP. 51-71.
- [24] Moazzami, D., Golshani, A., S. Akhondian, On the complexity of recognizing tenacious graphs, ARS Combinatoria, Vol CXXXI (January 2017), PP. 11-21.
- [25] Moazzami, D., Towards a measure of vulnerability, tenacity of a Graph, Journal of Algorithms and Computation, 48 (2016) PP. 149 - 154.
- [26] Piazza, B., F. Roberts, S. Stueckle, Edge-tenacious networks, Networks 25 (1995), no. 1, 7-17.

- [27] Piazza, B., S. Stueckle, A lower bound for edge-tenacity, Proceedings of the thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999) Congr. Numer. 137 (1999), 193-196.
- [28] Wang, Z.P., G. Ren, L.C. Zhao, Edge-tenacity in graphs. J. Math. Res. Exposition 24 (2004), no. 3, 405-410.
- [29] Wang, Z.P., G. Ren, A new parameter of studying the fault tolerance measure of communication networks—a survey of vertex tenacity theory. (Chinese) Adv. Math. (China) 32 (2003), no. 6, 641-652.
- [30] Wang, Z.P., G. Ren, C.R. Li, The tenacity of network graphs—optimization design. I. (Chinese) J. Liaoning Univ. Nat. Sci. 30 (2003), no. 4, 315-316.
- [31] Wang, Z.P., C.R. Li, G. Ren, L.C. Zhao, Connectivity in graphs—a comparative survey of tenacity and other parameters. (Chinese) J. Liaoning Univ. Nat. Sci. 29 (2002), no. 3, 237-240.
- [32] Wang, Z.P., C.R. Li, G. Ren, L.C. Zhao, The tenacity and the structure of networks. (Chinese) J. Liaoning Univ. Nat. Sci. 28 (2001), no. 3, 206–210.
- [33] Wu, Y., X.S. Wei, Edge-tenacity of graphs. (Chinese) Gongcheng Shuxue Xuebao 21 (2004), no. 5, 704-708.
- [34] Woodall, D. R., The binding number of a graph and its Anderson number, J. Combin. Theory B 15 (1973), 225-255.