Plane Bounded-Degree Spanners Among the Obstacles for the Points in Convex Position

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Abstract

Let S be a set of points in the plane that are in convex position. Let $\mathcal{O}$ be a set of simple polygonal obstacles whose vertices are in S. The visibility graph $Vis(S, \mathcal{O})$ is the graph which is obtained from the complete graph of S by removing all edges intersecting some obstacle of $\mathcal{O}$. In this paper, we show that there is a plane 5.19-spanner of the visibility graph $Vis(S, \mathcal{O})$ of degree at most 6. Moreover, we show that there is a plane 1.88-spanner of the visibility graph $Vis(S, \mathcal{O})$. These improve the stretch factor and the maximum degree of the previous results by A. van Renssen and G. Wong (*Theoretical Computer Science, 2021*) in the context of points in convex position.
Keywords: Plane spanner, Stretch factor, Shortest path, Computational Geometry.

1 Introduction

A *geometric graph* is a weighted graph such that each edge (p, q) of the graph is the straight-line between p and q, and the weight of the edge (p, q) is the Euclidean distance between p and q denoted by $|pq|$. Let S be a set of points in the plane, and let G be a geometric graph with the vertex set S. Let $t > 1$ be a real number. An induced subgraph H of G is called a *t-spanner* of G if for any two points $p, q \in S$, there is a path Q between p and q in H such that $|Q| \leq t\delta_G(p, q)$, where $|Q|$ is the length of the path Q and $\delta_G(p, q)$ is the length of the shortest path between p and q in G. The path Q is called a *t-path* between p and q in H. The minimum value of the real number t such that H is a t-spanner of G is called the *stretch factor* or *dilation* of H. If the geometric graph G is the complete graph of the point set S, then the t-spanner H is called the *t-spanner of S*. To study some algorithms for constructing spanners and their applications, we refer the reader to the book [7] due to by Narasimhan and Smid.

In the real world, there may be physical obstacles between the points that make it difficult to construct a t-spanner. Therefore, it is important to know how to construct a t-spanner in the presence of obstacles for the set of points. It is notable that the obstacles can take any shape. A number of research works have been done for the obstacles in the form of line segments [3–6]. Let $\mathcal{O}$ be a set of simple polygons whose vertices belong to the point set S. Assume that each vertex of S is the corner of at most one polygon of $\mathcal{O}$. Two points $p, q \in S$ is called *visible* if the straight line pq between them does not intersect any polygons of $\mathcal{O}$. Note that it is allowed that the line pq to be in contact with a vertex or be tangent to a side of a polygon. The graph with the vertex set S whose edge set consists of all visible edges with respect to $\mathcal{O}$ is called the *visibility graph* of S with respect to $\mathcal{O}$, and is denoted by $Vis(S, \mathcal{O})$. In [8], van Renssen and Wong prove that there is a plane 2-spanner of $Vis(S, \mathcal{O})$. They states that the degree of the proposed plane 2-spanner can be unbounded. To obtain a plane bounded-degree spanner of $Vis(S, \mathcal{O})$, they show that there is a plane 6-spanner of $Vis(S, \mathcal{O})$ of degree at most 7.

Assume that the points of S are placed in convex position. In [1] and [2], the authors propose two plane spanners for points in convex position. In [1], the authors show that there is a plane 1.88-spanner $H_{1.88}$ of S. In [2], the authors show that there is a plane 5.19-spanner $H_{5.19}$ of S of degree at most 3. Note that the degree of the proposed plane spanner in [1] can be unbounded. In this paper, using the constructions of $H_{1.88}$ and $H_{5.19}$, we show that there is a plane 1.88-spanner of $Vis(S, \mathcal{O})$, and there is a plane 5.19-spanner of degree at most 6. These improves the stretch factor and the degree of the previous results by van Renssen and Wong [8] in the context of points in convex position.

2 Preliminaries

In the following, we assume that S is a set of n points in the plane that are in convex position. Let $CH(S)$ be the boundary of the convex hull of S. In the following, the notations in [1] is used. For any two points $p, q \in S$, let $\delta^{cw}_{CH(S)}(p, q)$ and $\delta^{ccw}_{CH(S)}(p, q)$ be the clockwise and counter-clockwise paths from p to q along $CH(S)$, respectively. If

$$\min\left(\left|\delta^{cw}_{CH(S)}(p, q)\right|, \left|\delta^{cw}_{CH(S)}(p, q)\right|\right) \leq t|pq|,$$

then the point p is called a *t-good* point for q. If the point p is a t-good point of all points of S, then the point p is called a *t-good* point for S. A pair (p, q) with $p, q \in S$ is called a *diametral pair* of S, if (p, q) is the farthest pair of points of S, and the points p and q are called the *diametral points*, and the *diameter* of S is equal to $|pq|$. In [1], Biniaz et al., show that any diametral point is a 1.88-good for S. Based on this property, they proposed an algorithm that constructs a plane 1.88-spanner of S. The algorithm is as follows. At first, the algorithm adds $CH(S)$ to the edge set. Then, it finds a diametral point p of S. Next, it adds the edge connecting two neighbors of p on $CH(S)$ to the edge set. Then, the algorithm recursively applies the above operations on the point set $S\backslash\{p\}$. For more details, see Algorithm 2.1. Biniaz et al., [1] show that the output of Algorithm 2.1 is a triangulation of S with the stretch factor at most 1.88. Let $H_{1.88}$ be the triangulation of S generated by Algorithm 2.1. Hence, the following theorem due to by Biniaz et al., [1] holds.

Theorem 1. *The triangulation $H_{1.88}$ generated by* PLANESPANNER(S) *is a plane 1.88-spanner of S.*

It is not hard to see that the degree of $H_{1.88}$ can be unbounded.

Algorithm 2.1: PLANESPANNER(S) ([1])

input: A finite set S of points in the plane in convex position.
output: A plane 1.88-spanner $H_{1.88}$.
E :=the edge set of $CH(S)$;
$B := S$;
while $|B| \geq 4$ **do**
  $p :=$ a diametral point in B;
  q, r :=the two neighbors of p on $CH(S)$;
  $E := E \cup \{q, r\}$;
  $B := B\backslash\{p\}$;
end
return $H_{1.88} = (S, E)$

In 2017, Biniaz et al., presented an algorithm that constructs a plane 5.19-spanner of degree at most 3 for points in convex position [2]. Their algorithm works as follows (see [2]). The algorithm adds $CH(S)$ to the spanner. Then, a farthest pair (p, q) of points of S is selected. Next, the algorithm computes two convex chains by removing p and q from $CH(S)$. Now, the algorithm adds a matching between these two chains to the edge set. The matching is recursively computed as follows. At first, the closest pair of points between the two convex chains is added to the matching. Now, this closest pair splits the two convex chains into four convex chains. Now, the algorithm recurse on both sides and adds the closest pairs to the matching (see Algorithm 2.3). Let $H_{5.19}$ be the generated graph by Algorithm 2.3. Now, the following theorem due to by Biniaz et al., [2] holds.

Theorem 2. *The geometric graph $H_{5.19}$ generated by* DEG3PLANESPANNER(S) *is a plane $\frac{3+4\pi}{3}$-spanner (or 5.19-spanner) of S of degree at most 3.*

3 Main result

In this section, we present a plane 1.88-spanner of the visibility graph $Vis(S, \mathcal{O})$. Moreover, we present a plane 5.19-spanner of the visibility graph $Vis(S, \mathcal{O})$ of degree at most 6.

Since the points of S are in convex position and $\mathcal{O}$ consists of simple polygons, each member of $\mathcal{O}$ is a convex polygon, and therefore, the point set S can be divided to some subsets of points that are in

Algorithm 2.3: DEG3PLANESPANNER(S) ([2])

input: A non-empty finite set S of points in the plane that is in convex position
output: A plane degree-3 spanner of S.
$(p, q) :=$ a farthest pair of points of S;
$C_1, C_2 :=$ the two chains obtained by removing p and q from $CH(S)$;
$E' := CH(S) \cup$ MATCHING(C_1, C_2);
return $H_{5.19} = (S, E')$;

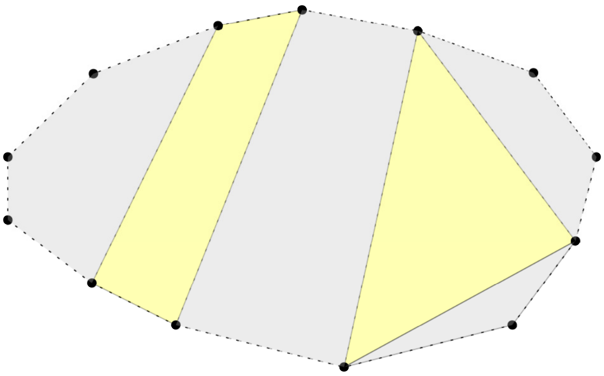

Figure 1: A point set with two obstacles in yellow color and four subsets in gray color.

convex position. Let $S_1, \ldots, S_k$ be such subsets (see Figure 1). Now, we have the following observation. The observation follows from this fact that the line segment between any two points on the convex hull of S cannot be outside the convex polygon.

Observation 3. *For any two subsets S_i and S_j, any two points p and q with $p \neq q$ and $p \in S_i$ and $q \in S_j$, are not visible unless there is point $u \in S$ such that u is a vertex of an obstacle and p, q and u are collinear.*

In Figure 2 two points x and y are not visible, and two points p and q are visible and collinear with the point u. Now, let $G_{1.88}$ be a geometric graph such that it is the union of the geometric graphs

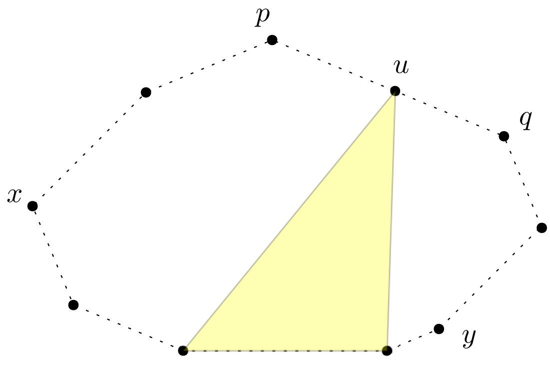

Figure 2: The yellow region is an obstacle. Two points x and y are not visible, and two points p and q are visible, and they are collinear with u.

PLANESPANNER(S_1), PLANESPANNER(S_2), …,PLANESPANNER(S_k). Let $G_{5.19}$ be a geometric graph such that it is the union of the geometric graphs DEG3PLANESPANNER(S_1), DEG3PLANESPANNER(S_2), …, DEG3PLANESPANNER(S_k). Now, we show that $G_{1.88}$ is a plane 1.88-spanner of $Vis(S, \mathcal{O})$, and we show that $G_{5.19}$ is a plane 5.19-spanner of $Vis(S, \mathcal{O})$ of degree at most 6.

Theorem 4. *The geometric graphs $G_{1.88}$ and $G_{5.19}$ are plane 1.88-spanner and plane 5.19-spanner of $Vis(S, \mathcal{O})$, respectively.*

Proof. First we prove the theorem for $G_{1.88}$. To prove the theorem for $G_{1.88}$, it is sufficient to prove that for any two visible points $p, q \in S$, there is a path between p and q in $G_{1.88}$ of length at most 1.88

times the length of the shortest path between p and q in $Vis(S, \mathcal{O})$. By Observation 3, it is sufficient to prove that for any i with $1 \leq i \leq k$, the graph generated by PlaneSpanner(S_i) is a 1.88-spanner of S_i. By Theorem 1, PlaneSpanner(S_i) is a 1.88-spanner of S_i. The proof of the theorem for $G_{5.19}$ is similar to the proof of the theorem for $G_{1.88}$ just we should apply Theorem 2. This completes the proof. □

Now, by Theorem 2, the degree of the plane spanner Deg3PlaneSpanner(S_i) is at most 3. Since every vertex of S is the corner of at most one polygon of $\mathcal{O}$, then clearly the degree of any vertex of S is at most 6. Hence, we have the following theorem.

Theorem 5. *The degree of the graph $G_{5.19}$ is at most 6.*

4 Conclusion

In this paper, we focused on the constructing of the plane spanners in the presence of obstacles for the points in convex position. We proposed two plane spanners of the visibility graph with the stretch factor at most 1.88 and 5.19. Moreover, we show that the proposed plane 5.19-spanner of the visibility graph has the degree at most 6.

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