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Sharp Upper bounds for Multiplicative Version of Degree Distance and Multiplicative Version of Gutman Index of Some Products of Graphs

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ABSTRACT

In 1994, degree distance of a graph was introduced by Dobrynin , Kochetova and Gutman. And Gutman proposed the Gutman index of a graph in 1994. In this paper, we introduce the concepts of multiplicative version of degree distance and the multiplicative version of Gutman index of a graph. We find the sharp upper bound for the multiplicative version of degree distance and multiplicative version of Gutman index of cartesian product of two connected graphs. And we compute the exact value for the cartesian product of two complete graphs. Using this result, we prove that our bound is tight. Also, we obtain the sharp upper bound for the multiplicative version of degree distance and the

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Abstract continued
multiplicative version of Gutman index of strong product of connected and complete graphs. And we observe the exact value for the strong product of two complete graphs. From this, we prove that our bound is tight.

1 Introduction

In this paper, all graphs considered are simple and connected graphs. We denote the vertex and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. $d_G(v)$ denotes the degree of a vertex v in G. The number of elements in the vertex set of a graph G is called the order of G and is denoted by $v(G)$. The number of elements in the edge set of a graph G is called the size of G and is denoted by $e(G)$. A graph with order n and size m is called a (n, m)-graph. For any $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. A graph G is complete, if every pair of its vertices are adjcent. A complete graph on n vertices is denoted by K_n.
The Cartesian product $[1]$of the graphs G_1 and G_2, denoted by $G_1 \Box G_2$ has the vertex set $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and $(u, x)(v, y)$ is an edge of $G_1 \Box G_2$ is $u = v$ and $xy \in E(G_2)$ or $uv \in E(G_1)$ and $x = y$. For two simple graphs G_1 and G_2, their strong product, denoted by $G_1 \boxtimes G_2$, has vertex set $V(G_1) \times V(G_2) = \{(u, v) : u \in V(G_1), v \in V(G_2)\}$ and $(u, x)(v, y)$ is an edge whenever $(i) u = v$ and $xy \in E(G_2)$, or $(ii) uv \in E(G_1)$ and $x = y$, or $(iii) uv \in E(G_1)$ and $xy \in E(G_2)$.
A topological index is a real number related to a structural graph of a molecule, which is invariant under graph isomorphism, that is, it does not depend on the labeling or pictorial representation of a graph. A topological index related to distance is called a "distance-based topological index". In 1947, H.Wiener $[9]$ introduced the first distance-based topological index which is named as Wiener index and it is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u, v).$$

The topological indices based on distances between vertices of a graph are widely used for characterizing molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds see $[6, 8]$.
There are some topological indices based on degrees known as the first and second Zagreb indices of molecular graphs. The first and second kinds of Zagreb indices are introduced by Gutman et al. in $[5]$. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of a graph G are defined as

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] = \sum_{v \in V(G)} d_G^2(v).$$

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

The degree distance was proposed by Dobrynin and Kochetova [3] and Gutman [4] as a weighted version of the Wiener index. The degree distance of G, denoted by $DD(G)$, is defined as

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v)] = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)[d_G(u) + d_G(v)].$$

Also in [4], Gutman defined the Schultz index of the second kind, which is now known as the Gutman index. The Gutman index of G, denoted by $Gut(G)$, is defined as

$$Gut(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) d_G(u) d_G(v) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v) d_G(u) d_G(v).$$

We introduce the multiplicative version of degree distance and the multiplicative version of Gutman index of graphs which are defined as

$$\begin{aligned} DD^*(G) &= \Big(\prod_{u,v \in V(G), u \neq v} d_G(u,v)[d_G(u) + d_G(v)] \Big)^{\frac{1}{2}} \\ Gut^*(G) &= \Big(\prod_{u,v \in V(G), u \neq v} d_G(u,v) d_G(u) d_G(v) \Big)^{\frac{1}{2}}. \end{aligned}$$

2 Basic Lemmas

Lemma 1. ***(Arithmetic Geometric inequality)***[2]
Let $x_1, x_2, ..., x_n$ be non-negative numbers. Then $\frac{x_1+x_2+...+x_n}{n} \geq \sqrt[n]{x_1 x_2 ... x_n}$

Lemma 2. [7] *(a) Let $w_{ij} = (u_i, v_j)$ and $w_{pq} = (u_p, v_q)$ be in $V(G_1 \Box G_2)$. Then $d_{G_1 \Box G_2}(w_{ij}, w_{pq}) = d_{G_1}(u_i, u_p) + d_{G_2}(v_j, v_q)$ and $d_{G_1 \Box G_2}(w_{ij}) = d_{G_1}(u_i) + d_{G_2}(v_j)$*
(c) Let x_{ij} denote the vertex (u_i, u_j) of $G \boxtimes K_r$. Now $d_{G \boxtimes K_r}(x_{ij}) = r d_G(u_i) + (r-1)$ and

$$d_{G \boxtimes K_r}(x_{ij}, x_{kp}) = \begin{cases} 1, & i = k, j \neq p \\ d_G(u_i, u_k), & i \neq k, j = p \\ d_G(u_i, u_k), & i \neq k, j \neq p \end{cases}$$

The degree of the vertex (u_i, v_j) of $V(G_1 \boxtimes G_2)$ is $d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i) d_{G_2}(v_j)$, That is $d_{G_1 \boxtimes G_2}(u_i, v_j) = d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i) d_{G_2}(v_j)$.

3 The Multiplicative Version of Degree Distance of Strong Product of Graphs.

In this section, we obtain the sharp upper bound of the multiplicative version of degree distance of $G \boxtimes K_r$.

Theorem 3. *Let G be a (n, m)graph. Then*

$$DD^*(G \boxtimes K_r) \quad \leq \quad \Big[\frac{4rm + 2n(r-1)}{n}\Big]^{\frac{nr(r-1)}{2}} \times \Big[\frac{2rDD(G) + 4(r-1)W(G)}{n(n-1)}\Big]^{\frac{nr(nr-r)}{2}}$$

Proof. Let $V(G) = \{u_0, u_1, u_2, ..., u_{n-1}\}$ and $V(K_r) = \{v_0, v_1, v_2, ..., v_{r-1}\}$.
Let $w_{ij}, w_{pq} \in V(G \boxtimes K_r)$, where $w_{ij} = (u_i, v_j)$ and $w_{pq} = (u_p, v_q)$.

$$\begin{aligned}
[DD^*(G \boxtimes K_r)]^2 &= \prod_{w_{ij}, w_{pq} \in V(G\boxtimes K_r), w_{ij} \neq w_{pq}} d_{G\boxtimes K_r}(w_{ij}, w_{pq})\Big[d_{G\boxtimes K_r}(w_{ij}) + d_{G\boxtimes K_r}(w_{pq})\Big] \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} d_{G\boxtimes K_r}(w_{ij}, w_{iq})\Big[d_{G\boxtimes K_r}(w_{ij}) + d_{G\boxtimes K_r}(w_{iq})\Big] \\
&\times \prod_{j=0}^{r-1} \prod_{i,p=0, i\neq p}^{n-1} d_{G\boxtimes K_r}(w_{ij}, w_{pj})\Big[d_{G\boxtimes K_r}(w_{ij}) + d_{G\boxtimes K_r}(w_{pj})\Big]
\end{aligned}$$

$$\times \prod_{i,p=0,i\neq p}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} d_{G\boxtimes K_r}(w_{ij},w_{pq})\Big[d_{G\boxtimes K_r}(w_{ij})+d_{G\boxtimes K_r}(w_{pq})\Big]$$

$= A\times B\times C,$ where A , B , C are terms of the above products taken in order.

$$A = \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} d_{G\boxtimes K_r}(w_{ij},w_{iq})\Big[d_{G\boxtimes K_r}(w_{ij})+d_{G\boxtimes K_r}(w_{iq})\Big]$$

$$= \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} 1\Big[rd_G(u_i)+(r-1)+rd_G(u_i)+(r-1)\Big] \text{ Lemma 2}$$

$$= \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} \Big[2rd_G(u_i)+2(r-1)\Big]$$

$$\leq \Big[\frac{1}{nr(r-1)} \sum_{i=0}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} \Big\{2rd_G(u_i)+2(r-1)\Big\}\Big]^{nr(r-1)}$$

$$= \Big[\frac{1}{nr(r-1)} \sum_{j,q=0,j\neq q}^{r-1} \Big\{2r\sum_{i=0}^{n-1} d_G(u_i)+2(r-1)\sum_{i=0}^{n-1} 1\Big\}\Big]^{nr(r-1)}$$

$$= \Big[\frac{1}{nr(r-1)} \sum_{j,q=0,j\neq q}^{r-1} \Big\{2\times 2rm+2(r-1)n\Big\}\Big]^{nr(r-1)}$$

$$= \Big[\frac{1}{nr(r-1)} \Big\{4rm+2(r-1)n\Big\} \sum_{j,q=0,j\neq q}^{r-1} 1\Big]^{nr(r-1)}$$

$$= \Big[\frac{1}{nr(r-1)} \Big\{4rm+2(r-1)n\Big\} r(r-1)\Big]^{nr(r-1)}$$

$$= \Big[\frac{4rm+2(r-1)n}{n}\Big]^{nr(r-1)}$$

$$B = \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_{G\boxtimes K_r}(w_{ij},w_{pj})\Big[d_{G\boxtimes K_r}(w_{ij})+d_{G\boxtimes K_r}(w_{pj})\Big]$$

$$= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big[rd_G(u_i)+(r-1)+rd_G(u_p)+(r-1)\Big] \text{ Lemma 2}$$

$$= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big[r\Big(d_G(u_i)+d_G(u_p)\Big)+2(r-1)\Big]$$

$$\leq \Big[\frac{1}{nr(n-1)} \sum_{j=0}^{r-1} \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big\{r\Big(d_G(u_i)+d_G(u_p)\Big)+2(r-1)\Big\}\Big]^{nr(n-1)}$$

$$
\begin{aligned}
&= \Big[\frac{1}{nr(n-1)} \sum_{j=0}^{r-1} \Big\{r \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big(d_G(u_i)+d_G(u_p)\Big) \\
&+ 2(r-1) \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big\}\Big]^{nr(n-1)} \\
&= \Big[\frac{1}{nr(n-1)}\Big\{2rDD(G) \sum_{j=0}^{r-1} 1 + 4(r-1)W(G) \sum_{j=0}^{r-1} 1\Big\}\Big]^{nr(n-1)} \\
&= \Big[\frac{1}{nr(n-1)}\Big\{2rDD(G)\times r + 4(r-1)W(G)\times r\Big\}\Big]^{nr(n-1)} \\
&= \Big[\frac{2rDD(G)+4(r-1)W(G)}{n(n-1)}\Big]^{nr(n-1)} \\
C &= \prod_{j,q=0,j\neq q}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_{G\boxtimes K_r}(w_{ij},w_{pq})\Big[d_{G\boxtimes K_r}(w_{ij})+d_{G\boxtimes K_r}(w_{pq})\Big] \\
&= \prod_{j,q=0,j\neq q}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big[rd_G(u_i)+(r-1)+rd_G(u_p)+(r-1)\Big] \\
&= \prod_{j,q=0,j\neq q}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big[r\Big(d_G(u_i)+d_G(u_p)\Big)+2(r-1)\Big] \\
&\leq \Big[\frac{1}{nr(n-1)(r-1)} \sum_{j,q=0,j\neq q}^{r-1} \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big\{r\Big(d_G(u_i) \\
&+ d_G(u_p)\Big)+2(r-1)\Big\}\Big]^{nr(n-1)(r-1)} \\
&= \Big[\frac{1}{nr(n-1)(r-1)} \sum_{j,q=0,j\neq q}^{r-1} \Big\{r \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big(d_G(u_i)+d_G(u_p)\Big) \\
&+ 2(r-1) \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big\}\Big]^{nr(n-1)(r-1)} \\
&= \Big[\frac{1}{nr(n-1)(r-1)}\Big\{2rDD(G) \sum_{j,q=0,j\neq q}^{r-1} 1 \\
&+ 4(r-1)W(G) \sum_{j,q=0,j\neq q}^{r-1} 1\Big\}\Big]^{nr(n-1)(r-1)} \\
&= \Big[\frac{1}{nr(n-1)(r-1)}\Big\{2rDD(G)\times r(r-1) + 4(r-1)W(G)\,r(r-1)\Big\}\Big]^{nr(n-1)(r-1)} \\
&= \Big[\frac{2rDD(G)+4(r-1)W(G)}{n(n-1)}\Big]^{nr(n-1)(r-1)}
\end{aligned}
$$

Hence we get the following upper bound.

$$\begin{aligned}
[DD^*(G \boxtimes K_r)]^2 &\leq \left[\frac{4rm + 2n(r-1)}{n}\right]^{nr(r-1)} \times \left[\frac{2rDD(G) + 4(r-1)W(G)}{n(n-1)}\right]^{nr(n-1)} \\
&\times \left[\frac{2rDD(G) + 4(r-1)W(G)}{n(n-1)}\right]^{nr(n-1)(r-1)} \\
DD^*(G \boxtimes K_r) &\leq \left[\frac{4rm + 2n(r-1)}{n}\right]^{\frac{nr(r-1)}{2}} \times \left[\frac{2rDD(G) + 4(r-1)W(G)}{n(n-1)}\right]^{\frac{nr(nr-r)}{2}}
\end{aligned}$$

□

Lemma 4.

$$DD^*(K_n \boxtimes K_r) = (2nr-2)^{\frac{nr(nr-1)}{2}}$$

Proof. The degree of every vertex in $K_n \boxtimes K_r$ is $r(n-1) + (r-1) = rn - 1$.
Hence $K_n \boxtimes K_r$ is a complete graph. Now

$$\begin{aligned}
[DD^*(K_n \boxtimes K_r)]^2 &= \prod_{w_{ij}, w_{pq} \in V(K_n \boxtimes K_r), w_{ij} \neq w_{pq}} d_{K_n \boxtimes K_r}(w_{ij}, w_{pq}) \Big[d_{K_n \boxtimes K_r}(w_{ij}) \\
&+ d_{K_n \boxtimes K_r}(w_{pq}) \Big]
\end{aligned}$$

$$\begin{aligned}
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} d_{K_n \boxtimes K_r}(w_{ij}, w_{iq}) \Big[d_{K_n \boxtimes K_r}(w_{ij}) + d_{K_n \boxtimes K_r}(w_{iq}) \Big] \\
&\times \prod_{j=0}^{r-1} \prod_{i,p=0, i\neq p}^{n-1} d_{K_n \boxtimes K_r}(w_{ij}, w_{pj}) \Big[d_{K_n \boxtimes K_r}(w_{ij}) + d_{K_n \boxtimes K_r}(w_{pj}) \Big] \\
&\times \prod_{i,p=0, i\neq p}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} d_{K_n \boxtimes K_r}(w_{ij}, w_{pq}) \Big[d_{K_n \boxtimes K_r}(w_{ij}) + d_{K_n \boxtimes K_r}(w_{pq}) \Big] \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} 1[nr-1+nr-1] \times \prod_{j=0}^{r-1} \prod_{i,p=0, i\neq p}^{n-1} 1[nr-1+nr-1] \\
&\times \prod_{i,p=0, i\neq p}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} 1[nr-1+nr-1] \\
&= (2nr-2)^{nr(r-1)} \times (2nr-2)^{nr(n-1)} \times (2nr-2)^{nr(r-1)(n-1)} \\
&= (2nr-2)^{nr(nr-1)}
\end{aligned}$$

$$\text{Hence} \quad DD^*(K_n \boxtimes K_r) = (2nr-2)^{\frac{nr(nr-1)}{2}} \tag{1}$$

□

Remark 5. *Using Lemma* 4, *we show that the upper bound in the Theorem* 3 *is sharp.*

$$\begin{aligned} \textit{Clearly} \quad DD(K_n) &= (2n-2)\frac{n(n-1)}{2} = n(n-1)^2 \\ W(K_n) &= \frac{n(n-1)}{2}, \quad \textit{and} \quad m = \frac{n(n-1)}{2} \end{aligned}$$

When $G = K_n$ *the upper bound in Theorem* 3 *becomes*

$$\begin{aligned} DD^*(K_n \boxtimes K_r) &\leq \Big[\frac{4rm + 2n(r-1)}{n}\Big]^{\frac{nr(r-1)}{2}} \\ &\times \Big[\frac{2rDD(K_n) + 4(r-1)W(K_n)}{n(n-1)}\Big]^{\frac{nr(n-1)}{2}} \\ &\times \Big[\frac{2rDD(K_n) + 4(r-1)W(K_n)}{n(n-1)}\Big]^{\frac{nr(n-1)(r-1)}{2}} \\ &= \Big[\frac{4r\frac{n(n-1)}{2} + 2n(r-1)}{n}\Big]^{\frac{nr(r-1)}{2}} \\ &\times \Big[\frac{2nr(n-1)^2 + 4(r-1)\frac{n(n-1)}{2}}{n(n-1)}\Big]^{\frac{nr(n-1)}{2}} \\ &\times \Big[\frac{2nr(n-1)^2 + 4(r-1)\frac{n(n-1)}{2}}{n(n-1)}\Big]^{\frac{nr(n-1)(r-1)}{2}} \end{aligned}$$

After simplification, we get

$$DD^*(K_n \boxtimes K_r) \leq [2rn-2]^{\frac{nr(nr-1)}{2}} \tag{2}$$

From (1) *and* (2), *we conclude that the upper bound is sharp.*

4 Multiplicative Version of Gutman Index of Strong product of graphs.

In this section, we present the multiplicative version of Gutman index of $G \boxtimes K_r$.

Theorem 6. *Let* G *be a* (n, m)*graph. Then*

$$\begin{aligned} Gut^*(G \boxtimes K_r) &\leq \Big[\frac{r^2M_1(G) + 4r(r-1)m + n(r-1)^2}{n}\Big]^{\frac{nr(r-1)}{2}} \\ &\times \Big[\frac{2r^2Gut(G) + 2r(r-1)DD(G) + 2(r-1)^2W(G)}{n(n-1)}\Big]^{\frac{nr(nr-r)}{2}} \end{aligned}$$

Proof.

$$[Gut^*(G \boxtimes K_r)]^2 = \prod_{w_{ij}, w_{pq} \in V(G), w_{ij} \neq w_{pq}} d_{G \boxtimes K_r}(w_{ij}, w_{pq}) d_{G \boxtimes K_r}(w_{ij}) d_{G \boxtimes K_r}(w_{pq})$$

$$
\begin{aligned}
&= \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} d_{G\boxtimes K_r}(w_{ij}, w_{iq}) d_{G\boxtimes K_r}(w_{ij}) d_{G\boxtimes K_r}(w_{iq}) \\
&\times \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_{G\boxtimes K_r}(w_{ij}, w_{pj}) d_{G\boxtimes K_r}(w_{ij}) d_{G\boxtimes K_r}(w_{pj}) \\
&\times \prod_{i,p=0,i\neq p}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} d_{G\boxtimes K_r}(w_{ij}, w_{pq}) d_{G\boxtimes K_r}(w_{ij}) d_{G\boxtimes K_r}(w_{pq}) \\
&= S_1 \times S_2 \times S_3, \quad \text{where } S_1\ , S_2 \text{ and } S_3 \text{ are terms of the above} \\
&\quad \text{products taken in order.}
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} d_{G\boxtimes K_r}(w_{ij}, w_{iq}) d_{G\boxtimes K_r}(w_{ij}) d_{G\boxtimes K_r}(w_{iq}) \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} 1\Big[\Big(rd_G(u_i) + (r-1)\Big)\Big(rd_G(u_i) + (r-1)\Big)\Big] \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} \Big[r^2 d_G^2(u_i) + 2r(r-1)d_G(u_i) + (r-1)^2\Big] \\
&\leq \Big[\frac{1}{nr(r-1)} \sum_{i=0}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} \Big\{r^2 d_G^2(u_i) + 2r(r-1)d_G(u_i) + (r-1)^2\Big\}\Big]^{nr(r-1)} \\
&= \Big[\frac{1}{nr(r-1)} \sum_{j,q=0,j\neq q}^{r-1} \Big\{r^2 \sum_{i=0}^{n-1} d_G^2(u_i) + 2r(r-1) \sum_{i=0}^{n-1} d_G(u_i) \\
&+ (r-1)^2 \sum_{i=0}^{n-1} 1\Big\}\Big]^{nr(r-1)} \\
&= \Big[\frac{1}{nr(r-1)} \sum_{j,q=0,j\neq q}^{r-1} \Big\{r^2 M_1(G) + 2r(r-1)2m + (r-1)^2 n\Big\}\Big]^{nr(r-1)} \\
&= \Big[\frac{r(r-1)}{nr(r-1)} \Big\{r^2 M_1(G) + 4r(r-1)m + n(r-1)^2\Big\}\Big]^{nr(r-1)} \\
&= \Big[\frac{r^2 M_1(G) + 4r(r-1)m + n(r-1)^2}{n}\Big]^{nr(r-1)}
\end{aligned}
$$

$$
\begin{aligned}
S_2 &= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_{G\boxtimes K_r}(w_{ij}, w_{pj}) d_{G\boxtimes K_r}(w_{ij}) d_{G\boxtimes K_r}(w_{pj}) \\
&= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_G(u_i, u_p)\Big[\Big(rd_G(u_i) + (r-1)\Big)\Big(rd_G(u_p) + (r-1)\Big)\Big] \\
&= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_G(u_i, u_p)\Big[r^2 d_G(u_i) d_G(u_p) \\
&+ r(r-1)d_G(u_i) + r(r-1)d_G(u_p) + (r-1)^2\Big]
\end{aligned}
$$

$$\leq \Big[\frac{1}{nr(n-1)}\sum_{j=0}^{r-1}\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big\{r^2 d_G(u_i)d_G(u_p)+r(r-1)d_G(u_i)$$

$$+r(r-1)d_G(u_p)+(r-1)^2\Big\}\Big]^{nr(n-1)}$$

$$= \Big[\frac{1}{nr(n-1)}\sum_{j=0}^{r-1}\Big\{r^2\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)d_G(u_i)d_G(u_p)$$

$$+ \ r(r-1)\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)d_G(u_i)$$

$$+ \ r(r-1)\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)d_G(u_p)+(r-1)^2\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big\}\Big]^{nr(n-1)}$$

$$= \Big[\frac{1}{nr(n-1)}\sum_{j=0}^{r-1}\Big\{r^2\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)d_G(u_i)d_G(u_p)$$

$$+ \ r(r-1)\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big(d_G(u_i)+d_G(u_p)\Big)$$

$$+ \ (r-1)^2\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big\}\Big]^{nr(n-1)}$$

$$= \Big[\frac{1}{nr(n-1)}\sum_{j=0}^{r-1}\Big\{2r^2 Gut(G)+2r(r-1)DD(G)+2(r-1)^2 W(G)\Big\}\Big]^{nr(n-1)}$$

$$= \Big[\frac{r}{nr(n-1)}\Big\{2r^2 Gut(G)+2r(r-1)DD(G)+2(r-1)^2 W(G)\Big\}\Big]^{nr(n-1)}$$

$$= \Big[\frac{2r^2 Gut(G)+2r(r-1)DD(G)+2(r-1)^2 W(G)}{n(n-1)}\Big]^{n(n-1)}$$

$$S_3 = \prod_{j,q=0,j\neq q}^{r-1}\prod_{i,p=0,i\neq p}^{n-1} d_{G\boxtimes K_r}(w_{ij},w_{pq})d_{G\boxtimes K_r}(w_{ij})d_{G\boxtimes K_r}(w_{pq})$$

$$= \prod_{j,q=0,j\neq q}^{r-1}\prod_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big[\Big(rd_G(u_i)+(r-1)\Big)\Big(rd_G(u_p)+(r-1)\Big)\Big]$$

$$= \prod_{j,q=0,j\neq q}^{r-1}\prod_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big[r^2 d_G(u_i)d_G(u_p)$$

$$+ \ r(r-1)d_G(u_i)+r(r-1)d_G(u_p)+(r-1)^2\Big]$$

$$\leq \Big[\frac{1}{nr(n-1)(r-1)}\sum_{j,q=0,j\neq q}^{r-1}\Big\{r^2\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)d_G(u_i)d_G(u_p)$$

$$+ \ r(r-1)\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)d_G(u_i)+r(r-1)\sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)d_G(u_p)$$

$$
\begin{aligned}
&+ \quad (r-1)^2 \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big\}\Big]^{nr(n-1)(r-1)} \\
&= \quad \Big[\frac{1}{nr(n-1)(r-1)} \sum_{j,q=0,j\neq q}^{r-1} \Big\{r^2 \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p) d_G(u_i) d_G(u_p) \\
&+ \quad r(r-1) \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big(d_G(u_i)+d_G(u_p)\Big) \\
&+ \quad (r-1)^2 \sum_{i,p=0,i\neq p}^{n-1} d_G(u_i,u_p)\Big\}\Big]^{nr(n-1)(r-1)} \\
&= \quad \Big[\frac{1}{nr(n-1)(r-1)} \sum_{j,q=0,j\neq q}^{r-1} \Big\{2r^2 Gut(G) + 2r(r-1)DD(G) \\
&+ \quad 2(r-1)^2 W(G)\Big\}\Big]^{nr(n-1)(r-1)} \\
&= \quad \Big[\frac{r(r-1)}{nr(n-1)(r-1)}\Big\{2r^2 Gut(G) + 2r(r-1)DD(G) \\
&+ \quad 2(r-1)^2 W(G)\Big\}\Big]^{nr(n-1)(r-1)} \\
&= \quad \Big[\frac{2r^2 Gut(G) + 2r(r-1)DD(G) + 2(r-1)^2 W(G)}{n(n-1)}\Big]^{n(n-1)(r-1)}
\end{aligned}
$$

Hence we get the following upper bound.

$$
\begin{aligned}
[Gut^*(G \boxtimes K_r)]^2 \quad &\leq \quad \Big[\frac{r^2 M_1(G) + 4r(r-1)m + n(r-1)^2}{n}\Big]^{nr(r-1)} \\
&\times \quad \Big[\frac{2r^2 Gut(G) + 2r(r-1)DD(G) + 2(r-1)^2 W(G)}{n(n-1)}\Big]^{nr(n-1)} \\
&\times \quad \Big[\frac{2r^2 Gut(G) + 2r(r-1)DD(G) + 2(r-1)^2 W(G)}{n(n-1)}\Big]^{nr(n-1)(r-1)}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad Gut^*(G \boxtimes K_r) \quad &\leq \quad \Big[\frac{r^2 M_1(G) + 4r(r-1)m + n(r-1)^2}{n}\Big]^{\frac{nr(r-1)}{2}} \\
&\times \quad \Big[\frac{2r^2 Gut(G) + 2r(r-1)DD(G) + 2(r-1)^2 W(G)}{n(n-1)}\Big]^{\frac{nr(nr-r)}{2}}
\end{aligned}
$$

□

Lemma 7.

$$
Gut^*(K_n \boxtimes K_r) \quad = \quad \big(nr-1\big)^{nr(nr-1)}
$$

Proof. The degree of every vertex in $K_n \boxtimes K_r$ is $r(n-1)+(r-1)=rn-1$. Hence $K_n \boxtimes K_r$ is a complete graph. Now

$$\begin{aligned}
[Gut^*(K_n \boxtimes K_r)]^2 &= \prod_{w_{ij},w_{pq}\in V(K_n\boxtimes K_r),w_{ij}\neq w_{pq}} d_{K_n\boxtimes K_r}(w_{ij},w_{pq})d_{K_n\boxtimes K_r}(w_{ij})d_{K_n\boxtimes K_r}(w_{pq})\\
&= \prod_{i=0}^{n-1}\prod_{j,q=0,j\neq q}^{r-1} d_{K_n\boxtimes K_r}(w_{ij},w_{iq})d_{K_n\boxtimes K_r}(w_{ij})d_{G\boxtimes K_r}(w_{iq})\\
&\times \prod_{j=0}^{r-1}\prod_{i,p=0,i\neq p}^{n-1} d_{K_n\boxtimes K_r}(w_{ij},w_{pj})d_{K_n\boxtimes K_r}(w_{ij})d_{G\boxtimes K_r}(w_{pj})\\
&\times \prod_{i,p=0,i\neq p}^{n-1}\prod_{j,q=0,j\neq q}^{r-1} d_{K_n\boxtimes K_r}(w_{ij},w_{pq})d_{K_n\boxtimes K_r}(w_{ij})d_{G\boxtimes K_r}(w_{pq})
\end{aligned}$$

$$\begin{aligned}
&= \prod_{i=0}^{n-1}\prod_{j,q=0,j\neq q}^{r-1} 1(nr-1)\times(nr-1) \times \prod_{j=0}^{r-1}\prod_{i,p=0,i\neq p}^{n-1} 1(nr-1)\times(nr-1)\\
&\times \prod_{i,p=0,i\neq p}^{n-1}\prod_{j,q=0,j\neq q}^{r-1} 1(nr-1)\times(nr-1)\\
&= \big(nr-1\big)^{2nr(r-1)} \times \big(nr-1\big)^{2nr(nr-1)} \times \big(nr-1\big)^{2nr(n-1)(r-1)}
\end{aligned}$$

$$\therefore\ Gut^*(K_n \boxtimes K_r) = \big(nr-1\big)^{nr(nr-1)} \tag{3}$$

□

Remark 8. *Using Lemma* 7*, we show that the upper bound in Theoerm* 6 *is sharp. Clearly*

$$\begin{aligned}
Gut(K_n) &= \frac{n(n-1)^3}{2},\ DD(K_n)=n(n-1)^2,\ M_1(K_n)=n(n-1)^2 \textit{and}\\
W(K_n) &= \frac{n(n-1)}{2}
\end{aligned}$$

When $G = K_n$, the upper bound in Theorem 6 becomes

$$\begin{aligned} Gut^*(K_n \boxtimes K_r) &\leq \Big[\frac{r^2M_1(K_n) + 4r(r-1)m + n(r-1)^2}{n}\Big]^{\frac{nr(r-1)}{2}} \\ &\times \Big[\frac{2r^2Gut(K_n) + 2r(r-1)DD(K_n) + 2(r-1)^2W(K_n)}{n(n-1)}\Big]^{\frac{nr(nr-r)}{2}} \\ &= \Big[\frac{r^2n(n-1)^2 + 4r(r-1)\frac{n(n-1)}{2} + n(r-1)^2}{n}\Big]^{\frac{nr(r-1)}{2}} \\ &\times \Big[\frac{2r^2 \times \frac{n(n-1)^3}{2} + 2r(r-1)n(n-1)^2 + 2(r-1)^2 \times \frac{n(n-1)}{2}}{n(n-1)}\Big]^{\frac{nr(nr-r)}{2}} \end{aligned}$$

Simplifing we get,

$$Gut^*(K_n \boxtimes K_r) \leq [nr-1]^{nr(nr-1)} \tag{4}$$

From (3) and (4), we conclude that the upper bound is sharp.

5 Multiplicative Version of degree distance of cartesian product of graphs.

In this section, we obtain the sharp upper bound of the multiplicative version of degree distance of $G_1 \square G_2$.

Lemma 9. *Let G be a graph with r-vertices. Let $V(G) = \{v_0, v_1, ..., v_{r-1}\}$. Then*

$$\sum_{j,q=0, j\neq q}^{r-1} d_G(v_q) = 2(r-1)e(G)$$

Proof:

$$\begin{aligned} \sum_{j,q=0, j\neq q}^{r-1} d_G(v_q) &= \sum_{j=0}^{r-1} \sum_{q=0, j\neq q}^{r-1} d_G(v_q) \\ &= \sum_{j=0}^{r-1} \Big[2e(G) - d(v_j)\Big] \\ &= 2e(G)\sum_{j=0}^{r-1} 1 - \sum_{j=0}^{r-1} d(v_j) \\ &= 2e(G)r - 2e(G) \\ &= 2e(G)(r-1) \end{aligned}$$

Theorem 10. *Let G_1 and G_2 be two graphs with order n and r respectively. Then*

$$\begin{aligned} DD^*(G_1 \Box G_2) &\leq \Big[\frac{8e(G_1)W(G_2) + 2nDD(G_2)}{nr(r-1)}\Big]^{\frac{nr(r-1)}{2}} \\ &\times \Big[\frac{2rDD(G_1) + 8W(G_1)e(G_2)}{rn(n-1)}\Big]^{\frac{nr(n-1)}{2}} \\ &\times \Big[\frac{1}{rn(n-1)(r-1)}\Big\{2r(r-1)DD(G_1) + 8W(G_1)(r-1)e(G_2) \\ &+ 8(n-1)e(G_1)W(G_2) + 2n(n-1)DD(G_2)\Big\}\Big]^{\frac{nr(n-1)(r-1)}{2}} \end{aligned}$$

Proof. Let $V(G_1) = \{u_0, u_1, u_2, ..., u_{n-1}\}$ and $V(G_2) = \{v_0, v_1, v_2, ..., v_{r-1}\}$.
Let $w_{ij}, w_{pq} \in V(G_1 \Box G_2)$, where $w_{ij} = (u_i, v_j)$ and $w_{pq} = (u_p, v_q)$.

$$\begin{aligned} [DD^*(G_1 \Box G_2)]^2 &= \prod_{w_{ij}, w_{pq} \in V(G_1 \Box G_2), w_{ij} \neq w_{pq}} d_{G_1 \Box G_2}(w_{ij}, w_{pq})\Big[d_{G_1 \Box G_2}(w_{ij}) + d_{G_1 \Box G_2}(w_{pq})\Big] \\ &= \prod_{i=0}^{n-1} \prod_{j,q=0, j \neq q}^{r-1} d_{G_1 \Box G_2}(w_{ij}, w_{iq})\Big[d_{G_1 \Box G_2}(w_{ij}) + d_{G_1 \Box G_2}(w_{iq})\Big] \\ &\times \prod_{j=0}^{r-1} \prod_{i,p=0, i \neq p}^{n-1} d_{G_1 \Box G_2}(w_{ij}, w_{pj})\Big[d_{G_1 \Box G_2}(w_{ij}) + d_{G_1 \Box G_2}(w_{pj})\Big] \\ &\times \prod_{i,p=0, i \neq p}^{n-1} \prod_{j,q=0, j \neq q}^{r-1} d_{G_1 \Box G_2}(w_{ij}, w_{pq})\Big[d_{G_1 \Box G_2}(w_{ij}) + d_{G_1 \Box G_2}(w_{pq})\Big] \\ &= J_1 \times J_2 \times J_3 \quad \text{where } J_1, J_2, J_3 \text{ are terms of the above product taken in order.} \end{aligned}$$

$$
\begin{aligned}
J_1 &= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} d_{G_1 \square G_2}(w_{ij}, w_{iq}) \Big[d_{G_1 \square G_2}(w_{ij}) + d_{G_1 \square G_2}(w_{iq}) \Big] \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} d_{G_2}(v_j, v_q) \Big[d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i) + d_{G_2}(v_q) \Big] \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j\neq q}^{r-1} d_{G_2}(v_j, v_q) \Big[2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_q) \Big] \\
&\leq \Big[\frac{1}{nr(r-1)} \sum_{i=0}^{n-1} \sum_{j,q=0, j\neq q}^{r-1} d_{G_2}(v_j, v_q) \Big\{ 2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_q) \Big\} \Big]^{nr(r-1)} \\
&= \Big[\frac{1}{nr(r-1)} \Big\{ 2 \sum_{i=0}^{n-1} d_{G_1}(u_i) \sum_{j,q=0, j\neq q}^{r-1} d_{G_2}(v_j, v_q) \\
&+ \sum_{i=0}^{n-1} 1 \sum_{j,q=0, j\neq q}^{r-1} d_{G_2}(v_j, v_q) \Big(d_{G_2}(v_j) + d_{G_2}(v_q) \Big) \Big\} \Big]^{nr(r-1)} \\
&= \Big[\frac{1}{nr(r-1)} \Big\{ 2 \times 2e(G_1) \times 2W(G_2) + n \times 2DD(G_2) \Big\} \Big]^{nr(r-1)} \\
&= \Big[\frac{8e(G_1)W(G_2) + 2nDD(G_2)}{nr(r-1)} \Big]^{nr(r-1)} \\
J_2 &= \prod_{j=0}^{r-1} \prod_{i,p=0, i\neq p}^{n-1} d_{G_1 \square G_2}(w_{ij}, w_{pj}) \Big[d_{G_1 \square G_2}(w_{ij}) + d_{G_1 \square G_2}(w_{pj}) \Big] \\
&= \prod_{j=0}^{r-1} \prod_{i,p=0, i\neq p}^{n-1} d_{G_1}(u_i, u_p) \Big[d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_p) + d_{G_2}(v_j) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big[d_{G_1}(u_i)+2d_{G_2}(v_j)+d_{G_1}(u_p)\Big] \\
&\leq \Big[\frac{1}{nr(n-1)} \sum_{j=0}^{r-1} \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big\{d_{G_1}(u_i)+d_{G_1}(u_p)+2d_{G_2}(v_j)\Big\}\Big]^{nr(n-1)} \\
&= \Big[\frac{1}{nr(n-1)}\Big\{\sum_{j=0}^{r-1} 1 \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big(d_{G_1}(u_i)+d_{G_1}(u_p)\Big) \\
&+ 2\sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p) \sum_{j=0}^{r-1} d_{G_2}(v_j)\Big\}\Big]^{nr(n-1)} \\
&= \Big[\frac{1}{nr(n-1)}\Big\{r\times 2DD(G_1)+2\times 2W(G_1)\times 2e(G_2)\Big\}\Big]^{nr(n-1)} \\
&= \Big[\frac{2rDD(G_1)+8e(G_2)W(G_1)}{nr(n-1)}\Big]^{nr(n-1)} \\
J_3 &= \prod_{i,p=0,i\neq p}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} d_{G_1\square G_2}(w_{ij},w_{pq})\Big[d_{G_1\square G_2}(w_{ij})+d_{G_1\square G_2}(w_{pq})\Big] \\
&= \prod_{i,p=0,i\neq p}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} \Big(d_{G_1}(u_i,u_p)+d_{G_2}(v_j,v_q)\Big)\Big[d_{G_1}(u_i)+d_{G_1}(u_p) \\
&+ d_{G_2}(v_j)+d_{G_2}(v_q)\Big] \\
&\leq \Big[\frac{1}{nr(n-1)(r-1)}\Big\{\sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_1}(u_i,u_p)\Big(d_{G_1}(u_i)+d_{G_1}(u_p) \\
&+ d_{G_2}(v_j)+d_{G_2}(v_q)\Big) + \sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q)\Big(d_{G_1}(u_i)+d_{G_1}(u_p) \\
&+ d_{G_2}(v_j)+d_{G_2}(v_q)\Big)\Big\}\Big]^{rn(n-1)(r-1)} \\
&= \Big[\frac{1}{nr(n-1)(r-1)}\Big\{\sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_1}(u_i,u_p)\Big(d_{G_1}(u_i)+d_{G_1}(u_p)\Big) \\
&+ \sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_1}(u_i,u_p)\Big(d_{G_2}(v_j)+d_{G_2}(v_q)\Big) \\
&+ \sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q)\Big(d_{G_1}(u_i)+d_{G_1}(u_p)\Big) \\
&+ \sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q)\Big(d_{G_2}(v_j)+d_{G_2}(v_q)\Big)\Big\}\Big]^{rn(n-1)(r-1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\frac{1}{nr(n-1)(r-1)}\Big\{ \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big(d_{G_1}(u_i)+d_{G_1}(u_p)\Big) \sum_{j,q=0,j\neq q}^{r-1} 1 \\
&+ \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p) \sum_{j,q=0,j\neq q}^{r-1} \Big(d_{G_2}(v_j)+d_{G_2}(v_q)\Big) \\
&+ \sum_{i,p=0,i\neq p}^{n-1} \Big(d_{G_1}(u_i)+d_{G_1}(u_p)\Big) \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q) \\
&+ \sum_{i,p=0,i\neq p}^{n-1} 1 \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q)\Big(d_{G_2}(v_j)+d_{G_2}(v_q)\Big) \Big\}\Big]^{rn(n-1)(r-1)} \\
&= \Big[\frac{1}{nr(n-1)(r-1)}\Big\{2DD(G_1)\times r(r-1) + 2W(G_1)\times 4(r-1)e(G_2) \\
&+ 4(n-1)e(G_1)\times 2W(G_2) + n(n-1)\times 2DD(G_2)\Big\}\Big]^{rn(n-1)(r-1)} \\
&\leq \Big[\frac{1}{nr(n-1)(r-1)}\Big\{2r(r-1)DD(G_1) + 8(r-1)W(G_1)e(G_2) \\
&+ 8W(G_2)(n-1)e(G_1) + 2n(n-1)DD(G_2)\Big\}\Big]^{rn(n-1)(r-1)}
\end{aligned}
$$

Substituting J_1, J_2 and J_3 we get,

$$
\begin{aligned}
DD^*(G_1\Box G_2) &\leq \Big[\frac{8e(G_1)W(G_2)+2nDD(G_2)}{nr(r-1)}\Big]^{\frac{nr(r-1)}{2}} \\
&\times \Big[\frac{2rDD(G_1)+8W(G_1)e(G_2)}{rn(n-1)}\Big]^{\frac{nr(n-1)}{2}} \\
&\times \Big[\frac{1}{rn(n-1)(r-1)}\Big\{2r(r-1)DD(G_1)+8W(G_1)(r-1)e(G_2) \\
&+ 8(n-1)e(G_1)W(G_2)+2n(n-1)DD(G_2)\Big\}\Big]^{\frac{nr(n-1)(r-1)}{2}}
\end{aligned}
$$

□

Lemma 11.

$$DD^*(K_n\Box K_r) = 2^{\frac{nr(r-1)(n-1)}{2}} \times (2n+2r-4)^{\frac{nr(nr-1)}{2}}$$

Proof.

$$
\begin{aligned}
[DD^*(K_n \Box K_r)]^2 &= \prod_{w_{ij},w_{pq}\in V(G_1\Box G_2),w_{ij}\neq w_{pq}} d_{G_1\Box G_2}(w_{ij},w_{pq})\Big[d_{G_1\Box G_2}(w_{ij}) \\
&+ d_{G_1\Box G_2}(w_{pq})\Big] \\
&= \prod_{i=0}^{n-1}\prod_{j,q=0,j\neq q}^{r-1} d_{K_n\Box K_r}(w_{ij},w_{iq})\Big[d_{K_n\Box K_r}(w_{ij})+d_{K_n\Box K_r}(w_{iq})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{j=0}^{r-1}\prod_{i,p=0,i\neq p}^{n-1} d_{K_n\Box K_r}(w_{ij},w_{pj})\Big[d_{K_n\Box K_r}(w_{ij})+d_{K_n\Box K_r}(w_{pj})\Big] \\
&\times \prod_{i,p=0,i\neq p}^{n-1}\prod_{j,q=0,j\neq q}^{r-1} d_{K_n\Box K_r}(w_{ij},w_{pq})\Big[d_{K_n\Box K_r}(w_{ij})+d_{K_n\Box K_r}(w_{pq})\Big] \\
&= \prod_{i=0}^{n-1}\prod_{j,q=0,j\neq q}^{r-1} 1[2(n-1)+2(r-1)] \\
&\times \prod_{j=0}^{r-1}\prod_{i,p=0,i\neq p}^{n-1} 1[2(n-1)+2(r-1)] \\
&\times \prod_{i,p=0,i\neq p}^{n-1}\prod_{j,q=0,j\neq q}^{r-1} (1+1)[2(n-1)+2(r-1)] \\
&= \Big(2n+2r-4\Big)^{nr(r-1)} \times \Big(2n+2r-4\Big)^{nr(n-1)} \\
&\times \quad 2^{nr(r-1)(n-1)} \times \Big(2n+2r-4\Big)^{nr(r-1)(n-1)}
\end{aligned}
$$

$$
\text{Hence } DD^*(K_n \Box K_r) = 2^{\frac{nr(r-1)(n-1)}{2}} \times (2n+2r-4)^{\frac{nr(nr-1)}{2}} \tag{5}
$$

□

Remark 12. *Using Lemma* 11*, we show that the upper bounds in the Theorem* 10 *is sharp.*

Clearly,

$$
\begin{aligned}
e(K_n) &= \frac{n(n-1)}{2},\ W(K_r) = \frac{r(r-1)}{2}, \\
DD(K_r) &= (2r-2)\frac{r(r-1)}{2} = r(r-1)^2 \\
DD(K_n) &= n(n-1)^2,\ W(K_n) = \frac{n(n-1)}{2},\ \textit{and}\ e(K_r) = \frac{r(r-1)}{2} \\
\textit{when}\ G_1 &= K_n, G_2 = K_r, \textit{the upper bounds in Theoerm 10 becomes} \\
DD^*(K_n \Box K_r) &\leq \Big[\frac{8e(K_n)W(K_r) + 2nDD(K_r)}{nr(r-1)}\Big]^{\frac{nr(r-1)}{2}} \\
&\times \Big[\frac{2rDD(K_n) + 8W(K_n)e(K_r)}{rn(n-1)}\Big]^{\frac{nr(n-1)}{2}} \\
&\times \Big[\frac{1}{rn(n-1)(r-1)}\Big\{2r(r-1)DD(K_n) + 8W(K_n)(r-1)e(K_r) \\
&+ 8(n-1)e(K_n)W(K_r) + 2n(n-1)DD(K_r)\Big\}\Big]^{\frac{nr(n-1)(r-1)}{2}} \\
&= \Big[\frac{1}{nr(r-1)}\Big\{8\ \frac{n(n-1)}{2}\frac{r(r-1)}{2} + 2nr(r-1)^2\Big\}\Big]^{\frac{nr(r-1)}{2}} \\
&\times \Big[\frac{1}{rn(n-1)}\Big\{2rn(n-1)^2 + 8\ \frac{n(n-1)}{2}\ \frac{r(r-1)}{2}\Big\}\Big]^{\frac{nr(n-1)}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[\frac{1}{rn(n-1)(r-1)}\Big\{2r(r-1)n(n-1)^2 \\
&+ 8(r-1)\ \frac{n(n-1)}{2}\ \frac{r(r-1)}{2} + 8\frac{n(n-1)}{2}\ \frac{r(r-1)}{2}(n-1) \\
&+ 2n(n-1)r(r-1)^2\Big\}\Big]^{\frac{nr(n-1)(r-1)}{2}}
\end{aligned}
$$

Simplifing we get,

$$DD^*(K_n \Box K_r) \leq 2^{\frac{nr(r-1)(n-1)}{2}} \times (2n+2r-4)^{\frac{nr(nr-1)}{2}} \tag{6}$$

From (5) *and* (6), *we conclude that the upper bound is sharp.*

Lemma 13. *Let* G *be graph with* r *vertices. Let* $V(G) = \{v_0, v_1, v_2, , \ldots, v_{r-1}\}$. *Then*

$$\sum_{j,q=0, j\neq q}^{r-1} d_G(v_j) d_G(v_q) = 4e^2(G) - M_1(G)$$

Proof.

$$
\begin{aligned}
\sum_{j,q=0,j\neq q}^{r-1} d_G(v_j)d_G(v_q) &= \sum_{j=0}^{r-1} \sum_{q=0,j\neq q}^{r-1} d_G(v_j)d_G(v_q) \\
&= \sum_{j=0}^{r-1} d_G(v_j) \quad \sum_{q=0,j\neq q}^{r-1} d_G(v_q) \\
&= \sum_{j=0}^{r-1} d_G(v_j)\Big[2e(G) - d_G(v_j)\Big] \\
&= 2e(G) \sum_{j=0}^{r-1} d_G(v_j) - \sum_{j=0}^{r-1} d_G^2(v_j) \\
&= 4e^2(G) - M_1(G)
\end{aligned}
$$

□

6 The Multiplicative version of Gutman index of Cartesian Product of Graphs.

In this section, we obtain the sharp upper bound of the multiplicative version of Gutman index of $G_1 \Box G_2$.

Theorem 14. *Let G_1 and G_2 be two graphs with order n and r respectively. Then*

$$
\begin{aligned}
Gut^*(G_1\Box G_2) &\leq \Big[\frac{2W(G_2)M_1(G_1) + 4e(G_1)DD(G_2) + 2nGut(G_2)}{nr(r-1)}\Big]^{\frac{nr(r-1)}{2}} \\
&\times \Big[\frac{2rGut(G_1) + 4DD(G_1)e(G_2) + 2W(G_1)M_1(G_2)}{rn(n-1)}\Big]^{\frac{nr(n-1)}{2}} \\
&\times \Big[\frac{1}{rn(n-1)(r-1)}\Big\{2r(r-1)Gut(G_1) + 4(r-1)e(G_2)DD(G_1) \\
&+ 2W(G_1)\Big(4e^2(G_2) - M_1(G_2)\Big) + 2\Big(4e^2(G_1) - M_1(G_1)\Big)W(G_2) \\
&+ 4DD(G_2)(n-1)e(G_1) + 2n(n-1)Gut(G_2)\Big\}\Big]^{\frac{nr(n-1)(r-1)}{2}}
\end{aligned}
$$

Proof. Let $V(G_1) = \{u_0, u_1, u_2, ..., u_{n-1}\}$ and $V(G_2) = \{v_0, v_1, v_2, ..., v_{r-1}\}$.

Let $w_{ij}, w_{pq} \in V(G_1 \Box G_2)$, where $w_{ij} = (u_i, v_j)$ and $w_{pq} = (u_p, v_q)$.

$$
\begin{aligned}
[Gut^*(G_1 \Box G_2)]^2 &= \prod_{w_{ij}, w_{pq} \in V(G_1 \Box G_2), w_{ij} \neq w_{pq}} d_{G_1 \Box G_2}(w_{ij}, w_{pq}) d_{G_1 \Box G_2}(w_{ij}) d_{G_1 \Box G_2}(w_{pq}) \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j \neq q}^{r-1} d_{G_1 \Box G_2}(w_{ij}, w_{iq}) d_{G_1 \Box G_2}(w_{ij}) d_{G_1 \Box G_2}(w_{iq}) \\
&\times \prod_{j=0}^{r-1} \prod_{i,p=0, i \neq p}^{n-1} d_{G_1 \Box G_2}(w_{ij}, w_{pj}) d_{G_1 \Box G_2}(w_{ij}) d_{G_1 \Box G_2}(w_{pj}) \\
&\times \prod_{i,p=0, i \neq p}^{n-1} \prod_{j,q=0, j \neq q}^{r-1} d_{G_1 \Box G_2}(w_{ij}, w_{pq}) d_{G_1 \Box G_2}(w_{ij}) d_{G_1 \Box G_2}(w_{pq}) \\
&= A_1 \times A_2 \times A_3, \text{ where } A_1,\ A_2,\ A_3 \text{ are terms of above} \\
&\quad \text{product taken in order.}
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= \prod_{i=0}^{n-1} \prod_{j,q=0, j \neq q}^{r-1} d_{G_1 \Box G_2}(w_{ij}, w_{iq}) d_{G_1 \Box G_2}(w_{ij}) d_{G_1 \Box G_2}(w_{iq}) \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j \neq q}^{r-1} d_{G_2}(v_j, v_q) \Big[d_{G_1}(u_i) + d_{G_2}(v_j) \Big] \Big[d_{G_1}(u_i) + d_{G_2}(v_q) \Big] \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j \neq q}^{r-1} d_{G_2}(v_j, v_q) \Big[d_{G_1}^2(u_i) + d_{G_1}(u_i) d_{G_2}(v_q) \\
&+ d_{G_2}(v_j) d_{G_1}(u_i) + d_{G_2}(v_j) d_{G_2}(v_q) \Big] \\
&\leq \Big[\frac{1}{nr(r-1)} \sum_{i=0}^{n-1} \sum_{j,q=0, j \neq q}^{r-1} d_{G_2}(v_j, v_q) \Big\{ d_{G_1}^2(u_i) + d_{G_1}(u_i) \Big(d_{G_2}(v_j) + d_{G_2}(v_q) \Big) \\
&+ d_{G_2}(v_j) d_{G_2}(v_q) \Big\} \Big]^{nr(r-1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\frac{1}{nr(r-1)}\Big\{\sum_{i=0}^{n-1} d^2_{G_1}(u_i) \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q) \\
&+ \sum_{i=0}^{n-1} d_{G_1}(u_i) \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q)\Big(d_{G_2}(v_j)+d_{G_2}(v_q)\Big) \\
&+ \sum_{i=0}^{n-1} 1 \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q) d_{G_2}(v_j) d_{G_2}(v_q)\Big\}\Big]^{nr(r-1)} \\
&\leq \Big[\frac{2W(G_2)M_1(G_1)+4e(G_1)DD(G_2)+2nGut(G_2)}{nr(r-1)}\Big]^{nr(r-1)} \\
A_2 &= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_{G_1\square G_2}(w_{ij},w_{pj}) d_{G_1\square G_2}(w_{ij}) d_{G_1\square G_2}(w_{pj}) \\
&= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big[d_{G_1}(u_i)+d_{G_2}(v_j)\Big]\Big[d_{G_1}(u_p)+d_{G_2}(v_j)\Big] \\
&= \prod_{j=0}^{r-1} \prod_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big[d_{G_1}(u_i)d_{G_1}(u_p)+d_{G_1}(u_i)d_{G_2}(v_j) \\
&+ d_{G_1}(u_p)d_{G_2}(v_j)+d^2_{G_2}(v_j)\Big] \\
&\leq \Big[\frac{1}{nr(n-1)}\sum_{j=0}^{r-1} \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big\{d_{G_1}(u_i)d_{G_1}(u_p)+d_{G_2}(v_j)\Big(d_{G_1}(u_i) \\
&+ d_{G_1}(u_p)\Big)+d^2_{G_2}(v_j)\Big\}\Big]^{nr(n-1)} \\
&= \Big[\frac{1}{nr(n-1)}\Big\{\sum_{j=0}^{r-1} 1 \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p) d_{G_1}(u_i) d_{G_1}(u_p) \\
&+ \sum_{j=0}^{r-1} d_{G_2}(v_j) \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big(d_{G_1}(u_i)+d_{G_1}(u_p)\Big) \\
&+ \sum_{j=0}^{r-1} d^2_{G_2}(v_j) \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big\}\Big]^{nr(n-1)} \\
&\leq \Big[\frac{2rGut(G_1)+4e(G_2)DD(G_1)+2W(G_1)M_1(G_2)}{rn(n-1)}\Big]^{nr(n-1)} \\
A_3 &= \prod_{i,p=0,i\neq p}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} d_{G_1\square G_2}(w_{ij},w_{pq}) d_{G_1\square G_2}(w_{ij}) d_{G_1\square G_2}(w_{pq}) \\
&= \prod_{i,p=0,i\neq p}^{n-1} \prod_{j,q=0,j\neq q}^{r-1} \Big(d_{G_1}(u_i,u_p)+d_{G_2}(v_j,v_q)\Big)\Big[d_{G_1}(u_i) \\
&+ d_{G_2}(v_j)\Big]\Big[d_{G_1}(u_p)+d_{G_2}(v_q)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big[\frac{1}{nr(n-1)(r-1)} \sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} \Big(d_{G_1}(u_i,u_p) \\
&+ d_{G_2}(v_j,v_q)\Big)\Big\{d_{G_1}(u_i)d_{G_1}(u_p) + d_{G_1}(u_i)d_{G_2}(v_q) + d_{G_2}(v_j)d_{G_1}(u_p) \\
&+ d_{G_2}(v_j)d_{G_2}(v_q)\Big\}\Big]^{rn(n-1)(r-1)} \\
&= \Big[\frac{1}{nr(n-1)(r-1)}\Big\{ \sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_1}(u_i,u_p)\Big(d_{G_1}(u_i)d_{G_1}(u_p) \\
&+ d_{G_1}(u_i)d_{G_2}(v_q) + d_{G_2}(v_j)d_{G_1}(u_p) + d_{G_2}(v_j)d_{G_2}(v_q)\Big) \\
&+ \sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_1}(v_j,v_q)\Big(d_{G_1}(u_i)d_{G_1}(u_p) + d_{G_1}(u_i)d_{G_2}(v_q) \\
&+ d_{G_2}(v_j)d_{G_1}(u_p) + d_{G_2}(v_j)d_{G_2}(v_q)\Big)\Big\}\Big]^{rn(n-1)(r-1)} \\
A_3 &\leq \Big[\frac{1}{nr(n-1)(r-1)}\Big\{A_{3,1} + A_{3,2}\Big\}\Big]^{nr(n-1)(r-1)}
\end{aligned}
$$

where $A_{3,1},\ A_{3,2}$ are terms of above sum taken in order.

$$
\begin{aligned}
\text{Let } A_{3,1} &= \sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_1}(u_i,u_p)\Big[d_{G_1}(u_i)d_{G_1}(u_p) + d_{G_1}(u_i)d_{G_2}(v_q) \\
&+ d_{G_2}(v_j)d_{G_1}(u_p) + d_{G_2}(v_j)d_{G_2}(v_q)\Big] \\
&= \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)d_{G_1}(u_i)d_{G_1}(u_p) \sum_{j,q=0,j\neq q}^{r-1} 1 \\
&+ \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p)\Big\{d_{G_1}(u_i) + d_{G_1}(u_p)\Big\} \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j) \\
&+ \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i,u_p) \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j)d_{G_2}(v_q) \\
&= 2r(r-1)Gut(G_1) + 2e(G_2)(r-1)2DD(G_1) + 2W(G_1)\big[4e^2(G_2) - M_1(G_2)\big] \\
&= 2r(r-1)Gut(G_1) + 4(r-1)e(G_2)DD(G_1) + 2W(G_1)\big[4e^2(G_2) - M_1(G_2)\big] \\
\text{Let } A_{3,2} &= \sum_{i,p=0,i\neq p}^{n-1} \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q)\Big[d_{G_1}(u_i)d_{G_1}(u_p) + d_{G_1}(u_i)d_{G_2}(v_q) \\
&+ d_{G_2}(v_j)d_{G_1}(u_p) + d_{G_2}(v_j)d_{G_2}(v_q)\Big] \\
&= \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i)d_{G_1}(u_p) \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q) \\
&+ \sum_{i,p=0,i\neq p}^{n-1} d_{G_1}(u_i) \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q)\Big(d_{G_2}(v_q) + d_{G_2}(v_j)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i,p=0,i\neq p}^{n-1} 1 \sum_{j,q=0,j\neq q}^{r-1} d_{G_2}(v_j,v_q) d_{G_2}(v_j) d_{G_2}(v_q) \\
&= 2\Big(4e^2(G_1) - M_1(G_1)\Big)W(G_2) + 4DD(G_2)(n-1)e(G_1) + 2n(n-1)Gut(G_2) \\
\therefore A_3 &\leq \Big[\frac{1}{rn(n-1)(r-1)}\Big\{2r(r-1)Gut(G_1) + 4(r-1)e(G_2)DD(G_1) \\
&+ 2W(G_1)\Big(4e^2(G_2) - M_1(G_2)\Big) + 2\Big(4e^2(G_1) - M_1(G_1)\Big)W(G_2) \\
&+ 4DD(G_2)(n-1)e(G_1) + 2n(n-1)Gut(G_2)\Big\}\Big]^{nr(n-1)(r-1)}
\end{aligned}
$$

Multiplying A_1, A_2 and A_3 we get,

$$
\begin{aligned}
Gut^*(G_1 \Box G_2) &\leq \Big[\frac{2W(G_2)M_1(G_1) + 4e(G_1)DD(G_2) + 2nGut(G_2)}{nr(r-1)}\Big]^{\frac{nr(r-1)}{2}} \\
&\times \Big[\frac{2rGut(G_1) + 4DD(G_1)e(G_2) + 2W(G_1)M_1(G_2)}{rn(n-1)}\Big]^{\frac{nr(n-1)}{2}} \\
&\times \Big[\frac{1}{rn(n-1)(r-1)}\Big\{2r(r-1)Gut(G_1) + 4(r-1)e(G_2)DD(G_1) \\
&+ 2W(G_1)\Big(4e^2(G_2) - M_1(G_2)\Big) + 2\Big(4e^2(G_1) - M_1(G_1)\Big)W(G_2) \\
&+ 4DD(G_2)(n-1)e(G_1) + 2n(n-1)Gut(G_2)\Big\}\Big]^{\frac{nr(n-1)(r-1)}{2}}
\end{aligned}
$$

□

Lemma 15.

$$Gut^*(K_n \Box K_r) = 2^{\frac{nr(n-1)(r-1)}{2}} \times \big(n + r - 2\big)^{nr(nr-1)}$$

Proof.

$$\begin{aligned}
[Gut^*(K_n \Box K_r)]^2 &= \prod_{w_{ij}, w_{pq} \in V(K_n \Box K_r), w_{ij} \neq w_{pq}} d_{K_n \Box K_r}(w_{ij}, w_{pq}) d_{K_n \Box K_r}(w_{ij}) d_{K_n \Box K_r}(w_{pq}) \\
&= \prod_{i=0}^{n-1} \prod_{j,q=0, j \neq q}^{r-1} d_{K_n \Box K_r}\Big(w_{ij}, w_{iq}\Big)\Big[d_{K_n \Box K_r}(w_{ij}) d_{K_n \Box K_r}(w_{iq})\Big] \\
&\times \prod_{j=0}^{r-1} \prod_{i,p=0, i \neq p}^{n-1} d_{K_n \Box K_r}\Big(w_{ij}, w_{pj}\Big)\Big[d_{K_n \Box K_r}(w_{ij}) d_{K_n \Box K_r}(w_{pj})\Big] \\
&\times \prod_{i,p=0, i \neq p}^{n-1} \prod_{j,q=0, j \neq q}^{r-1} d_{K_n \Box K_r}\Big(w_{ij}, w_{pq}\Big)\Big[d_{K_n \Box K_r}(w_{ij}) d_{K_n \Box K_r}(w_{pq})\Big] \\
&= \Big[(n+r-2)^2\Big]^{nr(r-1)} \times \Big[(n+r-2)^2\Big]^{nr(n-1)} \\
&\times 2^{nr(n-1)(r-1)} \times \Big[(n+r-2)^2\Big]^{nr(n-1)(r-1)}
\end{aligned}$$

After simplification, we get,

$$Gut^*(K_n \Box K_r) = 2^{\frac{nr(n-1)(r-1)}{2}} \times \Big(n+r-2\Big)^{nr(nr-1)} \tag{7}$$

□

Remark 16. *Using Lemma* 15*, we show that the upper bounds in the Theorem* 14 *is sharp. Clearly,*

$$\begin{aligned}
M_1(K_n) &= n(n-1)^2, \quad e(K_n) = \frac{n(n-1)}{2}, \\
M_2(K_n) &= \frac{n(n-1)}{2} \times (n-1)^2 = \frac{n(n-1)^3}{2}, \\
Gut(K_n) &= \frac{n(n-1)}{2} \times (n-1)^2 = \frac{n(n-1)^3}{2}, \\
DD(K_n) &= n(n-1)^2, \quad W(K_n) = \frac{n(n-1)}{2}.
\end{aligned}$$

When $G_1 = K_n$ and $G_2 = K_r$, the upper bound in Theorem 14 becomes

$$\begin{aligned}
Gut^*(K_n \Box K_r) &\leq \Big[\frac{2W(K_r)M_1(K_n) + 4e(K_n)DD(K_r) + 2nGut(K_r)}{nr(r-1)}\Big]^{\frac{nr(r-1)}{2}} \\
&\times \Big[\frac{2rGut(K_n) + 4DD(K_n)e(K_r) + 2W(K_n)M_1(K_r)}{rn(n-1)}\Big]^{\frac{nr(n-1)}{2}} \\
&\times \Big[\frac{1}{nr(n-1)(r-1)}\Big\{2r(r-1)Gut(K_n) + 4(r-1)e(K_r)DD(K_n) \\
&+ 2W(K_n)\Big(4e^2(K_r) - M_1(K_r)\Big) + 2\Big(4e^2(K_n) - M_1(K_n)\Big)W(K_r) \\
&+ 4DD(K_r)(n-1)e(K_n) + 2n(n-1)Gut(K_r)\Big\}\Big]^{\frac{nr(n-1)(r-1)}{2}}
\end{aligned}$$

$$\begin{aligned}
&= \Big[\frac{1}{nr(r-1)}\Big\{2n(n-1)^2 \times \frac{r(r-1)}{2} + 4\ \frac{n(n-1)}{2} r(r-1)^2 + 2n\ \frac{r(r-1)^3}{2}\Big\}\Big]^{\frac{nr(r-1)}{2}} \\
&\times \Big[\frac{1}{nr(n-1)}\Big\{2r\ \frac{n(n-1)^3}{2} + 2\ \frac{r(r-1)}{2} 2n(n-1)^2 + \frac{2n(n-1)}{2}\ r(r-1)^2\Big\}\Big]^{\frac{nr(r-1)}{2}} \\
&\times \Big[\frac{1}{nr(n-1)(r-1)}\Big\{2r(r-1)\ \frac{n(n-1)^3}{2} + 4(r-1)\ \frac{r(r-1)}{2}\ n(n-1)^2 \\
&+ 2\ \frac{n(n-1)}{2}\Big(4\ \frac{r^2(r-1)^2}{4} - r(r-1)^2\Big) + 2\frac{r(r-1)}{2}\Big(4\ \frac{n^2(n-1)^2}{4} - n(n-1)^2\Big) \\
&+ 4(n-1)\ \frac{n(n-1)}{2}\ r(r-1)^2 + 2n(n-1)\ \frac{r(r-1)^3}{2}\Big\}\Big]^{\frac{nr(n-1)(r-1)}{2}} \\
&= \Big[(n+r-2)^2\Big]^{\frac{nr(r-1)}{2}} \times \Big[(n+r-2)^2\Big]^{\frac{nr(n-1)}{2}} \\
&\times 2^{\frac{nr(n-1)(r-1)}{2}} \times \Big[(n+r-2)^2\Big]^{\frac{nr(n-1)(r-1)}{2}}
\end{aligned}$$

$$\textit{Hence}\quad Gut^*(K_n \Box K_r) = 2^{\frac{nr(n-1)(r-1)}{2}} \times \big(n+r-2\big)^{nr(nr-1)} \tag{8}$$

From (7) and (8), we conclude that the upper bound is sharp.

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