A comparison between the resolution and linear optimization of FREs defined by product t-norm and geometric mean operator

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Abstract

In this paper, a type of fuzzy system is firstly investigated whereby the feasible region is defined by the fuzzy relational equalities and the geometric mean as fuzzy composition. Some related basic and theoretical properties are derived and the feasible region is completely determined. Moreover, a comparison is made between this region and FRE defined by product t-norm. Finally, an example is described to illustrate the differences of these two FRE systems.

Keywords: Fuzzy relational equalities, fuzzy compositions, t-norm, geometric mean operator.

1 Introduction

In this paper, we study the following fuzzy system in which the constraints are formed as fuzzy relational equalities defined by geometric mean operator:

$$\begin{aligned} &\min \sum_{j=1}^{n} c_j x_j \\ &A \odot x = b \qquad (1) \\ &x \in [0,1]^n \end{aligned}$$

Where $I = \{1, 2, \ldots, m\}$, $J = 1, 2, \ldots, n$. $A = (a_{ij})_{m\times n}$ is a fuzzy matrix such that $0 \le a_{ij} \le 1$ ($\forall i \in I$ and $\forall j \in J$), $b = (b_i)_{m\times 1}$ is an m–dimensional fuzzy vector in $[0,1]^m$ (i.e., $0 \le b_i \le 1$, $\forall i \in I$) and "$\odot$" is the max −geometric composition, i.e., $x \odot y = \sqrt{xy}$.

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Furthermore, let $S(A, b)$ denote the feasible solutions sets of problem (1), that is, $S(A, b) = \{x \in [0, 1]^n : A \odot x = b\}$. By these notations, problem (1) can be also expressed as follows:

$$\max_{j \in J} \sqrt{a_{ij} x_j} = b_i \ , \ i \in I \qquad (2)$$
$$x \in [0, 1]^n$$

The theory of fuzzy relational equations (FRE) was firstly proposed by Sanchez and applied in problems of the medical diagnosis [39]. Nowadays, it is well known that many issues associated with a body knowledge can be treated as FRE problems [35]. In addition to the preceding applications, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [25, 33].

The solvability determination and the finding of solutions set are the primary (and the most fundamental) subject concerning with FRE problems. Actually, The solution set of FRE is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [5]. This non-convexity property is one of two bottlenecks making major contribution to the increase of complexity in problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. The other bottleneck is concerned with detecting the minimal solutions for FREs [2]. Markovskii showed that solving max-product FRE is closely related to the covering problem which is an NP-hard problem [32]. In fact, the same result holds true for a more general t-norms instead of the minimum and product operators [2, 3, 12, 13, 15, 16, 28, 29, 32].

Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [15,16,34,36,37,40,42,43,45,48,51]. Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI) [1214, 17, 18, 26, 50]. Li and Yang [26] studied a FRI with addition-min composition and presented an algorithm to search for minimal solutions. Ghodousian et al. [13] focused on the algebraic structure of two fuzzy relational inequalities $A\varphi x \leq b^1$ and $D\varphi x \leq b^2$, and studied a mixed fuzzy system formed by the two preceding FRIs, where φ is an operator with (closed) convex solutions.

The problem of optimization subject to FRE and FRI is one of the most interesting and ongoing research topic among the problems related to FRE and FRI theory [1, 8, 1116, 23, 27, 30, 38, 41, 46, 50]. Fang and Li [9] converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by branch and bound method using jump-tracking technique. In [23] an application of optimizing the linear objective with max-min composition was employed for the streaming media provider. Wu et al. [44] improved the method used by Fang and Li, by decreasing the search domain. The topic of the linear optimization problem was also investigated with max-product operation [20, 31]. Loetamonphong and Fang defined two sub-problems by separating negative and non-negative coefficients in the objective function and then obtained the optimal solution by combining those of the two sub-problems [31]. Also, in [20] some necessary conditions of the feasibility and simplification techniques were presented for solving FRE with max-product composition. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [46] and max-t-norm composition [15, 16, 21, 27, 41].

Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas [6, 10, 15, 16, 18, 24, 30, 47]. For example, Dempe and Ruziyeva [4] generalized the fuzzy linear optimization problem by considering fuzzy coefficients.

The optimization problem subjected to various versions of FRI could be found in the literature as well [1214, 17, 18, 49, 50]. Xiao et al. [50] introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities. Ghodousian et al. [12] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition.

The remainder of the paper is organized as follows. In section 2, basic properties of the feasible solutions set of (1) has been attained. It is shown that this region is determined as a union of the finite number of minimal solutions and a unique maximum solution. In section 3, an algorithm is presented to find all the optimal solutions for Problem (1). Finally, in section 4 an example is described to illustrate the algorithm. Moreover, a comparison is made between the feasible regions of product and geometric mean FREs.

2 Feasible region of Problem (1)

For each $i \in I$, define $S(a_i, b_i) = \{x \in [0,1]^n : \max_{j \in J} \sqrt{a_{ij} x_j} = b_i\}$. In other words, $S(a_i, b_i)$ denotes all the solutions satisfying the ith equation of (1). Based on this definition, the following lemma trivially holds true.

Lemma 1. Let $x \in [0,1]^n$. Then, $x \in S(a_i, b_i)$ iff the following two statements hold:
(a) For each $j \in J,\ x_j \le \min\{b_i^2/a_{ij}, 1\}$ (if $a_{ij} = b_i = 0,\ x_j \in [0,1]$); (b) There exist some $j_0 \in J$ such that $a_{ij_0} \ge b_i^2$ and $x_{j_0} = b_i^2/a_{ij_0}$ (if $a_{ij_0} = b_i = 0$, $x_{j_0} \in [0,1]$).

Remark 1. Suppose that $x \in S(a_i, b_i)$ and $a_{ij_0} < b_i^2$ for some $i \in I$ and $j_0 \in J$. According to Lemma 1, $x_{j_0} \leq 1$.

Corollary 1. Let $a_{ij_0} \geq b_i^2$, for some $i \in I$ and $j_0 \in J$. Also, suppose that $x' \in [0,1]^n$ such that $x'_{j_0} = b_i^2/a_{ij_0}$ (if $a_{ij_0} = b_i = 0,\ x'_{j_0} \in [0,1]$) and $x'_j = 0,\ \forall j \in J - \{j_0\}$. Then, $x' \in S(a_i, b_i)$.

Proof. Based on the assumptions, we have

$$\max_{j\in J}\{\sqrt{a_{ij}x'_j}\} = \max\{\max_{j\in J-\{j_0\}}\{\sqrt{a_{ij}x'_j}\}, \sqrt{a_{ij_0}x'_{j_0}}\} = \max\{0, \sqrt{a_{ij_0}x'_{j_0}}\} = \sqrt{a_{ij_0}x'_{j_0}} = b_i$$

that means $x' \in S(a_i, b_i)$. □

Corollary 2. For each fixed $i \in I$, let $J_i = \{j \in J : a_{ij} \geq b_i^2\}$. Then, $S(a_i, b_i) \neq \varnothing$ iff $J_i \neq \varnothing$.

Proof. Assume that $J_i \neq \varnothing$ and $j_0 \in J_i$. So, define $x' \in [0,1]^n$ such that $x'_{j_0} = b_i^2/a_{ij_0}$ and $x'_j = 0,\ \forall j \in J - \{j_0\}$. Now, from Corollary 1 we have $x' \in S(a_i, b_i)$. Conversely, let $S(a_i, b_i) \neq \varnothing$ and $x' \in S(a_i, b_i)$. Thus, according to Lemma 1 (b), there exist some $j_0 \in J$ such that $x'_{j_0} = b_i^2/a_{ij_0}$. Since $x' \in [0,1]^n$, therefore $x'_{j_0} = b_i^2/a_{ij_0} \leq 1$ that implies $j_0 \in J_i$. □

Definition 1. Suppose that $S(a_i, b_i) \neq \varnothing$. We define $\overline{X}(i) \in [0,1]^n$ such that

$$\overline{X}(i)_j = \begin{cases} b_i^2/a_{ij} & , a_{ij} \geq b_i^2 \\ 1 & , a_{ij} < b_i^2 \end{cases} \qquad , j \in J$$

where $\overline{X}(i)_j = 1$, if $a_{ij} = b_i = 0$.

Theorem 1. Suppose that $S(a_i, b_i) \neq \varnothing$. Then, $\overline{X}(i)$ is the maximum solution of $S(a_i, b_i)$.

Proof. Based on Lemma 1, $\overline{X}(i) \in S(a_i, b_i)$. Suppose that $x' \in S(a_i, b_i)$. So, from Lemma 1, $x'_j \leq b_i^2/a_{ij},\ \forall j \in J$. Therefore, $x'_j \leq b_i^2/a_{ij} = \overline{X}(i)_j,\ \forall j \in J_i$ and $x'_j \leq 1 = \overline{X}(i)_j$, $\forall j \in J - J_i$ (see Remark 1). Thus, $x'_j \leq \overline{X}(i)_j,\ \forall j \in J$. □

Definition 2. Let $i \in I$ and $S(a_i, b_i) \neq \varnothing$. For each $j \in J_i$, define $\underline{X}(i,j) \in [0.1]^n$ such that

$$\underline{X}(i,j)_k = \begin{cases} b_i^2/a_{ij} & , k = j \\ 0 & , otherwise \end{cases}$$

where $\underline{X}(i,j)_j = 0$, if $a_{ij} = b_i = 0$.

Remark 2. Suppose that $S(a_i, b_i) \neq \varnothing,\ j \in J_i$ and $b_i \neq 0$.Then, from Definitions 1 and 2 we have $\overline{X}(i)_j = \underline{X}(i,j)_j$.

Theorem 2. Suppose that $S(a_i, b_i) \neq \varnothing,\ j_0 \in J_i$ and $b_i \neq 0$. Then, $\underline{X}(i, j_0)$ is a minimal solution of $S(a_i, b_i)$.

Proof. From Corollary 1, $\underline{X}(i, j_0) \in S(a_i, b_i)$. Suppose that $x' \in S(a_i, b_i)$, $x' \leq \underline{X}(i, j_0)$ and $x' \neq \underline{X}(i, j_0)$. So, $x'_j \leq \underline{X}(i, j_0)_j$, $\forall j \in J$ and $x' \neq \underline{X}(i, j_0)$. Therefore, $x'_j = 0$, $\forall j \in J - \{j_0\}$ and $x'_{j_0} < b_i^2/a_{ij_0}$. However, in this case we will have

$$\max_{j\in J}\{\sqrt{a_{ij}x'_j}\} = \max\{\max_{j\in J-\{j_0\}}\{\sqrt{a_{ij}x'_j}\}, \sqrt{a_{ij_0}x'_{j_0}}\} = \sqrt{a_{ij_0}x'_{j_0}} < b_i$$

that contradicts $x' \in S(a_i, b_i)$. □

Corollary 3. Suppose that $S(a_i, b_i) \neq \varnothing,\ j_0 \in J_i$ and $b_i = 0$. Then, zero vector 0 is the unique minimum solution of $S(a_i, b_i)$.

Corollary 4. Let $x' \in S(a_i, b_i)$. There exists some $j_0 \in J_i$ such that $\underline{X}(i, j_0) \leq x'$.

Proof. Since $x' \in S(a_i, b_i)$, there exists at least some $j_0 \in J_i$ such that $x'_{j_0} = b_i^2/a_{ij_0}$ (Lemma 1). So, according to Definition 2, we have $\underline{X}(i, j_0) \leq x'$. □

Corollary 5. $S(a_i, b_i) = \bigcup_{j\in J_i} [\underline{X}(i, j), \overline{X}(i)]$

Proof. let $x' \in S(a_i, b_i)$. From Theorem 1, $x' \leq \overline{X}(i)$. On the other hand, from Corollary 4, there exists some $j_0 \in J_i$ such that $\underline{X}(i, j_0) \leq x'$. Therefore, $x' \in [\underline{X}(i, j_0), \overline{X}(i)]$ that means $x' \in \bigcup_{j\in J_i} [\underline{X}(i, j), \overline{X}(i)]$. Conversely, let $x' \in \bigcup_{j\in J_i} [\underline{X}(i, j), \overline{X}(i)]$. So, there exists some $j_0 \in J_i$ such that $x' \in [\underline{X}(i, j_0), \overline{X}(i)]$. Thus, for each $j \in J$, $x'_j \leq \overline{X}(i)_j$ and therefore, from Definition 2 we conclude that $x_j \leq \min\{b_i^2/a_{ij}, 1\}$, $\forall j \in J$ (*). On the other hand, since $\underline{X}(i, j_0)_{j_0} \leq x'_{j_0} \leq \overline{X}(i)_{j0}$ and $\underline{X}(i, j_0)_{j_0} = \overline{X}(i)_{j_0}$ (see Remark 2), then $x'_{j_0} = \underline{X}(i, j_0)_{j_0} = \overline{X}(i)_{j_0} = b_i^2/a_{ij_0}$ (**). Hence, (*), (**) and Lemma 1 necessitate that $x' \in S(a_i, b_i)$. □

Definition 3. Let $\overline{X}(i)$ be as in Definition 1, $\forall i \in I$. We define $\overline{X} = \min_{i\in I}\{\overline{X}(i)\}$.

Definition 4. Let $e : I \to \bigcup_{i\in I} J_i$ so that $e(i) \in J_i,\ \forall i \in I$, and let E be the set of all vectors e . For the sake of convenience, we represent each $e \in E$ as an $m$dimensional vector $e = [j_1, j_2, \ldots, j_m]$ in which $j_k = e(k),\ k = 1, 2, \ldots, m$.

Definition 5. Let $e = [j_1, j_2, \ldots, j_m] \in E$. We define $\underline{X}(e) \in [0, 1]^n$ such that $\underline{X}(e)_j = \max_{i\in I}\{\underline{X}(i, e(i))_j\} = \max_{i\in I}\{\underline{X}(i, j_i)_j\},\ \forall j \in J$.

The following theorem indicates that the feasible region of problem 1 is completely found by the finite number of closed convex cells.

Theorem 3. Let $S(A,b) = \{x \in [0,1]^n : A \odot x = b\}$. Then, $S(A,b) = \bigcup\limits_{e\in E}[\underline{X}(e), \overline{X}]$.

Proof. Since $S(A,b) = \bigcap\limits_{i\in I} S(a_i, b_i)$, from Corollary 5 and Definition 4 we have

$$\begin{aligned} S(A,b) &= \bigcap_{i\in I}\bigcup_{j\in J_i}[\underline{X}(i,j), \overline{X}(i)] = \bigcup_{e\in E}\bigcap_{i\in I}[\underline{X}(i,e(i)), \overline{X}(i)] \\ &= \bigcup_{e\in E}[\max_{i\in I}\{\underline{X}(i,e(i))\}, \min_{i\in I}\{\overline{X}(i)\}] \\ &= \bigcup_{e\in E}[\underline{X}(e), \overline{X}] \end{aligned}$$

where the last equality is obtained from Definitions 3 and 5. □

Corollary 6. $S(A,b) \neq \varnothing$ iff $\overline{X} \in S(a,b)$.

3 Resolution of Problem (1)

It is easy to prove that $\overline{X}$ is the optimal solution for min $\{Z_1 = \sum_{j=1}^n c_j^- x_j : \ A \odot x = b, x \in [0,1]^n\}$, and the optimal solution for $\min\{Z_2 = \sum_{j=1}^n c_j^+ x_j : \ A \odot x = b, \ x \in [0,1]^n\}$ is $\underline{X}(e^*)$ for some $e^* \in E$, where $c_j^+ = \max\{c_j, 0\}$ and $c_j^- = \min\{c_j, 0\}$ for $j = 1, 2, \ldots, n$ [9, 13, 19, 28]. According to the foregoing results, the following theorem shows that the optimal solution of Problem (1) can be obtained by the combination of $\overline{X}$ and $\underline{X}(e^*)$.

Theorem 4. Suppose that $S(A,b) \neq \varnothing$, and $\overline{X}$ and $\underline{X}(e^*)$ are the optimal solutions of sub-problems Z_1 and Z_2 , respectively. Then, $c^T x^*$ is the lower bound of the optimal objective function in (1), where $x^* \in [0,1]^n$ is defined as follows:

$$x_j^* = \begin{cases} \overline{X}_j & c_j < 0 \\ \underline{X}(e^*)_j & c_j \geq 0 \end{cases} \qquad (3)$$

for $j = 1, 2, \ldots, n$.

Proof. Let $x \in S(A,b)$. Then, from Theorem 3 we have $x \in \bigcup_{e\in E}[\underline{X}(e), \overline{X}]$. Therefore, for each $j \in J$ such that $c_j \geq 0$, inequality $x_j^* \leq x_j$ implies $c_j^+ x_j^* \leq c_j^+ x_j$. In addition, for each $j \in J$ such that $c_j < 0$, inequality $x_j^* \geq x_j$ implies $c_j^- x_j^* \leq c_j^- x_j$. Hence, $\sum_{j=1}^n c_j x_j^* \leq \sum_{j=1}^n c_j x_j$. □

Corollary 7. Suppose that $S(A,b) \neq \varnothing$. Then, x^* as defined in (3), is the optimal solution of problem (1).

Proof. According to the definition of vector x^*, we have $\underline{X}(e^*)_j \leq x^*_j \leq \overline{X}_j$, $\forall j \in J$, which implies $x^* \in \bigcup_{e \in E} [\underline{X}(e), \overline{X}] = S(A,b)$. □

The following theorem shows the difference between the feasible solution sets of $\{x \in [0,1]^n : \ A \circ x = b\}$ and $\{x \in [0,1]^n : \ A \odot x = b\}$, where $\circ$ and $\odot$ denote product t-norm and geometric operator, respectively.

Theorem 5. Let $\overline{X}_P$ and $\overline{X}_G$ be the maximum solutions of $\{x \in [0,1]^n : \ A \circ x = b\}$ and $\{x \in [0,1]^n : \ A \odot x = b\}$, respectively. Then, $\overline{X}_G \leq \overline{X}_P$.

Proof. From Definitions 1 and 3, for each $j \in J$ we have $(\overline{X}_G)_j = \min_{i \in I}\{\overline{X}(i)_j\}$ where $\overline{X}(i)_j = b_i^2/a_{ij}$ if $a_{ij} \geq b_i^2$ and $\overline{X}(i)_j = 1$ if $a_{ij} < b_i^2$. In a similar way, $(\overline{X}_P)_j = \min_{i \in I}\{\overline{X}_P(i)_j\}$ where $\overline{X}_P(i)_j = b_i/a_{ij}$ if $a_{ij} \geq b_i$ and $\overline{X}_P(i)_j = 1$ if $a_{ij} < b_i$. Now, let $j \in J$. If $a_{ij} \geq b_i$ (and hence, $a_{ij} \geq b_i^2$), then $\overline{X}(i)_j = b_i^2/a_{ij} \leq b_i/a_{ij} = \overline{X}_P(i)_j$. In other case, if $a_{ij} < b_i$ and $a_{ij} < b_i^2$, then $\overline{X}_P(i)_j = \overline{X}(i)_j = 1$. Finally, if $a_{ij} < b_i$ and $a_{ij} \geq b_i^2$, then $\overline{X}(i)_j = b_i^2/a_{ij} \leq 1 = \overline{X}_P(i)_j$. Thus, $\overline{X}(i)_j \leq \overline{X}_P(i)_j$, $\forall i \in I$, which means $\overline{X}_G \leq \overline{X}_P$. □

We now summarize the preceding discussion as an algorithm.

Algorithm 1 (optimization of problem (1))

Given problem (1):

1. Compute $J_i = \{j \in J : \ a_{ij} \geq b_i\}$ for each $i \in I$.
2. If $J_i = \varnothing$ for some $i \in I$, then stop; $S(a_i, b_i)$ (and therefore $S(A,b)$) is empty (Corollary 2).
3. Compute $\overline{X}(i)$ for each $i \in I$ (Definition 1).
4. Compute $\overline{X}$ (Definition 3).
5. If $\overline{X} \notin S(A,b)$, then stop; $S(A,b)$ is empty (Corollary 6).
6. Find solutions $\underline{X}(e)$, $\forall e \in E$ (Definition 5).
7. By pairwise comparison, find the minimal solutions between all $\underline{X}(e)$, $\forall e \in E$.
8. Find the optimal solution $\underline{X}(e^*)$ for the sub-problem Z_2.
9. Find the optimal solution x^* for the problem (1) by (3) (Theorem 4).

4 Numerical example

Example 1. Consider the following linear optimization problem (1):

$$\min Z = 2.2137x_1 + 4.0759x_2 - 2.3332x_3 + 4.5737x_4 + 7.7457x_5$$

$$\begin{bmatrix} 0.0867 & 0.1033 & 0.4944 & 0.8909 & 0.7440 \\ 0.1361 & 0.1200 & 0.1345 & 0.9777 & 0.5085 \\ 0.7667 & 0.4271 & 0.7446 & 0.9593 & 0.2505 \\ 0.7075 & 0.3601 & 0.8795 & 0.5472 & 0.5059 \\ 0.2276 & 0.8096 & 0.3275 & 0.1386 & 0.6991 \end{bmatrix} \odot x = \begin{bmatrix} 0.7000 \\ 0.6675 \\ 0.8039 \\ 0.8422 \\ 0.8687 \end{bmatrix}$$

$$x \in [0,1]^5$$

Step 1: In this example, we have $J_1 = \{3, 4, 5\}$, $J_2 = \{4, 5\}$, $J_3 = \{1, 3, 4\}$, $J_4 = \{3\}$ and $J_5 = \{2\}$.

Step 2: Since $J_i \neq \varnothing$ for each $i \in I$, we continue the algorithm.

Step 3: By Definition 1, we have $\overline{X}(1) = [1, 1, 0.9907, 0.5498, 0.6583]$, $\overline{X}(2) = [1, 1, 1, 0.4558, 0.8763]$, $\overline{X}(3) = [0.8430, 1, 0.8680, 0.6737, 1]$, $\overline{X}(4) = [1, 1, 0.8066, 1, 1]$ and $\overline{X}(5) = [1, 0.9322, 1, 1, 1]$.

Step 4: From Definition 3, $\overline{X} = [0.8430, 0.9322, 0.8066, 0.4558, 0.6583]$.

Step 5: Since $\overline{X} \in S(A, b)$, set $S(A, b)$ is feasible.

Step 6 and 7: Note that $|E| = 18$, that is, there are 18 solutions $\underline{X}(e)$ that may be minimal solutions of the feasible region. By pairwise comparison, it turns out that the feasible region has only one minimal solution. This unique minimal solution is generated by $e = [5, 4, 1, 3, 2]$ as follows:

$$\underline{X}(e) = [0.8430, 0.9322, 0.8066, 0.4558, 0.6583]$$

Step 8: Vector $\underline{X}(e^*) = \underline{X}(e)$ is the optimal solution of the sub-problem Z_2 that is obtained by $e^* = e$.

Step 9: The optimal solution of Problem (1) is resulted as $x^* = [0.8430, 0.9322, 0.8066, 0.4558, 0.6583]$ with optimal objective value $Z^* = 10.9675$.

Conclusion

In this paper, we proposed an algorithm to solve the linear programming subjected to the fuzzy relational equalities defined by geometric operator. Based on the structural properties of geometric operator, the feasible solutions set was completely determined. It was shown that the feasible can be write by the unique maximum solution and a finite number of minimal solutions. Based on the foregoing results, an algorithm was presented to find the optimal solution of the problem. As future works, we aim at testing our algorithm in other type of fuzzy systems and linear optimization problems whose constraints are defined as FRE with other averaging operators.

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