# On the J-Tightness of Graphs

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**Abstract**

We introduce a new invariant vulnerability parameter named J-Tightness or J(G) for graphs. As a stability measure, its properties along with comparisons to other parameters of a graph are proposed. We show how it is calculated for complete graphs and cycles. We show that J-Tightness better fits the properties of vulnerability measures and can be used with more confidence to assess the vulnerability of any classes of graphs.

**Keywords:** Network Vulnerability, Connectivity, Binding Number, Scattering Number, Rapture Degree, Integrity, Toughness, Tenacity, J-Tightness

AMS Subject Classification: 05C78

## 1 Introduction

One way of measuring the stability of a communication network (connections, communications, or paths) is through the ease (or the cost) with which one can disrupt the network. A lot of work has been done regarding the proposition of new vulnerability measurement parameters and the relationship between these parameters as well as the original properties of graphs [55]. The very first efforts on vulnerability parameters of graphs go back to the 1960s [22]. All of these parameters act as functions whose inputs are graphs and outputs are real numbers. In other words, these parameters map graph samples to real numbers. With this operation, the graphs are comparable in terms of vulnerability with the comparison of mapped numbers. All of these parameters seek to find the Achilles heel or the weakest and most vulnerable point of the graph. Therefore, the function that is defined for these vulnerability parameters is often a minimization/maximization function. The most important parameters that have been proposed to measure vulnerability so far are Connectivity, Binding Number, Scattering Number, Rapture Degree, Integrity, Toughness, and Tenacity.

Any of these parameters have approached the vulnerability problems from a certain point of view considering different characteristics of graphs. Some consider the integrity, toughness, tenacity, and binding number as members of a class of vulnerability parameters of a graph that are often used to study network reliability [59]. The term reliability intends to positively consider the properties of graphs, while vulnerability tries to show weakness. However, in some other studies, these parameters have fallen into three categories including cutting, covering, and closeness perspectives [7]. These three categories mostly highlight the approaches to which the graphs are analyzed. In the following sections, some of these parameters are illustrated. Parameters should behave in the case of different graphs so that if logically one graph is weaker than another graph, the vulnerability calculated by that parameter for the first graph is smaller than the second one. Despite all these parameters, can we introduce another parameter that behave more logically? In this paper, a new parameter for calculating graph and network vulnerabilities will be presented and we will discuss the criteria required for a good vulnerability parameter.

Throughout the paper, we use Bondy and Murty [23] for terminology and notation. For a graph G, by $\omega(G)$ we denote the number of components of $G$, and $\tau(G)$ the order of the largest component of $G$. We shall use $\lfloor x \rfloor$ for the largest integer not larger than $x$ and $\lceil x \rceil$ the smallest integer not smaller than $x$. The organization of the rest of this paper is as follows. Section 2 provides an overview of the popular vulnerability parameters, Section 3 covers the characteristics of any desired vulnerability parameter. In Section 4 the J-Tightness of graphs is introduced. The subsequent theorems and lemmas elaborated in Section 5, and Section 6 concludes the paper.

# 2 Vulnerability Parameters

Many studies have proposed various vulnerability parameters have to assess the vulnerability of graphs and networks so far. In this section, we will study these parameters and some of their properties. As stated in the previous section, depending on how we approach the vulnerability problem, these parameters fall into three categories, including cutting, covering, and closeness [7].

## 2.1 Connectivity

Beineke and Harary [22] first proposed the connectivity parameter based on Mengers theorem [1, 33, 48]. The connectivity of a non-complete graph $G$ is defined by:

$$\kappa(G) = \min\{|F| : F \subset V(G), \omega(G - F) > 1\} \tag{1}$$

and that of the complete graph $K_n$ is defined as $n - 1$.

The connectivity gives the minimum cost to disrupt the network, but it does not consider the remaining components after a disruption. One can say that this disruption is further successful if the network contains more disconnected components and much more successful if, in addition, these components are small. A review of the connectivity measure is covered in [47]. Since then, many studies have tried to provide a more detailed view of this parameter and its relationship with some other properties of graphs. In [56], Whitney has shown that $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for any graph $G$. The undirected graphs are the most common ones that are being studied since they are less complex than directed graphs. However, the directed graphs have not been totally neglected, as an example, Geller and Harary have shown that if $D$ is a digraph, then $\kappa(D) \leq \lambda(D) \leq \delta(D)$ [32]. Other detailed properties identified so far can be found in the relevant surveys in this regard including the studies in [38, 44]. In [51], algorithmic aspects of connectivity is discussed.

### 2.2 Binding Number

The binding number of graphs was first introduced in 1973 by D. R. Woodall [57]. According to Woodall, the binding number of a graph $G$ is defined by:

$$bind(G) = \min_{A \in F(G)} \left\{ \frac{N(A)}{|A|} \right\} \tag{2}$$

where $F = \{S \subseteq V(G) : S \neq \varnothing, N(S) \neq V(G)\}$ [57]. Goddard showed in [34] that the binding number $bind(G)$ for claw-free graphs $G$ is as follows: Let $G$ be a claw-free graph of order $n$. If the connectivity of $G$ is at least $\delta - 1$ and $n \neq \delta + 2$, then $bind(G) = (n-1)/(n-\delta)$. Cunningham has shown that the binding number of a graph is computable in polynomial time [30].

### 2.3 Scattering Number

The scattering number of a non-complete connected graph $G$ was first proposed by Jung [41] and is defined by:

$$s(G) = \max \{\omega(G - F) - |F| : F \subset V(G), \omega(G - F) > 1\} \tag{3}$$

Jung proposed the scattering number as "additive dual" for the concept of toughness. Several findings regarding scattering number have been discussed throughout several papers and it has been computed for several types of graphs [3, 61]. It is proved that the computing complexity of scattering number is NP-complete [60].

### 2.4 Rapture Degree

In 2005, Li et. al [45] introduced the rupture degree of a graph $G$ which is defines as:

$$r(G) = \max \{\omega(G - F) - |F| - \tau(G - F) : F \subset V(G), \omega(G - F) > 1\} \tag{4}$$

And rupture degree of $K_n$ is defined as $1 - n$. The rupture degree of special classes of graphs have been studied in [42,43,45]; for instance, he rupture degree of the star $K_{1,n}$ $(n \geq 3)$ is $n-3$. If $G_1$ and $G_2$ are two connected graphs of order $n_1$ and $n_2$ respectively, then $r(G_1+G_2) = max\{r(G_1)-n_2, r(G_2)-n_1\}$. If $G_n$ is a gear graph, then $r(G_n) = 0$. The rupture degree of graphs has been proved to be NP-Complete [43].

### 2.5 Integrity

The integrity of a graph was first introduced by Barefoot et. al. [8,9] as:

$$I(G) = \min_{F \subset V(G)} \{|F| + \tau(G-F)\} \tag{5}$$

In [6] the authors have covered several relationships regarding the integrity of a graph that has been identified. For instance, the integrity of a complete graph $K_p$ is $p$; the integrity of any complete multipartite graph of order $p$ and largest partite set or order $r$ is $p-r+1$. The boundaries of integrity and the conditions under which the boundaries are reached for integrity are covered in [35]. As a result, for a graph $G$ of order $p$: $I(G) = 1$ if and only if $G$ is null; $I(G) = 2$ if and only if all nontrivial components of $G$ are edges or the only nontrivial component is a star; $I(G) = p-1$ if and only if $G$ is not complete and $G$ has girth at least 5; $I(G) = p$ if and only if $G$ is complete [35]. The computation complexity of integrity has been proved to be NP-complete in [28].

### 2.6 Toughness

Toughness is another popular vulnerability parameter of graphs which was first proposed by Chavatal in 1973 [27]. The toughness of graph $G$ is defined by:

$$t(G) = \min \left\{ \frac{|F|}{\omega(G-F)} : F \subset V(G), \omega(G-F) > 1 \right\} \tag{6}$$

Some properties of toughness are covered in [27]. Some instances are mentioned here: let $G$ and $H$ be two graphs such that $G \subset H$, then $t(G) \leq t(H)$. The toughness of some special types of graphs are calculated by Chvtal. For example, Chvtal has proved that for any complete multipartite graph $K_{m,n}$, if $m \leq n$, then $t(K_{m,n}) = m/n$. The toughness of Cartesian product of two complete graphs is $t(K_m \times K_n) = \frac{1}{2}(m+n)-1$ such that $m, n \geq 2$. Many further researched had been conducted to calculate the toughness of diversity classes of graphs [2, 10, 14, 16–18, 36, 52, 54]. The toughness of directed graphs is covered in [21] and it is denoted by $\overrightarrow{t}$. As an example, for any graph of order $p$, $\overrightarrow{t}(G) \leq \overrightarrow{\kappa}(G)$; $\overrightarrow{t}(G) \leq \overline{V}(G)/(p-\overline{V}(G))$; if $G$ is 2-edge connected, then $\overrightarrow{t} \geq 1/(p-1)$ [21]. In addition, he has discussed some important findings of graphs regarding their toughness; for instance, it is proved that every Hamiltonian graph is 1-tough. The famous conjecture for toughness is that there exists $t_0$ such that every $t_0$-tough graph is Hamiltonian [27]; he has also conjectured that $t_0$ must be 2. The Hamiltonian properties of 2-tough graphs were discussed by Bauer et. al [11] and finally, the conjecture was dismissed by Bauer et. al. in 2000 [12]. The computation complexity of toughness has been discussed in several types of research [15, 19, 20] and it has been proved to be NP-hard.

## 2.7 Tenacity

Cozzens et.al first proposed the Tenacity of a non-complete connected graph $G$ in [29] and is defined as:

$$T(G) = \min\left\{\frac{|F| + \tau(G-F)}{\omega(G-F)} : F \subset V(G), \omega(G-F) > 1\right\} \tag{7}$$

This parameter is also amongst the most popular ones and many studies have been launched to explore the properties and values for certain types of graphs [4, 5, 26, 39, 46, 49, 50, 53]. This parameter has also been proved to be NP-hard [37]. This parameter has been presented in various formats so far. The mixed Tenacity $T_m$ is defined as:

$$T_m(G) = \min\left\{\frac{|F| + \tau(G-F)}{\omega(G-F)} : F \subset E(G), \omega(G-F) > 1\right\} \tag{8}$$

The edge-analogs of these concepts are defined similarly; see [8, 9, 27, 29]. The values of the vulnerability parameters, based on the calculation logic, lay in a variety of ranges. This makes it difficult to compare the values for different graphs. In this regard, normalized toughness and normalized tenacity of graphs were introduced by Javan et. al [40] as below, respectively:

$$t_{v_N}(G) = \frac{2}{n-1} \times \min_{F\subset V(G)}\left\{\frac{|F|}{\omega(G-F)} : \omega(G-F) > 1\right\} \tag{9}$$

$$t_{e_N}(G) = \frac{2}{n-1} \times \min_{F\subset E(G)}\left\{\frac{|F|}{\omega(G-F)} : \omega(G-F) > 1\right\} \tag{10}$$

$$T_{v_N}(G) = \frac{1}{n} \times \min_{F\subset V(G)}\left\{\frac{|F| + \tau(G-F)}{\omega(G-F)} : \omega(G-F) > 1\right\} \tag{11}$$

$$T_{e_N}(G) = \frac{2}{n-1} \times \min_{F\subset E(G)}\left\{\frac{|F| + \tau(G-F)}{\omega(G-F)} : \omega(G-F) > 1\right\} \tag{12}$$

# 3 Characteristics of an efficient vulnerability Parameter

Based on the content mentioned in Section 2, it is clear that the connectivity of a graph reflects the difficulty in breaking down a network into several pieces. This invariant is often too weak since it does not consider the remaining components after the corresponding graph is disconnected. Unlike the connectivity, each of the other vulnerability measures, i.e. toughness, scattering number, integrity, tenacity, and rupture degree, reflect not only the difficulty in breaking down the network but also the damage that has been caused. Further, we can see that the tenacity and rupture degree are the two most advanced ones among these parameters when measuring the stability of networks. To compare these parameters better and overcome the controversies of advantages of each over the other, the authors in [25] have discussed some desirable characteristics of vulnerability measures. Some of these characteristics are:

- ***Comparability:*** The values given by the parameters to any graph must be comparable. It means that if the first graph seems to be less vulnerable than the second graph, the value of the vulnerability parameter for the first graph must be greater than the second graph. To prove the existence of this property mathematically, we must show that if $G \subset H$ then $F(G) \leq F(H)$.

- ***Monotonicity:*** The parameter values must be monotone (either increasing or decreasing). It means that the value of the vulnerability parameter must change from the minimum value to the maximum value with approximately equal steps.

- ***Distinguishability:*** The measure must be global enough so that its values could distinguish between the two graphs. It means that the vulnerability parameter values must be different for different graphs.

- ***Unambiguity:*** The value of the vulnerability parameter for a particular graph should not have different values by two calculation methods. For example, the $K_3$ may be considered as $C_3$, and if we calculate the value of vulnerability in two cases, we may have different values of the vulnerability parameter for one graph.

- ***Normalized:*** The values should be in a finite and bounded range of real numbers (i.e. [0,1]).

- ***Computational Complexity:*** The vulnerability parameter should be computed in polynomial time for any graph.

Connectivity, tenacity, and toughness are amongst the most popular parameters. However, their disadvantage is that they do not meet some of the characteristics mentioned above. For example, these parameters provide diverse ranges of values that make it difficult to distinguish between two graphs regarding their vulnerability (i.e. connectivity, tenacity, or toughness). Connectivity works only on the number of vertices that the graph gets disconnected by removing them, so its comparability is poor since this parameter does not consider the structure. The binding number works better than connectivity, so it should be better in evaluations, but since its relation is the number of neighbors of set $A$ divided by the cardinality of set $A$, it still seems not to have very good comparability, but it is better than connectivity. Rapture degree is better than binding number and connectivity because the number of components created as well as the largest component are both considered in the formula. The same is for integrity because it considers the largest component in the formula and thus it is better. Scattering number works better than connectivity because it takes into account the number of components, but because it does not relate to the largest component, it is weaker than rapture degree. Toughness works better than the rest because it uses the number of components created and this number of components is used in the denominator. The same is for tenacity. Normalized toughness and normalized tenacity are also like toughness and tenacity.

For connectivity, the harder the graph is, the better the value of Monotonicity, and therefore it performs linear concerning the hardness or size of the graph. Binding number also has a good monotonicity because it uses division in its formula and that is why we consider it as strong. Other rapture degree, Integrity, and scattering number are weaker than connectivity and do not have good monotonicity. Tenacity and toughness also perform better than other parameters because they use division in their relationship and consider the remaining components.

Connectivity performs very poorly regarding distinguishability because it is difficult to compare. Binding number, like connectivity, have poor distinguishability because they are not highly dependent on the graph structure. Distinguishability of Integrity and rapture degree are the same because they consider the largest component in their formula, better show the distinction between graphs. Scattering number, because it does not have the largest component, it performs weaker than rapture degree and integrity. Toughness and tenacity, both because use the division operator and consider the largest component, perform better distinguishability.

In connectivity we have ambiguity because its value is not defined for all graphs such as $C_3 = K_3$ and $P_2 = K_2$ i.e.. The binding number for a complete graph also has a problem, which must be defined for a complete graph in the above special cases. Rapture degree and the remaining parameters are specified for the complete graphs and their value is not infinite and they give the same values for certain cases. None of the parameters are normalized except normalized tenacity and normalized toughness. It is proved that the binding number has polynomial computational complexity and the others are all NP-hard [10,13,18,20,24,37,58]. All these comparisons are given in Table 1. However we are going to present a novel parameter i.e. J-Tightness which meets these characteristics to the most.

Table 1: Comparison of vulnerability parameter characteristics. The letters S, M, and W are used for Strong, Medium, and Weak, respectively.

| | $\kappa$ | $bind$ | $r$ | $I$ | $s$ | $t$ | $T$ | $t_N$ | $T_N$ |
|---|---|---|---|---|---|---|---|---|---|
| Comparability | W | M | S | S | M | S | S | S | S |
| Monotonicity | S | S | M | M | M | S | S | S | S |
| Distinguishibility | W | W | M | M | W | M | S | M | S |
| Unambiguity | No | No | Yes | Yes | Yes | No | No | Yes | Yes |
| Normalized | No | No | No | No | No | No | No | Yes | Yes |
| Complexity | NP-Hard | P | NP-Hard | NP-Hard | NP-Hard | NP-Hard | NP-Hard | NP-Hard | NP-Hard |

# 4 The J-Tightness of Graphs

The purpose of the network is to establish communication between the nodes. Therefore, we can investigate the network vulnerability based on the amount of communication lost in the network after manipulation. In this section, we introduce J-Tightness and Edge-J-Tightness, and explain how their values can be calculated for several types of graphs. As the J-Tightness of a graph, one must assess the vulnerability of a graph to disruption by losing nodes or edges while taking into account all possible connections (edges) as well as the remaining components. To distinguish this parameter from the Tightness in set theory, the name J-Tightness has been used for this graph parameter. The J-Tightness and Edge-J-Tightness of graph $G$ are defined respectivley by:

$$J(G) = \min_{F \subseteq V(G)} \left\{ \frac{2 \times \varepsilon \times |F|}{\nu \times \left( \binom{\nu}{2} - \sum_{i=1}^{\omega(G-F)} \binom{c_i}{2} \right)} \right\} \tag{13}$$

$$J_e(G) = \min_{F \subseteq E(G)} \left\{ \frac{2 \times \varepsilon \times |F|}{\nu \times (\nu - 1) \times \left( \binom{\nu}{2} - \sum_{i=1}^{\omega(G-F)} \binom{c_i}{2} \right)} \right\} \tag{14}$$

where:

- $\nu = |V(G)|$ and $\varepsilon = |E(G)|$
- $c_i$ is the number of vertices in $i^{th}$ component of $G - F$
- $F$ is the cutset (edges/vertices)
- $\binom{\nu}{2} - \sum_{i=1}^{\omega(G-F)} \binom{c_i}{2}$ is the number of removed paths in $G - F$

# 5 Theorems and Lemmas

We prove some basic theorems and lemmas about J-Tightness and Edge-J-Tightness. These proofs will help to examine the properties of the parameters mentioned in Section 3.

**Lemma 1.** *If $G$ is a spanning subgraph of $H$, then $J(H) \geq J(G)$, $J_e(H) \geq J_e(G)$.*

*Proof.* Let $G$ be a spanning subgraph of $H$, and $F$ is a subset of $V(H)/E(H)$ which achieves $J(H)/J_e(H)$ respectively, then we have

$$\sum_{i=1}^{\omega(G-F)} \binom{c_i}{2} \leq \sum_{i=1}^{\omega(H-F)} \binom{c_i}{2} \tag{15}$$

thus

$$\binom{\nu}{2} - \sum_{i=1}^{\omega(G-F)} \binom{c_i}{2} \geq \binom{\nu}{2} - \sum_{i=1}^{\omega(H-F)} \binom{c_i}{2} \tag{16}$$

therefore

$$J(H) \geq J(G), J_e(H) \geq J_e(G). \tag{17}$$

□

The comparability of the J-Tightness and Edge-J-Tightness parameters can be deduced from lemma 1.

**Proposition 1.** *If $G$ is an empty graph with $|V(G)| = \nu > 1$, then $J(G) = 0$ and $J_e(G) = 0$.*

**Proposition 2.** *If $G$ is a complete graph, then $J(G) = 1$ and $J_e(G) = 1$.*

Proposition 1 and 2 together with lemma 1 indicate that J-Tightness and Edge-J-Tightness are normalized.

**Lemma 2.** *Suppose $c_i$ is the number of vertices of the $i^{th}$ component of graph $C_n$ by deleting $k$ edge. With constant $|f| = k$, the maximum value for the denominator of definition 14 is obtained when the difference in the number of vertices of the two components is not more than one, or in other words $\forall i, j$ : $|c_i - c_j| \leq 1$.*

*Proof.* For each pair $p$ and $q$ from created components of the graph $C_n$ by removing $k$, we have:

$$\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_i c_j = \left( \sum_{\substack{i=1 \\ i\neq p \\ i\neq q}}^{k-1} \sum_{\substack{j=i+1 \\ j\neq p \\ j\neq q}}^{k} c_i c_j \right) + \left( c_p \sum_{\substack{i=1 \\ i\neq p \\ i\neq q}}^{k} c_i \right) + \left( c_q \sum_{\substack{j=1 \\ j\neq p \\ j\neq q}}^{k} c_j \right) + c_p c_q \tag{18}$$

Suppose

$$S = \sum_{\substack{i=1 \\ i\neq p \\ i\neq q}}^{k} c_i, \qquad A = \sum_{\substack{i=1 \\ i\neq p \\ i\neq q}}^{k-1} \sum_{\substack{j=1 \\ j\neq p \\ j\neq q}}^{k} c_i c_j \tag{19}$$

Then the relationship will be simplified as follows:

$$\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} c_i c_j = A + Sc_p + Sc_q + c_p c_q = A + S(c_p + c_q) + c_p c_q \tag{20}$$

The values of $A$, $S$, and $c_p + c_q$ are constant. Thus the maximum value for the above relationship occurs when the $c_p c_q$ is maximal. Therefore, the quantities of $c_p$ and $c_q$ in discrete scale had to be as close as possible, in other words, the difference in the number of vertices in each component pair is at most equal to one. So, the maximum value for the above relationship happens when we divide the vertices equal between components as far as possible. □

**Theorem 1.** *For any Cycle graph with n vertices,*

$$J_e(C_n) = \frac{4}{(n-1)\times\left(\binom{n}{2}-\binom{\lfloor\frac{n}{2}\rfloor}{2}-\binom{\lceil\frac{n}{2}\rceil}{2}\right)}. \tag{21}$$

*Proof.* It can be easily verified that by deleting $k$ edges from the graph $C_n$ with $k>1$, the graph is divided into $k$ components. In other words, $\omega(G-F)=k$, we consider the definition 14 for the $k$ constant value. Using lemma 2, we will examine Edge-J-Tightness for cycle graphs. Considering the $k$ component of the graph $G$ after deleting the $k$ edges, assume that $r$ be the divide remaining $n$ vertices to the $k$ components. In this case we will have:

$$r = mod(n,k), \quad x = \lfloor\frac{n}{k}\rfloor \Rightarrow r = n-kx \tag{22}$$

To minimize the amount of Edge-J-Tightness, the relation 14 will be converted as follows:

$$J_e(G) = \frac{2}{n-1}\times\min_{2\le k\le n}\left\{\frac{k}{\binom{n}{2}-\sum_{i=1}^{r}\binom{x+1}{2}-\sum_{i=r+1}^{k}\binom{x}{2}}\right\}$$

$$= \frac{2}{n-1}\times\min_{2\le k\le n}\left\{\frac{k}{\binom{n}{2}-r\binom{x+1}{2}-(k-r)\binom{x}{2}}\right\}$$

$$= \frac{2}{n-1}\times\min_{2\le k\le n}\left\{\frac{k}{\binom{n}{2}-(n-kx)\binom{x+1}{2}-(k-n+kx)\binom{x}{2}}\right\}$$

$$= \frac{2}{n-1}\times\min_{2\le k\le n}\left\{\frac{2k}{(n^2-n)-(n-kx)(x^2+x)-(k-n+kx)(x^2-x)}\right\}$$

$$= \frac{2}{n-1}\times\min_{2\le k\le n}\left\{\frac{2k}{n^2-n-2nx+kx^2+kx}\right\}$$

$$= \frac{2}{n-1}\times\min_{2\le k\le n}\left\{\frac{2k}{n^2-n-2n\lfloor\frac{n}{k}\rfloor+k\lfloor\frac{n}{k}\rfloor^2+k\lfloor\frac{n}{k}\rfloor}\right\}$$

By deriving above relation, in continuous form, we have

$$f(x) = \frac{2k}{n^2-n-\frac{2n^2}{k}+\frac{n^2}{k}+n} = \frac{2k^2}{kn^2-n^2}$$

$$\frac{df(k)}{dk} = \frac{4k(kn^2-n^2)-2n^2k^2}{(kn^2-n^2)^2}$$

$$\frac{df(k)}{dk} = 0$$
$$\Rightarrow 4k(kn^2 - n^2) - 2n^2k^2 = 0$$
$$\Rightarrow 2n^2k^2 - 4kn^2 = 0$$
$$\Rightarrow k^2 - 2k = 0$$
$$\Rightarrow k(k-2) = 0$$

To minimize the relationship, $k$ can take two values of 0 and 2. There is no 0 value in the range, therefore $k = 2$.
Thus, in order to obtain the Edge-J-Tightness for cycles, it is enough to divide the graph into two equal components, by removing only two edges. Therefore, the value of the Edge-J-Tightness for cycle graphs with $n$ vertices ($n \geq 3$) will be as follows:

$$J_e(C_n) = \frac{4}{(n-1) \times \left(\binom{n}{2} - \binom{\lfloor \frac{n}{2} \rfloor}{2} - \binom{\lceil \frac{n}{2} \rceil}{2}\right)}.$$

□

With the same calculation, we can have the following Corollaries.

**Corollary 1.** *If $P_n$ is a path with $n$ vertices then:*

$$J_e(P_n) = \frac{2}{n \times \left(\binom{n}{2} - \binom{\lfloor \frac{n}{2} \rfloor}{2} - \binom{\lceil \frac{n}{2} \rceil}{2}\right)}.$$

*It is shown that by removing k edges, the graph become k components maximum. In the Path, by removing k edges the graph will be converted to at most $k+1$ components. The rest of the relationships are provable as in Theorem 1.*

**Corollary 2.** *If $C_n$ is a cycle with $n$ vertices ($n > 3$) then:*

$$J(C_n) = \frac{4}{\binom{n}{2} - \binom{\lfloor \frac{n-2}{2} \rfloor}{2} - \binom{\lceil \frac{n-2}{2} \rceil}{2}}.$$

*Vertex-tightness, same as edge-tightness is minimized when we remove 2 almost opposite vertices.*

**Corollary 3.** *If $P_n$ is a path with $n$ vertices then:*

$$J(P_n) = \frac{2(n-1)}{n \times \left(\binom{n}{2} - \binom{\lfloor \frac{n-1}{2} \rfloor}{2} - \binom{\lceil \frac{n-1}{2} \rceil}{2}\right)}.$$

*By removing one vertex, the graph splits into two almost same components.*

**Corollary 4.** $J_e(P_2) = J_e(K_2) = 1$.

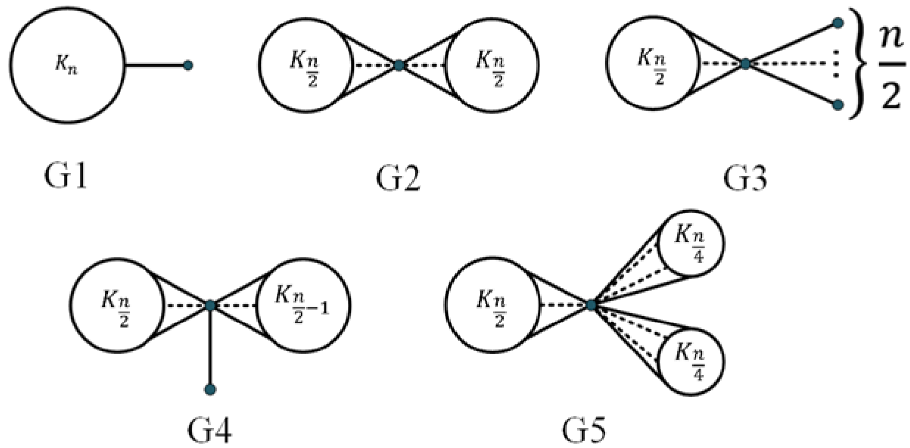


Figure 1: Five different graphs with $n+1$ vertices

**Corollary 5.** $J_e(C_3) = J_e(K_3) = 1$.

**Corollary 6.** $J(P_2) = J(K_2) = 1$.

Corollaries 4, 5, and 6 are resulted from corollaries 1, 2, and 3 by substituting $n = 2$ and $n = 3$. Corollaries 1 to 6 show that the definitions introduced for Edge-J-Tightness, unlike the Toughness and Tenacity, are not ambiguous for $P_2$, $K_2$ and $C_3$, $K_3$. J-Tightness is also unambiguous for $P_2$, $K_2$, and J-Tightness for $C_n$, $n \leq 3$ is not defined.

Table 2 shows the vulnerability of the graphs in Figure 1 based on the Connectivity, Integrity, Toughness, Tenacity, and J-Tightness parameters. All of these graphs have $n+1$ vertices. We expect a suitable parameter to compute different values for these graphs, or in other words, to distinguish between these graphs. In Table 2, Connectivity or $\kappa$ does not distinguish any of the example graphs. In other words, speaking of $\kappa = 1$ does not identify which type of graph is meant. The edge-analog of this parameter has the same problem (i.e., it does not distinguish among $G1$, $G3$, and $G4$). The Integrity parameter does not distinguish the vulnerabilities of $G2$, $G3$, $G4$, and $G5$, either. The same happens for its edge-analog that does not determine unique values for $G1$, $G2$, $G4$, and $G5$. The Tenacity does not distinguish $G4$ and $G5$. However, the Mixed-Tenacity identifies all graphs and shows better performance. According to this table, the parameter J-Tightness and edge-J-Tightness both distinguish all the graphs and meet the desired properties of vulnerability parameters.

To investigate the monotonicity property of the new parameters, we need graphs whose order of vulnerability is logically known. The authors in [40] studied this property on Harary graphs. In this article, we will use Harary graphs to examine this property in new parameters. Table 3 shows the vulnerability for several Harary graphs based on definitions 13 and 14. Figure 2 shows the growth

Table 2: The value of vulnerability parameters for the graphs shown in Figure 1

| Graph | $\kappa$ | $\kappa_e$ | $I$ | $I_e$ | $t$ | $t_e$ | $T$ | $T_m$ | $J$ | $J_e$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $G1$ | 1 | 1 | $n$ | $n+1$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{n}{2}$ | $\frac{n+1}{2}$ | $\frac{n^2-n+2}{2n^2+n-1}$ | $\frac{n^2 2+2}{n^3+n^2}$ |
| $G2$ | 1 | $\frac{n}{2}$ | $\frac{n}{2}+1$ | $n+1$ | $\frac{1}{2}$ | $\frac{n}{4}$ | $\frac{n+2}{4}$ | $\frac{n^2+2n+4}{4n+4}$ | $\frac{2n+4}{n^2+5n+4}$ | $\frac{1}{n+1}$ |
| $G3$ | 1 | 1 | $\frac{n}{2}+1$ | $n$ | $\frac{2}{n+2}$ | $\frac{1}{2}$ | 1 | $\frac{n^2+6n+8}{8n+8}$ | $\frac{2n+12}{3n^2+9n+6}$ | $\frac{n+6}{4n^2+4n}$ |
| $G4$ | 1 | 1 | $\frac{n}{2}+1$ | $n+1$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{n+2}{6}$ | $\frac{n^2+8}{4n+4}$ | $\frac{2n^2+8}{n^3+7n^2+2n-4}$ | $\frac{n^2+4}{2n^3+2n^2}$ |
| $G5$ | 1 | $\frac{n}{4}$ | $\frac{n}{2}+1$ | $n+1$ | $\frac{1}{3}$ | $\frac{n}{8}$ | $\frac{n+2}{6}$ | $\frac{3n^2+8n+16}{16n+16}$ | $\frac{6n+16}{5n^2+21n+16}$ | $\frac{3n+8}{6n^2+14n+8}$ |

trend of this rate. In this figure, it is clear that the growth of the J-Tightness is smoother than the growth of the Edge-J-Tightness. In other words, on the linear scale, the J-Tightness shows better behavior, but the Edge-J-Tightness performs better on an exponential scale. Although, both show good behavior in the term of monotonicity.

Table 3: J-Tightness and Edge-J-Tightness of some Harary graphs

| Parameter | $H_{2,6}=C_6$ | $H_{3,6}$ | $H_{4,6}$ | $H_{5,6}=K_6$ |
|---|---|---|---|---|
| $J$ | $\frac{4}{13}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | 1 |
| $J_e$ | $\frac{4}{45}$ | $\frac{3}{10}$ | $\frac{2}{5}$ | 1 |

# 6 Conclusion

In this paper, we tried to introduce a new parameter to assess the vulnerability of the graphs. According to the Theorem, lemmas, and corollaries discussed above, it is clear that the J-Tightness meets the desired characteristics of the vulnerability parameters. The computational complexity of the parameter is estimated to be NP-Complete because it seems that to calculate these parameters, we need to examine all vertex or edge combinations. However, this property should be studied further as an open problem. The comparison of J-Tightness to the connectivity, integrity, tenacity, and toughness parameters indicates that J-Tightness can be a utilizable measure for evaluating graph stability.

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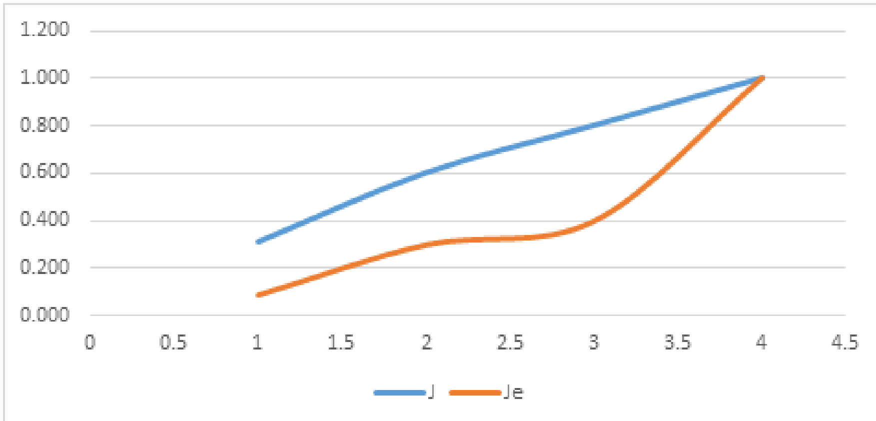


Figure 2: J-Tightness and Edge-J-Tightness of Harary graphs $H_{2,6}$ to $H_{5,6}$


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