Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

Pair Difference Cordiality of Some Snake and Butterfly Graphs

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ABSTRACT

Let $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels. Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of

Keyword: Triangular snake, Alternate triangular snake, Quadrilatral Snake, Alternate Quadrilatral Snake, Butter fly

ARTICLE INFO

Article history:
Research paper
Received 15, October 2020
Received in revised form 10, April 2021
Accepted 12 May 2021
Available online 01, June 2021

AMS subject Classification: 05C78.

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Abstract continued: edges labeled with 1 and number of edges not labeled with 1. A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph. In this paper we investigate the pair difference cordial labeling behavior of some snake and butterfly graphs.

1 Introduction

In this paper we consider only finite, undirected and simple graphs.The concept of cordial labeling was introduced by Cachit[1].The notion of pair diference cordial labeling of a graph was introduced and studied some properties of pair difference cordial labeling in [4].The pair difference cordial labeling behavior of several graphs like path, cycle, star, wheel etc have been investigated in [4].In this paper, we study the pair difference cordiality of some snake and butterfly graphs.Terms not defined here are follow from Gallian[2] and Harary[3].

2 Preliminaries

Definition 2.1. Two even cycles of the same order say C_n, sharing a common vertex with m pendent edges attached at the common vertex is called a butterfly graph $By_{m,n}$.Define $V(By_{m,n}) = \{v_i, u_i : 1 \leq i \leq n\} \cup \{w_j : 1 \leq j \leq m\}$ and $E(By_{m,n}) = \{u_1 w_i : 1 \leq i \leq m\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\}$ where u_1 is identifying with v_1.

Definition 2.2. The triangular snake T_n is obtained from the path P_n by replacing each edge of the path by a triangle C_3.Let $V(T_n) = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n-1\}$ and $E(T_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i, u_i v_{i+1} : 1 \leq i \leq n-1\}$.There are $2n-1$ vertices and $3n-3$ edges.

Definition 2.3. The alternate triangular snake $A(T_n)$ is obtained from the path $u_1 u_2 \cdots u_n$ by joining u_i and u_{i+1} (alternatively) to new vertex v_i. That is every alternate edge of a path is replaced by C_3.Now we define the vertex set and edge set of $A(T_n)$ as follows.

Type 1. The edge $u_1 u_2$ lies on the triangle and the edge $u_{n-1} u_n$ lies on the triangle.In this case n is even.Let $V(A(T_n)) = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq \frac{n}{2}\}$ and $E(A(T_n)) = \{u_{2i} u_{2i+1}, u_{2i} v_j, u_{2i-1} v_j : 1 \leq i, j \leq \frac{n}{2}\}$.There are $\frac{3n}{2}$ vertices and $2n-1$ edges.

Type 2. The edge $u_1 u_2$ not lies on the triangle and the edge $u_{n-1} u_n$ not lies on the triangle.Here clearly n is even.Let $V(A(T_n)) = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq \frac{n-2}{2}\}$ and $E(A(T_n)) = \{u_{2i} v_j, u_{2i+1} v_j : 1 \leq i, j \leq \frac{n-2}{2}\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$.There are $\frac{3n-2}{2}$ vertices and $2n-3$ edges.

Type 3. The edge $u_1 u_2$ not lies on the triangle and the edge $u_{n-1} u_n$ lies on the triangle.In this type n is odd.Let $V(A(T_n)) = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq \frac{n-1}{2}\}$ and $E(A(T_n)) = \{u_{2i} v_j, u_{2i+1} v_j : 1 \leq i, j \leq \frac{n-1}{2}\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$.There are $\frac{3n-1}{2}$

vertices and $2n-2$ edges.

Definition 2.4. The quadrilateral snake Q_n is obtained from the path P_n by replacing each edge of the path by a cycle C_4.Let $V(Q_n) = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le n-1\}$ and $E(Q_n) = \{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_iv_i, v_iw_i, w_iu_{i+1} : 1 \le i \le n-1\}$.There are $3n-2$ vertices and $4n-4$ edges.
Definition 2.5. The alternate quadrilateral snake $A(Q_n)$ is obtained from the path $u_1u_2\cdots u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i, w_i respectively and then joining v_i and w_i. That is every alternate edge of a path is replaced by a cycle C_4.Now we define the vertex set and edge set of $A(Q_n)$ as follows.

Type 1. The edge u_1u_2 lies on the quadrilateral and the edge $u_{n-1}u_n$ lies on the quadrilateral.In this type n is even. Let $V(A(Q_n)) = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le \frac{n}{2}\}$ and $E(A(Q_n)) = \{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_{2i-1}v_i, v_iw_i, w_iu_{2i} : 1 \le i \le \frac{n}{2}\}$.There are $2n$ vertices and $\frac{5n-2}{2}$ edges.

Type 2. The edge u_1u_2 not lies on the quadrilateral and the edge $u_{n-1}u_n$ lies on the quadrilateral.In this case n is odd. Let $V(A(Q_n)) = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le \frac{n-1}{2}\}$ and $E(A(Q_n)) = \{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_{2i}v_i, v_iw_i, w_iu_{2i+1} : 1 \le i \le \frac{n-1}{2}\}$.There are $2n-1$ vertices and $\frac{5n-5}{2}$ edges.

Type 3. The edge u_1u_2 not lies on the quadrilateral and the edge $u_{n-1}u_n$ not lies on the quadrilateral.Here clearly n is even. Let $V(A(Q_n)) = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le \frac{n-2}{2}\}$ and $E(A(Q_n)) = \{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_{2i}v_i, v_iw_i, w_iu_{2i+1} : 1 \le i \le \frac{n-2}{2}\}$.There are $2n-2$ vertices and $\frac{5n-8}{2}$ edges.

3 Butterfly Graphs

Theorem 3.1 The butterfly graph $By_{m,n}$ is a pair difference cordial if $m = 2, 3, 4, 5, 6$ and for all values of $n \ge 3$.

Proof. Let us consider the vertex set and edge set from the definition 2.1.
Case 1. $m = 2$.
Define the pair difference labeling $f : V(By_{2,n}) \longrightarrow \{\pm 1, \pm 2, \cdots, \pm(n)\}$.There are four cases arises.
Subcase 1. $n \equiv 0 \pmod 4$.
First assign the label 2 to the vertex u_1. Assign the labels $3, 4$ to the vertices u_2, u_3 respectively and assign the labels $6, 5$ respectively to the vertices u_4, u_5.Next assign the labels $7, 8$ respectively to the vertices u_6, u_7 and assign the labels $10, 11$ respectively to the vertices u_8, u_9.Proceeding like this until we reach the vertex u_{n-1}.Now assign the labels $f(v_i) = -f(u_i, 2 \le i \le n-1)$.Finally assign the labels $1, -2, 1, -1$ respectively to the vertices u_n, v_n, w_1, w_2.

Subcase 2. $n \equiv 1 \pmod 4$.
As in subcase 1, assign the labels to the vertices $u_i, (1 \leq i \leq n-1)$, $v_i(2 \leq i \leq n-1)$ and w_1, w_2.Finally assign the labels $2, -2$ respectively to the vertices u_n, v_n.
Subase 3. $n \equiv 2 \pmod 4$.
As in subcase 1, assign the labels to the vertices $u_i, (1 \leq i \leq n)$, $v_i, (2 \leq i \leq n)$ and w_1, w_2.
Subcase 4. $n \equiv 3 \pmod 4$.
As in subcase 2, assign the labels to the vertices $u_i, (1 \leq i \leq n), v_i(2 \leq i \leq n-4), v_n, w_1$ and w_2.Finally assign the labels $-n+2, -n+1, -n$ to the vertices $v_{n-3}, v_{n-2}, v_{n-1}$ respectively.
This vertex labeling in all the four cases gives that $\Delta_{f_1} = n+1 = \Delta_{f_1^c}$.

Case 2. $m = 3$.
Define the pair difference labeling $f : V(By_{3,n}) \longrightarrow \{\pm1, \pm2, \cdots, \pm(n+1)\}$.There are four cases arises.
Subcase 1. $n \equiv 0 \pmod 4$.
First assign the label 2 to the vertex u_1. Assign the labels $3, 4$ to the vertices u_2, u_3 respectively and assign the labels $6, 5$ respectively to the vertices u_4, u_5.Next assign the labels $7, 8$ respectively to the vertices u_6, u_7 and assign the labels $10, 11$ respectively to the vertices u_8, u_9.Proceeding like this until we reach the vertex u_n.Now assign the labels $f(v_i) = -f(u_i), (2 \leq i \leq n)$.Finally assign the labels $1, -1, -2$ respectively to the vertices w_1, w_2, w_3.
Subcase 2. $n \equiv 1 \pmod 4$.
As in subcase 1, assign the labels to the vertices $u_i, (1 \leq i \leq n)$, $v_i(2 \leq i \leq n-1)$ and w_1, w_2, w_3.
Subase 3. $n \equiv 2 \pmod 4$.
As in subcase 1, assign the labels to the vertices $u_i, (1 \leq i \leq n)$, $v_i, (2 \leq i \leq n)$ and w_1, w_2, w_3.
Subcase 4. $n \equiv 3 \pmod 4$.
As in subcase 1, assign the labels to the vertices $u_i, (1 \leq i \leq n)$, $v_i, (2 \leq i \leq n)$ and w_1, w_2, w_3.

Case 3. $m = 4$.
Define the pair difference labeling $f : V(By_{4,n}) \longrightarrow \{\pm1, \pm2, \cdots, \pm(n+1)\}$.Similar to case 2, assign the labels to the vertices $u_i, 1 \leq i \leq n$,$v_i, 2 \leq i \leq n$ and w_1, w_2, w_3.There are two cases arises.
Subcase 1. n is even.
Finally assign label -2 to the vertex w_4.
Subcase 2. n is odd.
Finally assign label 1 to the vertex w_4.

Case 4. $m = 5$.
Define the pair difference labeling $f : V(By_{5,n}) \longrightarrow \{\pm1, \pm2, \cdots, \pm(n+2)\}$.There are

two cases arises.
Subcase 1. n is even.
Similar to case 2, assign the labels to the vertices $u_i, 1 \le i \le n$,$v_i, 2 \le i \le n$ and w_1, w_2, w_3.Finally assign the labels $n+2, -n-2$ respectively to the vertices w_4, w_5.
Subcase 2. n is odd.
Similar to case 2, assign the labels to the vertices $u_i, 1 \le i \le n$,$v_i, 2 \le i \le n-3$ and w_1, w_2, w_3.Finally assign the labels $-n+2, -n+1, -n$ respectively to the vertices v_{n-2}, v_{n-1}, v_n and assign the labels $n+2, -n-2$ respectively to the vertices w_4, w_5 .

Case 5. $m = 6$.
Define the pair difference labeling $f : V(By_{6,n}) \longrightarrow \{\pm1, \pm2, \cdots, \pm(n+2)\}$.As in case 4, assign the labels to the vertices $u_i, 1 \le i \le n$,$v_i, 2 \le i \le n$ and w_1, w_2, w_3, w_4, w_5.There are two cases arises.
Subcase 1. n is even.
Finally assign label -2 to the vertex w_6.
Subcase 2. n is odd.
Finally assign label 1 to the vertex w_6.

□

4 Some Snake Graphs

Theorem 4.1. The triangular snake T_n is pair difference cordial if and only if $n \ge 3$.

Proof. Let us take the vertex set and edge set from the definition 2.4.There are three cases arises.

Case 1. $n = 2$.
Clearly $T_2 \cong C_3$.Hence T_2 is not pair difference cordial[6].

Case 2. $n = 3$.
Assign the labels $1, 2, -1$ to the vertices u_1, u_2, u_3 respectively.Next assign the labels $2, -2$ to the vertices v_1, v_2 respectively.

Case 3. $n > 3$.
There are four cases arises.
Subcase 1. $n \equiv 0 \pmod 4$.
Assign the labels $1, 3, 5, \cdots, n-1$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n}{2}}$ and assign the labels $-1, -3, -5, \cdots, -(n-1)$ to the vertices $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, u_{\frac{n+6}{2}}, \cdots, u_n$.Now we consider the vertices $v_j, 1 \le j \le n-1$.
Next assign the labels $2, 4, 6, \cdots, n-2$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-2}{2}}$ respectively and now assign the label -2 to the vertex $v_{\frac{n+2}{2}}$.Assign the labels $-6, -4$ repectively to the vertices $v_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}$ and assign the labels $-10, -8$ to the vertices $v_{\frac{n+8}{2}}, v_{\frac{n+10}{2}}$ respectively.Now

we assign the labels $-14, -12$ repectively to the vertices $v_{\frac{n+12}{2}}, v_{\frac{n+14}{2}}$.Proceeding like this until we reach the vertex v_{n-1}.Finally assign the label $n-2$ to the vertex $v_{\frac{n}{2}}$.

Subcase 2. $n \equiv 1 \pmod 4$.
Assign the labels $1, 3, 5, \cdots, n-1$ to the vertices $u_2, u_3, u_4, \cdots, u_{\frac{n+1}{2}}$ respectively. Assign the labels $-1, -3, -5, \cdots, -(n-1)$ to the vertices $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$.Now consider the vertices $v_j, 1 \leq j \leq n-2$.
Next assign the labels $2, 4, 6, \cdots, n-1$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-1}{2}}$ respectively and now assign the label -2 to the vertex $v_{\frac{n+1}{2}}$.Assign the labels $-6, -4$ repectively to the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}$ and Assign the labels $-10, -8$ to the vertices $v_{\frac{n+7}{2}}, v_{\frac{n+9}{2}}$ respectively.Now we assign the labels $-14, -12$ repectively to the vertices $v_{\frac{n+11}{2}}, v_{\frac{n+13}{2}}$.Proceeding like this until we reach the vertex v_{n-1}.Finally assign the label 2 to the vertices u_n.

Subcase 3. $n \equiv 2 \pmod 4$.
As in case 1, assign the labels to the vertices $u_i, 1 \leq i \leq n$, $v_j, 1 \leq j \leq \frac{n-2}{2}$ and $\frac{n+2}{2} \leq j \leq n-1$.Finally assign the label 2 to the vertex $v_{\frac{n}{2}}$.

Subcase 4. $n \equiv 3 \pmod 4$.
As in case 2, assign the labels to the vertices $u_i, 1 \leq i \leq n$, $v_j, 1 \leq j \leq n-1$.
The Table 1 given below establish that this vertex labeling f is a pair difference cordial of T_n for all values of $n \geq 3$.

Nature of n	Δ_{f_1}	$\Delta_{f_1^c}$
$n \equiv 0 \pmod 4$	$\frac{3n-2}{2}$	$\frac{3n-4}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n-3}{2}$	$\frac{3n-3}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n-2}{2}$	$\frac{3n-4}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n-3}{2}$	$\frac{3n-3}{2}$

Table 1:

□

Theorem 4.2. The alternate triangular snake $A(T_n)$ is pair difference cordial if the edge u_1u_2 lies on the triangle and the edge $u_{n-1}u_n$ lies on the triangle for all even $n \geq 4$.

Proof. The vertex set and edge set taken from the definition 2.5.There are two cases arises.
Case 1. $n \equiv 0 \pmod 4$.

Define the map $f : V(A(T_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n}{4}\}$ by

$$
\begin{aligned}
f(u_{3i+1}) &= 3i+1, & 0 \le i \le \frac{n-4}{4},\\
f(u_{3i+2}) &= 3i+2, & 0 \le i \le \frac{n-4}{4},\\
f(u_{3i+3}) &= -(3i+1), & 0 \le i \le \frac{n-4}{4},\\
f(u_{3i+4}) &= -(3i+2), & 0 \le i \le \frac{n-4}{4},\\
f(v_{2i+1}) &= 3i+3, & 0 \le i \le \frac{n-4}{4},\\
f(v_{2i+2}) &= -(3i+3), & 0 \le i \le \frac{n-4}{4}.
\end{aligned}
$$

This vertex labeling gives the pair difference cordial labeling of alternate triangular snake $A(T_n)$.
Hence $\Delta_{f_1} = n, \Delta_{f_1^c} = n-1$.

Case 2. $n \equiv 2 \pmod 4$.
Define the map $f : V(A(T_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n-2}{4}\}$ by

$$
\begin{aligned}
f(u_{3i+1}) &= 3i+1, & 0 \le i \le \frac{n-6}{4},\\
f(u_{3i+2}) &= 3i+2, & 0 \le i \le \frac{n-6}{4},\\
f(u_{3i+3}) &= -(3i+1), & 0 \le i \le \frac{n-6}{4},\\
f(u_{3i+4}) &= -(3i+2), & 0 \le i \le \frac{n-6}{4},\\
f(v_{2i+1}) &= 3i+3, & 0 \le i \le \frac{n-6}{4},\\
f(v_{2i+2}) &= -(3i+3), & 0 \le i \le \frac{n-6}{4}.
\end{aligned}
$$

Finally assign the labels $-\frac{3n-2}{4}, \frac{3n-2}{4}, \frac{3n-6}{2}$ respectively to the vertices $u_{n-1}, u_n, v_{\frac{n}{2}}$.
This vertex labeling gives the pair difference cordial labeling of alternate triangular snake $A(T_n), n \ge 4$.
Hence $\Delta_{f_1} = n-1, \Delta_{f_1^c} = n$.

□

Theorem 4.3. The alternate triangular snake $A(T_n)$ is pair difference cordial if the edge u_1u_2 not lies on the triangle and the edge $u_{n-1}u_n$ not lies on the triangle for all even $n \ge 4$.

Proof. Consider the vertex set and edge set as in definition 2.5.There are two cases arises.

Case 1. $n \equiv 0 \pmod 4$.
Define the map $f : V(A(T_n)) \rightarrow \{\pm 1, \pm 2, \cdots, \pm \frac{3n-4}{4}\}$ by $f(u_1) = 1, f(u_n) = -1, f(u_2) = 2, f(u_3) = 3, f(u_4) = -2, f(u_5) = -3, f(v_1) = 4, f(v-2) = -4$,

$$
\begin{aligned}
f(u_{4i+2}) &= f(u_{4i-2}) + 3, && 1 \le i \le \frac{n-4}{4},\\
f(u_{4i+3}) &= f(u_{4i-1}) + 3, && 1 \le i \le \frac{n-4}{4},\\
f(u_{4i+4}) &= f(u_{4i}) - 3, && 1 \le i \le \frac{n-4}{4},\\
f(u_{4i+5}) &= f(u_{4i+1}) - 3, && 1 \le i \le \frac{n-4}{4},\\
f(v_{2i+1}) &= f(v_{2i-1}) + 3, && 1 \le i \le \frac{n-4}{4},\\
f(v_{2i}) &= f(v_{2i-2}) - 3, && 1 \le i \le \frac{n-4}{4},
\end{aligned}
$$

Finally assign the labels $-n+4, n-4$ to the vertices $u_{n-2}, v_{\frac{n-2}{2}}$ and then assign the label -2 to the vertex u_{n-1}.
Hence $\Delta_{f_1} = n-2, \Delta_{f_1^c} = n-1$.

Case 2. $n \equiv 2 \pmod 4$.
Define the map $f : V(A(T_n)) \rightarrow \{\pm 1, \pm 2, \cdots, \pm \frac{3n-2}{4}\}$ by $f(u_1) = 1, f(u_n) = -1, f(u_2) = 2, f(u_3) = 3, f(u_4) = -2, f(u_5) = -3, f(v_1) = 4, f(v-2) = -4$,

$$
\begin{aligned}
f(u_{4i+2}) &= f(u_{4i-2}) + 3, && 1 \le i \le \frac{n-6}{4},\\
f(u_{4i+3}) &= f(u_{4i-1}) + 3, && 1 \le i \le \frac{n-6}{4},\\
f(u_{4i+4}) &= f(u_{4i}) - 3, && 1 \le i \le \frac{n-6}{4},\\
f(u_{4i+5}) &= f(u_{4i+1}) - 3, && 1 \le i \le \frac{n-6}{4},\\
f(v_{2i+1}) &= f(v_{2i-1}) + 3, && 1 \le i \le \frac{n-6}{4},\\
f(v_{2i}) &= f(v_{2i-2}) - 3, && 1 \le i \le \frac{n-6}{4},
\end{aligned}
$$

This vertex labeling establish the pair difference cordial labeling of alternate triangular snake $A(T_n), n \ge 4$.
Hence $\Delta_{f_1} = n-1, \Delta_{f_1^c} = n-2$.

□

Theorem 4.4. The alternate triangular snake $A(T_n)$ is pair difference cordial if the edge u_1u_2 not lies on the triangle and the edge $u_{n-1}u_n$ lies on the triangle for all odd $n \ge 3$.

Proof. Take the vertex set and edge set from definition 2.5.There are two cases arises.
Case 1. $n \equiv 1 \pmod 4$.
Define the map $f : V(A(T_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n-3}{4}\}$ by $f(u_1) = 1, f(u_2) = -2, f(u_3) = -3, f(u_4) = 1, f(u_5) = 2, f(v_1) = -4, f(v_2) = 3$

$$\begin{aligned}
f(u_{4i+2}) &= f(u_{4i-2}) - 3, && 1 \leq i \leq \frac{n-5}{4}, \\
f(u_{4i+3}) &= f(u_{4i-1}) - 3, && 1 \leq i \leq \frac{n-5}{4}, \\
f(u_{4i+4}) &= f(u_{4i}) + 3, && 1 \leq i \leq \frac{n-5}{4}, \\
f(u_{4i+5}) &= f(u_{4i+1}) + 3, && 1 \leq i \leq \frac{n-5}{4}, \\
f(v_{2i+1}) &= f(v_{2i-1}) - 3, && 1 \leq i \leq \frac{n-9}{4}, \\
f(v_{2i}) &= f(v_{2i-2}) + 3, && 1 \leq i \leq \frac{n-5}{4},
\end{aligned}$$

$$f(v_{\frac{n-3}{2}}) = -\frac{3n-3}{4} = f(u_{n-2}).$$

Hence $\Delta_{f_1} = n - 1, \Delta_{f_1^c} = n - 1$.

Case 2. $n \equiv 3 \pmod 4$.
Define the map $f : V(A(T_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n-1}{4}\}$ by $f(u_1) = -1, f(u_2) = -2, f(u_3) = -3, f(u_4) = 1, f(u_5) = 2, f(v_1) = -4, f(v_2) = 3$

$$\begin{aligned}
f(u_{4i+2}) &= f(u_{4i-2}) - 3, && 1 \leq i \leq \frac{n-3}{4}, \\
f(u_{4i+3}) &= f(u_{4i-1}) - 3, && 1 \leq i \leq \frac{n-7}{4}, \\
f(u_{4i+4}) &= f(u_{4i}) + 3, && 1 \leq i \leq \frac{n-7}{4}, \\
f(u_{4i+5}) &= f(u_{4i+1}) + 3, && 1 \leq i \leq \frac{n-7}{4}, \\
f(v_{2i+1}) &= f(v_{2i-1}) - 3, && 1 \leq i \leq \frac{n-7}{4}, \\
f(v_{2i}) &= f(v_{2i-2}) + 3, && 1 \leq i \leq \frac{n-7}{4}, \\
f(u_{n-1}) &= f(u_{n-3}) + 3 && f(u_n) = f(u_{n-2}) + 3, \\
f(v_{\frac{n-1}{2}}) &= -f(u_n).
\end{aligned}$$

Hence $\Delta_{f_1} = n - 1, \Delta_{f_1^c} = n - 1$.
This vertex labeling gives the pair difference cordial labeling of alternate triangular snake $A(T_n), n \geq 3$.

□

Theorem 4.5 The quadrilateral snake Q_n is pair difference cordial i $n\ geq2$.

Proof. Take the vertex set and edge set from definition 2.6.There are three cases arises.
Case 1. $n = 2$.
Clearly $T_2 \cong C_4$.Hence T_2 is pair difference cordial [6].

Case 2. $n = 3$.
Assign the labels $1, 2, -1$ to the vertices u_1, u_2, u_3 respectively.Next assign the labels $1, -3$ to the vertices v_1, v_2 respectively.Finally assign the labels $3, -2$ respectively to the vertices w_1, w_2.

Case 3. $n > 3$.
There are two cases arises.

Case 1. n is even.
Define the map $f : V(Q_n) \rightarrow \{\pm1, \pm2, \cdots, \pm\frac{3n-2}{2}\}$ by $f(u_1) = 1, f(u_2) = 2, f(v_1) = 4, f(w_1) = 3$

$$
\begin{aligned}
f(u_i) &= f(u_{i-1}) + 3, && 3 \leq i \leq \frac{n}{2},\\
f(v_i) &= f(v_{i-1}) + 3, && 2 \leq i \leq \frac{n-2}{2},\\
f(w_i) &= f(w_{i-1}) + 3, && 2 \leq i \leq \frac{n-2}{2},\\
f(u_{\frac{n+2i}{2}}) &= -f(u_i), && 1 \leq i \leq \frac{n}{2},\\
f(v_{\frac{n+2i}{2}}) &= -f(v_i), && 1 \leq i \leq \frac{n-2}{2},\\
f(w_{\frac{n+2i}{2}}) &= -f(w_i), && 1 \leq i \leq \frac{n-2}{2},\\
f(u_n) &= -f(u_{\frac{n}{2}}),\\
f(v_{\frac{n}{2}}) &= f(v_{\frac{n-2}{2}}) + 1,\\
f(w_{\frac{n}{2}}) &= -f(w_{\frac{n-2}{2}}) - 1.
\end{aligned}
$$

Case 2. n is odd.
Define the map $f : V(Q_n) \rightarrow \{\pm1, \pm2, \cdots, \pm\frac{3n-1}{2}\}$ by $f(u_1) = 1, f(u_2) = 2, f(v_1) =$

4, $f(w_1) = 3$

$$
\begin{aligned}
f(u_i) &= f(u_{i-1}) + 3, && 3 \leq i \leq \frac{n+1}{2},\\
f(v_i) &= f(v_{i-1}) + 3, && 2 \leq i \leq \frac{n-3}{2},\\
f(w_i) &= f(w_{i-1}) + 3, && 2 \leq i \leq \frac{n-3}{2},\\
f(u_{\frac{n+1+2i}{2}}) &= -f(u_{i+1}), && 1 \leq i \leq \frac{n-1}{2},\\
f(v_{\frac{n-1+2i}{2}}) &= -f(v_i), && 1 \leq i \leq \frac{n-3}{2},\\
f(w_{\frac{n-1+2i}{2}}) &= -f(w_i), && 1 \leq i \leq \frac{n-3}{2},\\
f(v_{\frac{n-1}{2}}) &= f(v_{\frac{n-3}{2}}), && f(v_{\frac{n+1}{2}}) = -f(u_{\frac{n+1}{2}}),\\
f(w_{\frac{n-1}{2}}) &= f(v_{\frac{n-1}{2}}) + 2, && f(w_{\frac{n+1}{2}}) = -f(w_{\frac{n-1}{2}}).
\end{aligned}
$$

Hence from this vertex labeling the quadrilateral snake Q_n is pair difference cordial fo all values of $n \geq 2$.
. □

Theorem 4.6. The alternate quadrilateral snake $A(Q_n)$ is pair difference cordial if the edge u_1u_2 lies on the quadrilateral and the edge $u_{n-1}u_n$ lies on the quadrilateral for all even $n \geq 4$.

Proof. Consider the vertex set and edge set in definition 2.7.There are two cases arises.
Case 1. $n \equiv 0 \pmod 4$.
Define the map $f : V(A(Q_n)) \to \{\pm 1, \pm 2, \cdots, \pm n\}$ by

$$
\begin{aligned}
f(u_{2i+2}) &= 4i + 1, && 0 \leq i \leq \frac{n-4}{4},\\
f(u_{2i+3}) &= 4i + 2, && 0 \leq i \leq \frac{n-4}{4},\\
f(v_i) &= 4i + 4, && 0 \leq i \leq \frac{n-4}{4},\\
f(w_i) &= 4i + 3, && 0 \leq i \leq \frac{n-4}{4},\\
f(u_{\frac{n+2i}{2}}) &= -f(u_i), && 1 \leq i \leq \frac{n}{2},\\
f(v_{\frac{n+4i}{4}}) &= -f(w_i), && 1 \leq i \leq \frac{n}{4},\\
f(w_{\frac{n+4i}{4}}) &= -f(v_i), && 1 \leq i \leq \frac{n}{4}.
\end{aligned}
$$

Case 2. $n \equiv 2 \pmod 4$.
Define the map $f : V(A(Q_n)) \to \{\pm 1, \pm 2, \cdots, \pm n\}$ by

$$
\begin{aligned}
f(u_{2i+2}) &= 4i+1, & 0 \le i \le \frac{n+2}{4},\\
f(u_{2i+3}) &= 4i+2, & 0 \le i \le \frac{n+2}{4},\\
f(v_i) &= 4i+4, & 0 \le i \le \frac{n-6}{4},\\
f(w_i) &= 4i+3, & 0 \le i \le \frac{n-6}{4},\\
f(u_{\frac{n+2i}{2}}) &= -f(u_i), & 1 \le i \le \frac{n-2}{2},\\
f(v_{\frac{n+4i+2}{4}}) &= -f(w_i), & 1 \le i \le \frac{n-2}{4},\\
f(w_{\frac{n+4i+2}{4}}) &= -f(v_i), & 1 \le i \le \frac{n-2}{4}.
\end{aligned}
$$

Hence from this vertex labeling the alternate quadrilateral snake $A(Q_n)$ is pair difference cordial if the edge u_1u_2 lies on the quadrilateral and the edge $u_{n-1}u_n$ lies on the quadrilateral for all even $n \ge 4$. □

Theorem 4.7. The alternate quadrilateral snake $A(Q_n)$ is pair difference cordial if the edge u_1u_2 not lies on the quadrilateral and the edge $u_{n-1}u_n$ lies on the quadrilateral for all odd $n \ge 3$.

Proof. Let us take the vertex set and edge set from definition 2.7.There are two cases arises.
Case 1. $n \equiv 1 \pmod 4$.
Define the map $f : V(A(Q_n)) \to \{\pm 1, \pm 2, \cdots, \pm n-1\}$ by

$$
\begin{aligned}
f(u_{2i+2}) &= 4i+1, & 0 \le i \le \frac{n-5}{4},\\
f(u_{2i+3}) &= 4i+2, & 0 \le i \le \frac{n-5}{4},\\
f(v_i) &= 4i+4, & 0 \le i \le \frac{n-5}{4},\\
f(w_i) &= 4i+3, & 0 \le i \le \frac{n-5}{4},\\
f(u_{\frac{n+2i+3}{2}}) &= -f(u_i), & 1 \le i \le \frac{n-1}{2},\\
f(v_{\frac{n+4i-1}{4}}) &= -f(w_i), & 1 \le i \le \frac{n-1}{4},\\
f(w_{\frac{n+4i-1}{4}}) &= -f(v_i), & 1 \le i \le \frac{n-1}{4},\\
f(u_1) &= n-1.
\end{aligned}
$$

Case 2. $n \equiv 3 \pmod 4$.
Define the map $f : V(A(Q_n)) \to \{\pm 1, \pm 2, \cdots, \pm n-1\}$ by

$$
\begin{aligned}
f(u_{2i+2}) &= 4i+1, & 0 \le i \le \frac{n-3}{4},\\
f(u_{2i+3}) &= 4i+2, & 0 \le i \le \frac{n-3}{4},\\
f(v_i) &= 4i+4, & 0 \le i \le \frac{n-7}{4},\\
f(w_i) &= 4i+3, & 0 \le i \le \frac{n-7}{4},\\
f(u_{\frac{n+2i+3}{2}}) &= -f(u_i), & 1 \le i \le \frac{n-3}{2},\\
f(v_{\frac{n+4i+1}{4}}) &= -f(w_i), & 1 \le i \le \frac{n-3}{4},\\
f(w_{\frac{n+4i+1}{2}}) &= -f(v_i), & 1 \le i \le \frac{n-3}{4},\\
f(v_{\frac{n+1}{4}}) = -f(u_{\frac{n+1}{2}}), f(w_{\frac{n+1}{4}}) &= -f(u_{\frac{n+3}{2}}), \quad f(u_1) & = n-1.
\end{aligned}
$$

This vertex labeling gives that the alternate quadrilateral snake $A(Q_n)$ is pair difference cordial if the edge u_1u_2 not lies on the quadrilateral and the edge $u_{n-1}u_n$ lies on the quadrilateral for all odd $n \geq 3$.

□

Theorem 4.8. The alternate quadrilateral snake $A(Q_n)$ is pair difference cordial if the edge u_1u_2 not lies on the quadrilateral and the edge $u_{n-1}u_n$ not lies on the quadrilateral for all even $n \geq 4$.

Proof. Take vertex set and edge set from definition 2.7. There are two cases arises.
Case 1. $n \equiv 0 \pmod 4$.

Define the map $f : V(A(Q_n)) \to \{\pm 1, \pm 2, \cdots, \pm n-1\}$ by

$$
\begin{aligned}
f(u_{2i+2}) &= 4i+1, && 0 \le i \le \frac{n-4}{4},\\
f(u_{2i+3}) &= 4i+2, && 0 \le i \le \frac{n-4}{4},\\
f(v_i) &= 4i+4, && 0 \le i \le \frac{n-4}{4},\\
f(w_i) &= 4i+3, && 0 \le i \le \frac{n-4}{4},\\
f(u_{\frac{n+2i+2}{2}}) &= -f(u_i), && 1 \le i \le \frac{n-4}{2},\\
f(v_{\frac{n+2i}{4}}) &= -f(w_i), && 1 \le i \le \frac{n-8}{4},\\
f(w_{\frac{n+2i}{4}}) &= -f(v_i), && 1 \le i \le \frac{n-8}{4},\\
f(u_1) &= n-1, && f(u_n) = -n+1,\\
f(v_{\frac{n-2}{2}}) &= -f(v_{\frac{n-4}{4}}), && f(w_{\frac{n-2}{2}}) = -f(w_{\frac{n-4}{4}})\\
, f(v_{\frac{n}{4}}) &= f(u_{\frac{n}{2}}), && f(w_{\frac{n}{4}}) = f(u_{\frac{n+2}{2}}).
\end{aligned}
$$

Case 2. $n \equiv 2 \pmod 4$.
Define the map $f : V(A(Q_n)) \to \{\pm 1, \pm 2, \cdots, \pm n-1\}$ by

$$
\begin{aligned}
f(u_{2i+2}) &= 4i+1, && 0 \le i \le \frac{n-6}{4},\\
f(u_{2i+3}) &= 4i+2, && 0 \le i \le \frac{n-6}{4},\\
f(v_i) &= 4i+4, && 0 \le i \le \frac{n-6}{4},\\
f(w_i) &= 4i+3, && 0 \le i \le \frac{n-6}{4},\\
f(u_{\frac{n+2i}{2}}) &= -f(u_i), && 1 \le i \le \frac{n-6}{2},\\
f(v_{\frac{n+2i}{4}}) &= -f(w_i), && 1 \le i \le \frac{n-6}{4},\\
f(w_{\frac{n+2i}{4}}) &= -f(v_i), && 1 \le i \le \frac{n-6}{4},
\end{aligned}
$$

$$
f(u_1) = n-1, \quad f(u_n) = -n+1,
$$
$$
f(v_{\frac{n-2}{2}}) = -f(v_{\frac{n-2}{4}}), \quad f(w_{\frac{n-2}{2}}) = -f(w_{\frac{n-2}{4}}).
$$

Thus this vertex labeling gives that the alternate quadrilateral snake $A(Q_n)$ is pair difference cordial if the edge u_1u_2 not lies on the quadrilateral and the edge $u_{n-1}u_n$ not lies on the quadrilateral for all even $n \ge 4$.

□

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