Fuzzy Cumulative Distribution Function and its Properties

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Abstract

The statistical methods based on cumulative distribution function is a start point for many parametric or nonparametric statistical inferences. However, there are many practical problems that require dealing with observations/parameters that represent inherently imprecise. However, Hesamian and Taheri (2013) was extended a concept of fuzzy cumulative distribution function. Applying a common notion of fuzzy random variables, they extended a vague concept of fuzzy cumulative distribution function. However, the main properties of the proposed method has not yet been considered in fuzzy environment. This paper aims to extend the classical properties of the fuzzy cumulative distribution function in fuzzy environment.

keywords: Cumulative distribution function, fuzzy random variable, fuzzy parameter, ranking method, convergence, divergence to infinity.
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1 Introduction

The cumulative distribution function (**C.D.F.**) is the probability that the random variable X takes a value less than or equal to a specified value x. It plays an essential rule in many fields of statistical methods such as: Kolmogorov-Smirnov test, functional statistics, estimation of conditional quantiles and etc. In the classical version of such methods, the model information such as random variable or parameters are generally assumed to be crisp (precise) quantities. But, in the real world, such elements may be imprecisely observed or reported. Therefore, there is need to model such imprecise information and extend the conventional statistical inferences in fuzzy environment. Since the introduction of the fuzzy st theory, the fuzzy statistical methods have been successfully developed (for parametric cases, see for instance, [2, 3, 4, 5, 6, 27, 24, 30] and see [1, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 20, 23] for nonparametric cases).

The present paper aims to extend the classical properties of a well-established fuzzy cumulative distribution function based on fuzzy random variables and fuzzy parameters introduced by Hesamian and Taheri [14]. To do this, after introducing a concept of convergence/ divergence on the space of fuzzy numbers, the a classical properties of a **C.D.F.** are investigated for a fuzzy cumulative distribution function.

The rest of this paper is organized as follows: in the next section, we briefly review the classical cumulative distribution function. In addition, we recall some necessary concepts related to fuzzy numbers and fuzzy random variables. Some concept of converging or diverging a sequence of fuzzy numbers are also introduced in this section. Section 3 extends the classical properties of a cumulative distribution function for a fuzzy cumulative distribution function to fuzzy environments with fuzzy data and fuzzy parameters at a crisp as well as at a fuzzy point. Finally, a brief conclusion is provided in Section 4.

2 Some Preliminaries

2.1 Classical cumulative distribution function

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A set of probability measures $P_{\boldsymbol{\theta}}$ on $(\Omega, \mathcal{A})$ where $\boldsymbol{\theta} \in \Theta$ is said to be a parametric family if $\Theta \subset \mathbb{R}^p$ for some fixed positive integer p. The set Θ is called the p-dimensional parameter space. For a random variable $X : \Omega \to \mathbb{R}$, the cumulative distribution function of X at $x \in \mathbb{R}$ is defined as follows:

$$F_X(x) = P\{X \leq x\} = \int_{\{\omega\in\Omega | X(\omega)\leq x\}} dP(\omega).$$

It should be noted that every cumulative distribution function F_X is non-decreasing and right-continuous. Furthermore, $\lim_{x\to-\infty} F_X(x) = 0$, $\lim_{x\to\infty} F_X(x) = 1$.

2.2 Fuzzy numbers and fuzzy random variables

A fuzzy set $\widetilde{A}$ of $\mathbb{X}$ (the universal set) is defined by its membership function $\widetilde{A} : \mathbb{X} \to [0, 1]$. The set $\widetilde{A}[\alpha] := \{x \in \mathbb{X} : \widetilde{A}(x) \geq \alpha\}$ is called the α-level set (or α-cut) of the fuzzy set $\widetilde{A}$, for each $\alpha \in (0, 1]$ [29]. The set $\widetilde{A}[0]$ is also defined equal to the closure of the set $\{x \in \mathbb{X} : \widetilde{A}(x) > 0\}$. A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ (the real line) is called a fuzzy number if it is normal, i.e. there exists a unique $x^*_{\widetilde{A}} \in \mathbb{R}$ with $\widetilde{A}(x^*_{\widetilde{A}}) = 1$, and for every $\alpha \in [0, 1]$, the set $\widetilde{A}[\alpha]$ is a non-empty compact interval in $\mathbb{R}$. This interval will be denoted by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. A fuzzy number $\widetilde{A}$ is called a LR-fuzzy number if there exist real numbers a, l and r with $l, r \geq 0$, and strictly decreasing and continuous functions $L, R : [0, 1] \to [0, 1]$ such that

$$\widetilde{A}(x) = \begin{cases} L(\frac{a-x}{l}) & a - l \leq x \leq a, \\ R(\frac{x-a}{r}) & a < x \leq a + r, \\ 0 & x \in \mathbb{R} - [a - l, a + r]. \end{cases}$$

In this case $\widetilde{A}$ is denoted simply by $(a; l, r)_{LR}$. The most common used LR-fuzzy numbers in many real applications are the so-called triangular fuzzy numbers in which the shape functions L and R are given by $L(x) = R(x) = 1 - x$, for all $x \in [0, 1]$. The membership function of triangular fuzzy number, denoted by $\widetilde{A} = (a; l, r)_T$, is given by

$$\widetilde{A}(x) = \begin{cases} \frac{x-a+l}{l} & a - l \leq x \leq a, \\ \frac{a+r-x}{r} & a \leq x \leq a + r, \\ 0 & x \in \mathbb{R} - [a - l, a + r]. \end{cases}$$

A well-known ordering of fuzzy numbers, used in the sections below for investigating the properties of a fuzzy cumulative distribution function.

Definition 2.1 *[29] For $\widetilde{A}, \widetilde{B} \in \mathcal{F}(\mathbb{R})$, we say that $\widetilde{A} \preceq (\prec)\widetilde{B}$ if $\widetilde{A}^L_\alpha \leq (<)\widetilde{B}^L_\alpha$ and $\widetilde{A}^U_\alpha \leq (<)\widetilde{B}^U_\alpha$ for any $\alpha \in [0,1]$.*

Here, we also recall a common distance in space of fuzzy numbers so-called the Hausdorff distance measure [26].

Definition 2.2 *For two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, the Hausdorff distance between $\widetilde{A}$ and $\widetilde{B}$, denoted by $D_H(\widetilde{A}, \widetilde{B})$, is defined as follows*

$$D_H(\widetilde{A}, \widetilde{B}) = \sup_{\alpha \in [0,1]} \max\{|\widetilde{A}^L_\alpha - \widetilde{B}^L_\alpha|, |\widetilde{A}^U_\alpha - \widetilde{B}^U_\alpha|\}.$$

We shall use D_H to investigate the properties of the proposed fuzzy cumulative distribution function in the next section.

Definition 2.3 *Let $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy numbers.*

1. *$\{\widetilde{A}_n\}_{n\in\mathbb{N}}$ is said to converges to $\widetilde{Z} \in \mathbb{F}(\mathbb{R})$, briefly $\widetilde{A}_n \to \widetilde{Z}$, if $\lim_{n\to\infty} D_H(\widetilde{A}_n, \widetilde{Z}) = 0$.*
2. *This sequence is called increasing (respectively, decreasing) if $\widetilde{A}_n \preccurlyeq \widetilde{A}_{n+1}$ (respectively, $\widetilde{A}_{n+1} \preccurlyeq \widetilde{A}_n$), for all $n \in \mathbb{N}$. The sequence is said to diverge to infinity, denoted by $\widetilde{A}_n \to +\infty$, if for all fuzzy numbers $\widetilde{M} \in \mathbb{F}(\mathbb{R})$ there is $N \in \mathbb{N}$ such that $\widetilde{M} \preccurlyeq \widetilde{A}_n$, for all $n \geq N$. The notion $\widetilde{A}_n \to -\infty$ is defined similarly.*

Here, we recall a common definition of fuzzy random variables [19, 20, 22, 25].

Definition 2.4 *Given a probability space $(\Omega, \mathbb{A}, \mathbf{P})$, a fuzzy random variable is defined to be a Borel measurable mapping $\widetilde{X} : \Omega \to \mathbb{F}(\mathbb{R})$ such that for any $\alpha \in [0,1]$, the α-cut $\widetilde{X}[\alpha]$ is a random variable, i.e. $\widetilde{X}[\alpha] : \Omega \to \mathfrak{H}(\mathbb{R})$ ($\mathfrak{H}(\mathbb{R})$ is the class of nonempty compact intervals) [25]. It is a Borel measurable function with respect to the Borel σ-field generated by the topology associated with the (ordinary) Hausdorff metric on $\mathfrak{H}(\mathbb{R})$.*

3 Fuzzy cumulative distribution function

In this section, we first recall a common notion of fuzzy cumulative distribution function introduced Hesamian and Taheri [14].

Definition 3.1 *Let X be fuzzy random variable. The membership function of the fuzzy cumulative distribution function (**F.C.D.F.**) of $\widetilde{X}$ at $\widetilde{x} \in \mathbb{F}(\mathbb{R})$ is defined by:*

$$\mu_{\widetilde{F}_{\widetilde{X}}(\widetilde{x})}(y) = \sup\left\{\alpha \in [0,1] : y \in [F_{\widetilde{X}^U_\alpha}(\widetilde{x}^L_\alpha), F_{\widetilde{X}^L_\alpha}(\widetilde{x}^U_\alpha)]\right\}. \tag{3.1}$$

Remark 3.1 *In cases where the fuzzy point of $\widetilde{x}$ reduces to the exact value x, then the α-cuts of the **F.C.D.F.** reduce as:*

$$(\widetilde{F}_{\widetilde{X}}(x))[\alpha] = [F_{\widetilde{X}^U_\alpha}(x), F_{\widetilde{X}^L_\alpha}(x)]. \tag{3.2}$$

*In addition, if $\widetilde{X}$ reduces to the ordinary random variable X, then $(\widetilde{F}_{\widetilde{X}}(x))^L_\alpha = (\widetilde{F}_{\widetilde{X}}(x))^L_\alpha(x) = F_X(x)$ which is the classical **C.D.F.***

Remark 3.2 *According to [9, 20, 28], it should be noted that $\widetilde{X}$ can be regarded as a vague perception of an ordinary random variable X. Therefore, one may imagine that* ***F.C.D.F.*** *$\widetilde{F}_{\widetilde{X}}$ is also a vague perception of F_X. In this regard, let $\{F_{X,\underline{\theta}} : \underline{\theta} \in \Theta \subseteq \mathbb{R}^p\}$ be a family of continuous parametric* ***C.D.F.*** *Therefore, based on a set of fuzzy parameters $\underline{\widetilde{\theta}} : \Theta \to (\mathbb{F}(\mathbb{R}))^p, p \geq 1$, we can reconstruct a* ***F.C.D.F.*** *as:*

$$(\widetilde{F}_{\widetilde{X}})^L_\alpha = \inf_{\underline{\theta}\in\underline{\widetilde{\theta}}[\alpha]} F_{X,\underline{\theta}},\ (\widetilde{F}_{\widetilde{X}})^U_\alpha = \sup_{\underline{\theta}\in\underline{\widetilde{\theta}}[\alpha]} F_{X,\underline{\theta}}. \tag{3.3}$$

Example 3.1 *Let $\widetilde{X}$ be a fuzzy random variable distributed according to exponential distribution with fuzzy parameter $\widetilde{\lambda} \in \mathbb{F}(\mathbb{R})$, $supp(\widetilde{\lambda}) \subseteq (0, \infty)$. By (3.3), we then get:*

$$\mu_{\widetilde{F}_{\widetilde{X}}(x)}(y) = \sup_{\alpha\in[0,1]} \alpha I(y \in [1 - \exp(\frac{-x}{\widetilde{\lambda}^U_\alpha}), 1 - \exp(\frac{-x}{\widetilde{\lambda}^L_\alpha})]). \tag{3.4}$$

For instance, assume that $\widetilde{\lambda} = (80, 100, 120)_T$. For this case, 3-dimensional curve of ***F.C.D.F.*** *is plotted in Fig. 1.*

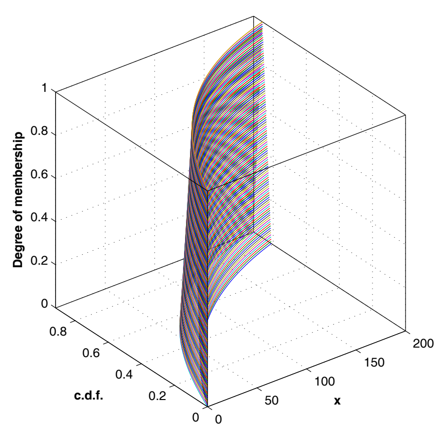

Fig. 1. F.C.D.F. in Example 3.1.

Example 3.2 *Assume that $\widetilde{X}$ distributed according to normal distribution with fuzzy $\widetilde{\mu} \in \mathcal{F}(\mathbb{R})$ and fuzzy variance $\widetilde{\sigma^2} \in \mathcal{F}((0, \infty))$. The membership degree of* ***F.C.D.F.*** *of $\widetilde{X}$ at $\widetilde{x} \in \mathcal{F}(\mathbb{R})$ can be evaluated as:*

$$\mu_{\widetilde{F}_{\widetilde{X}}(x)}(y) = \sup\{\alpha \in [0, 1] : y \in [\Phi(\frac{\widetilde{x}^L_\alpha - \widetilde{\mu}^U_\alpha}{\widetilde{\sigma}^U_\alpha}), \Phi(\frac{\widetilde{x}^U_\alpha - \widetilde{\mu}^L_\alpha}{\widetilde{\sigma}^L_\alpha})]\}. \tag{3.5}$$

For example, let $\widetilde{\mu} = (8, 10, 12)_T$ and $\widetilde{\sigma^2} = (3, 4, 5)_T$. Then, the membership function of the ***F.C.D.F.*** *at two fuzzy point $\widetilde{x}_1 = (2, 3, 5)_T$ and $\widetilde{x}_2 = (7, 10, 13)_T$ are drown in Fig. 2. In addition, the membership function of the* ***F.C.D.F.*** *at $x \in \mathbb{R}$ can be evaluated as:*

$$\mu_{\widetilde{F}_{\widetilde{X}}(x)}(y) = \sup\{\alpha \in [0, 1] : y \in [\Phi(\frac{x - \widetilde{\mu}^U_\alpha}{\widetilde{\sigma}^U_\alpha}), \Phi(\frac{x - \widetilde{\mu}^L_\alpha}{\widetilde{\sigma}^L_\alpha})]\}, \tag{3.6}$$

where Φ denotes the ***C.D.F.*** *of standard normal distribution. Moreover, for $\widetilde{\mu} = (8, 10, 12)_T$ and $\widetilde{\sigma^2} = (3, 4, 5)_T$, the 3-dimensional curve of* ***F.C.D.F.*** *is shown in Fig. 3.*

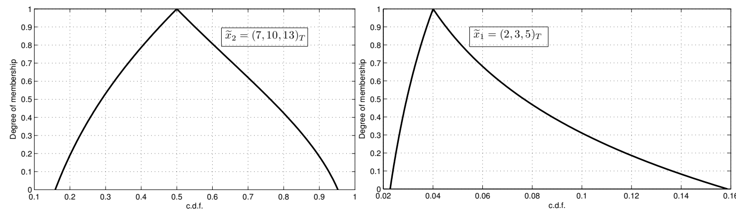

Fig. 2. F.C.D.F. at $\widetilde{x}_1 = (2, 3, 5)_T$ and $\widetilde{x}_2 = (7, 10, 13)_T$ in Example 3.2.

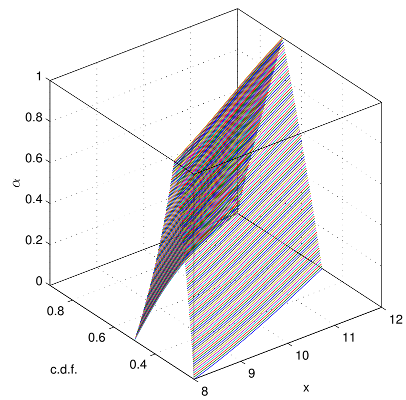

Fig. 3. F.C.D.F. in Example 3.2.

Example 3.3 *Assume that $\widetilde{X}$ follows the double exponential distribution (with **C.D.F.** of $F_X(x) = 1 - e^{-\frac{x-\theta}{\sigma}},\ x > \theta$) where $\widetilde{\theta} \in \mathcal{F}(\mathbb{R})$ and $\widetilde{\sigma} \in \mathcal{F}((0, \infty))$. Then, the membership function of **F.C.D.F.** at $x \in \mathbb{R}$ can be given as follows:*

$$\mu_{\widetilde{F}_{\widetilde{X}}(x)}(y) = \sup_{\alpha \in [0,1]} \alpha I(y \in [1 - e^{-\frac{\widetilde{x}^L_\alpha - \widetilde{\theta}^U_\alpha}{\widetilde{\sigma}^U_\alpha}}, 1 - e^{-\frac{\widetilde{x}^U_\alpha - \widetilde{\theta}^L_\alpha}{\widetilde{\sigma}^L_\alpha}}]). \tag{3.7}$$

*Let $\widetilde{\theta} = (-1, 0, +1)_T$ and $\widetilde{\sigma} = (0.25.0.50, 0.75)_T$ for instance. The membership function of **F.C.D.F.** at $\widetilde{x}_1 = (0.5, 1, 1.5)_T$ is then drown in Fig. 4. In addition, its membership function at $x \in \mathbb{R}$ can be evaluated as follows:*

$$\mu_{\widetilde{F}_{\widetilde{X}}(x)}(y) = \sup\{\alpha \in [0, 1] : y \in [1 - e^{-\frac{x - \widetilde{\theta}^U_\alpha}{\widetilde{\sigma}^U_\alpha}}, 1 - e^{-\frac{x - \widetilde{\theta}^L_\alpha}{\widetilde{\sigma}^L_\alpha}}]\}. \tag{3.8}$$

*In this case, the 3-dimensional curve of **F.C.D.F.** is shown in Fig. 5.*

It should be noted that the conventional properties of the underline **F.C.D.F.** was not investigate in fuzzy environment. Here, we are going to investigate its properties in fuzzy environment.

Proposition 3.1 *Let $\widetilde{X}$ be a fuzzy random variable on a probability space $(\Omega, \mathcal{A}, P)$. Then:*

(i) For $\widetilde{t}, \widetilde{s} \in \mathcal{F}(\mathbb{R})$ if $\widetilde{t} \preceq \widetilde{s}$ then $\widetilde{F}_{\widetilde{X}}(\widetilde{t}) \preceq \widetilde{F}_{\widetilde{X}}(\widetilde{s})$.

(ii) For $\widetilde{t} \in \mathcal{F}(\mathbb{R})$, if $\{\widetilde{t}_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of fuzzy numbers with $\widetilde{t}_n \to \widetilde{t}$ then $\widetilde{F}_{\widetilde{X}}(\widetilde{t}_n) \to \widetilde{F}_{\widetilde{X}}(\widetilde{t})$.

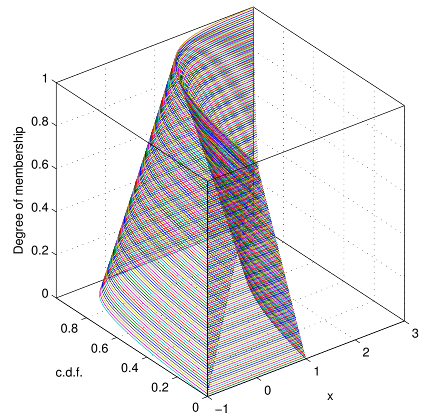

Fig. 4. F.C.D.F. in Example 3.3.

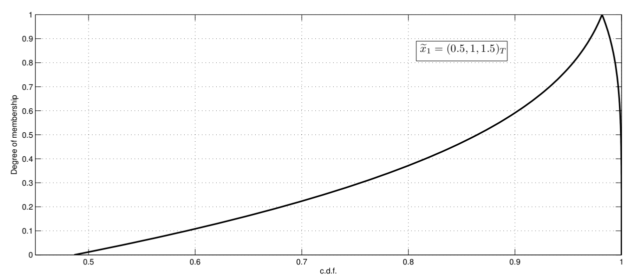

Fig. 5. F.C.D.F. at $\widetilde{x}_1 = (0.5, 1, 1.5)_T$ in Example 3.3.

(iii) If $\{\widetilde{t}_n\}_{n\in\mathbb{N}}$ is an increasing sequence of fuzzy numbers with $\widetilde{t}_n \to +\infty$ then $\widetilde{F}_{\widetilde{X}}(\widetilde{t}_n) \to I\{1\}$.

(iv) If $\{\widetilde{t}_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of fuzzy numbers with $\widetilde{t}_n \to -\infty$ then $\widetilde{F}_{\widetilde{X}}(\widetilde{t}_n) \to I\{0\}$.

where I denotes the indicator function.

Proof. Assume $\widetilde{s} \preceq \widetilde{t}$. So the proof of part (i) is clear by the monotonic property of a ordinary **C.D.F.** and the ranking method introduced in Section 2.2. To prove part (ii), let $\{\widetilde{t}_n\}_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{F}(\mathbb{R})$ which converges to $\widetilde{t}$. Then, based on the classical properties of **C.D.F.**, we have $\lim_{n\to+\infty} \widetilde{F}_{\widetilde{X}^U_\alpha}((\widetilde{t}_n)^L_\alpha) = \widetilde{F}_{\widetilde{X}^U_\alpha}(\widetilde{t}^L_\alpha)$ and $\lim_{n\to+\infty} \widetilde{F}_{\widetilde{X}^L_\alpha}((\widetilde{t}_n)^U_\alpha) = \widetilde{F}_{\widetilde{X}^L_\alpha}(\widetilde{t}^U_\alpha)$ for any $\alpha \in [0,1]$ and so $\lim_{n\to+\infty} D_H(\widetilde{F}_{\widetilde{X}}(\widetilde{t}_n), \widetilde{F}_{\widetilde{X}}(\widetilde{t})) = 0$. For part (iii), suppose $\{\widetilde{t}_n\}_{n\in\mathbb{N}}$ is an increasing sequence of fuzzy numbers with $\widetilde{t}_n \to +\infty$. According to the definition of divergence to $+\infty$, therefore for any fuzzy numbers $\widetilde{M}$ there is $N \in \mathbb{N}$ with $\widetilde{M} \preceq \widetilde{t}_n$, for all $n \geq N$. Applying the classical properties of a **C.D.F.**, for any $\alpha \in [0,1]$, it is concluded that $\lim_{n\to+\infty} \widetilde{F}_{(\widetilde{X})^U_\alpha}((\widetilde{t}_n)^L_\alpha) = \lim_{n\to+\infty} \widetilde{F}_{(\widetilde{X})^L_\alpha}((\widetilde{t}_n)^U_\alpha) = 1$ and so $\lim_{n\to+\infty} D_H(\widetilde{F}_{\widetilde{X}}(\widetilde{t}_n), I\{1\})) = 0$. Part (iv) can be proven in a similar manner.

Remark 3.3 *If fuzzy points are reduced to crisp ones, then the above proposition reduces as follows. Let $\widetilde{X}$ be a fuzzy random variable on a probability space $(\Omega, \mathcal{A}, P)$. Then:*

(i) For $t, s \in \mathbb{R}$ if $t < s$ then $\widetilde{F}_{\widetilde{X}}(t) \preceq \widetilde{F}_{\widetilde{X}}(s)$.

(ii) For $t \in \mathbb{R}$, if $\{t_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers with $t_n \to t$ then $\widetilde{F}_{\widetilde{X}}(t_n) \to \widetilde{F}_{\widetilde{X}}(t)$.

(iii) If $\{t_n\}_{n\in\mathbb{N}}$ is an increasing sequence of fuzzy numbers with $t_n \to +\infty$ then $\widetilde{F}_{\widetilde{X}}(t_n) \to I\{1\}$.

(iv) If $\{t_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers with $t_n \to -\infty$ then $\widetilde{F}_{\widetilde{X}}(t_n) \to I\{0\}$.

4 Conclusion

This paper investigated the classical properties of a common fuzzy cumulative distribution with fuzzy random variables and fuzzy parameters. For this purpose, applying a concept of convergence or divergence a sequence of fuzzy fuzzy numbers, the classical properties of a **C.D.F.** were extended at crisp or fuzzy points. Some numerical examples are provided to clarify the discussions in this paper.

References

[1] Arefi, M., Viertl, R. and Taheri, S. M., Fuzzy density estimation. Metrika, 75 (2012) 5–22.

[2] Ayyub, B. M., Gupta, M. M., Uncertainty analysis in engineering and sciences: Fuzzy logic, statistics, and neural network approach. Boston: Kluwer Academic Publishers, 1997.

[3] Bandemer, H. and Näther, W. Fuzzy Data Analysis. Kluwer Academic Publisher, Dordrecht, 1992.

[4] Bertoluzza, C., Gil, M. A. and Ralescu, D. A. Statistical Modeling, Analysis and Management of Fuzzy Data. Heidelberg, New York, 2002.

[5] Buckley, J. J. Fuzzy Statistics, Studies in Fuzziness and Soft Computing, Springer-Verlag, Berlin, 2006.

[6] Coppi, R., Gil, M.A. and Kiers, H., The fuzzy approach to statistical analysis. Computational Statatistics and Data Analysis, 51 (2006) 1-14.

[7] Denœux, T., Masson, M. H. and Herbert, P. H., Non-parametric rank-based statistics and significance tests for fuzzy data. Fuzzy Sets and Systems, 153 (2005) 1-28.

[8] Gebhardt, MÁ, J. and Gil, R. Kruse, Concepts of fuzzy-valued statistics. In: (R Slowinski, Ed.) Fuzzy Sets in Decision Analysis, Operations Research and Statistics, Kluwer, NewYork, 1998, PP. 311-347

[9] Grzegorzewski, P. Statistical inference about the median from vague data. Control and Cybernetics, 27 (1998) 447-464.

[10] Grzegorzewski, P., Distribution-free tests for vague data, In: (Lopez-Diaz M, et al (Eds.)) Soft Methodology and Random Information Systems. Springer, Heidelberg, 2004, PP. 495-502.

[11] Grzegorzewski, P., Two-sample median test for vague data. In: Proceedings of the 4th Conf European Society for Fuzzy Logic and Technology-Eusflat, Barcelona, June 2005, PP. 621-626.

[12] Grzegorzewski, P. A bi-robust test for vague data. In: Proceedings of the 12th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU2008, L Magdalena, M Ojeda-Aciego, JL Verdegay (Eds), Spain, Torremolinos (Malaga), June 2008, PP. 138-144.

[13] Grzegorzewski, P., K-sample median test for vague data. International Journal of Intelligent Systems, 24 (2009) 529-539.

[14] Hesamian, G. and Taheri, S. M., Fuzzy empirical distribution function: Properties and application. Kybernetika, 49 (2013) 962-982.

[15] Hesamian, G. and Chachi, J., Two-sample KolmogorovSmirnov fuzzy test for fuzzy random variables. Statistical Papers, 56 (2015) 61-82.

[16] Hryniewicz, O., Goodman-Kruskal γ measure of dependence for fuzzy ordered categorical data, Computational Statatistics and Data Analysis, 51 (2006) 323-334.

[17] Hung, K., Valdik, K., Wu, B. and Gang, X. Computing statistics under interval and fuzzy uncertainty. Studies in Computational Inteligence, Springer, Berlin, 2011.

[18] Kahraman, C., Bozdag, C. F. and Ruan, D. Fuzzy sets approaches to statistical parametric and non-parametric tests. International Journal of Intelligent Systems, 19 (2004) 1069-1078.

[19] Klement, E. P., Puri, M. L. and Ralescu, D. A., Limit theorems for fuzzy random variables, Mathematical Proceedings of the Cambridge Philosophical Society, London Ser. A, 407 (1986) 171–182.

[20] Kruse, R. and Meyer, K. D., Statistics with Vague Data. Reidel Publishing Company, Dordrecht, Netherlands, 1987.

[21] Lee, K. H., First Course on Fuzzy Theory and Applications, Springer-Verlag, Berlin, 2005.

[22] Li, S. and Ogura, Y., Strong laws of large numbers for independent fuzzy set-valued random variables. Fuzzy Sets and Systems, 157 (2006) 2569–2578.

[23] Lin, P., Wu, B. and Watada, J., Kolmogorov-Smirnov two sample test with continuous fuzzy data, Advances in Intelligent and Soft Computing, 68 (2010) 175-186.

[24] Nguyen, H. T. and Wu, B. Fundamentals of statistics with fuzzy data. Springer, New York, 2006.

[25] Puri, M. L. and Ralescu, D. A., Fuzzy random variables. Journal of Mathematical Analysis and Applications, 361 (1986) 409–422.

[26] Rodriguez-López, J. and Romaguera, S. The Hausdorff fuzzy metric on compact sets. Fuzzy Sets and Systems, 147 (2004) 273-283.

[27] Ross, T., Booker, J. and Parkinson, W., Fuzzy Logic and Probability Applications. Bridging the Gap, Society for Industerial and Applied Mathematics, Philadelphia, PA, 2003.

[28] Shapiro, A. F., Fuzzy random variables. Insurance: Mathematics and Economics, 44 (2009) 307–314.

[29] Wu, H. C., Statistical hypotheses testing for fuzzy data, Fuzzy Sets and Systems, 175 (2005) 30-56.

[30] Viertl, R., Statistical Methods for Fuzzy Data. Wiley, Chichester, 2011.