P_3-Rainbow Edge Colouring of Digraphs

Mahdieh Hasheminezhad*

Department of Computer Science, Yazd University, Yazd, Iran
Combinatorial and Geometric Algorithms Lab, Yazd University, Yazd, Iran

Abstract

An edge coloring of a digraph D is called a P_3-rainbow edge coloring if the edges of any directed path of D with length 2 are colored with different colors. It is proved that for a P_3-rainbow edge coloring of a digraph D, at least $\lceil log_2 \chi(D) \rceil$ colors are necessary and $2 \lceil log_2 \chi(D) \rceil \}$ colors are enough. One can determine in linear time if a digraph has a P_3-rainbow edge coloring with 1 or 2 colors. In this paper, it is proved that determining that a digraph has a P_3-rainbow edge coloring with 3 colors is an NP-complete problem even for planar digraphs. Moreover, it is shown that $\lceil log_2 \chi(D) \rceil$ colors is necessary and sufficient for a P_3-rainbow edge coloring of a transitive orientation digraph D.

Keywords: `rainbow coloring`, `planar digraphs`, `template-driven rainbow coloring`, `transitive digraph`, `dichromatic index`

Introduction

Consider an edge (vertex) colored graph G. A subgraph of G is rainbow-colored if none pair of its edges (vertices) are colored with the same color. A considerable number of researches have been done on the rainbow coloring of graphs and digraphs. There are different types of problems on rainbow coloring. Some works are on rainbow connectivity [5, 11, 12, 23] and look for the minimum number of colors for coloring a graph such that for each pair of vertices x and y of the graph, there is a rainbow $x-y$-path.

Some other researches give some conditions for having some special rainbow subgraphs [3, 6, 7, 13, 15, 20, 22] and some others look for the maximum number

*Corresponding author
Email address: `hasheminezhad@yazd.ac.ir` (Mahdieh Hasheminezhad)

of colors can be used for coloring of a graph such that the graph does not have any rainbow subgraph isomorphic to some given graph [14, 18, 19, 24].

Considering a graph T, an edge (vertex) coloring of a graph G is a template-driven rainbow coloring if each copy of T in G is rainbow-colored. In this type of rainbow coloring, graph T is considered as a template, and graph G is considered as host. It is mentioned in [4] that for efficient design of parallel algorithms on multiprocessor architectures with memory banks, simultaneous access to a specified subgraph of a graph data structure by multiple processors requires that the data items belonging to the subgraph reside in distinct memory banks. Such "conflict-free" access to parallel memory systems and other applied problems motivate the study of template-driven rainbow coloring of a graph. There are some researches on template-driven rainbow coloring by considering special structures for templates and hosts. In [10], the host graph G is a q-ary tree or binomial tree, and the template T is a path or a subtree. In [1], the host graph is two-dimensional arrays, circular lists, and complete trees and the templates are paths. In [9], the host is a tori or a hypercube graph and templates are stars. In the most recent research [4], the host is an interval graph and the templates are cycles.

In this paper, a template-driven rainbow edge (vertex) coloring with graph T as the template is called a T-rainbow edge (vertex) coloring. We consider P_3 as template graph and digraphs as host. The minimum number of colors used for P_3-rainbow edge coloring of a digraph is called dichoromatic index of the digraph. It is proved that for a digraph D, dichoromatic index of D is not less than $\lceil log_2\chi(D)\rceil$ and it is not more than $2\lceil log_2\chi(D)\rceil\}$ [17]. In [25] some results are given on dichoromatic index of complete graphs, tournaments and product of graphs. In [2] the dichoromatic index of digraphs is studied where the out-degree and in-degree of the digraph is bounded.

In the first section we give some preliminaries to state the problem and results. In the second section, it is determined that how many colors is necessary and sufficient for a P_3-rainbow coloring of a transitive orientation digraph. In the third section, the digraphs with P_3 rainbow edge coloring with 2 colors are characterized and in section 4, we prove that characterizing digraphs with P_3-rainbow edge coloring with 3 colors is NP-complete even for planar graphs. Section 5 concludes the paper.

1. Preliminaries

Consider an edge (vertex) colored graph G. A subgraph of G is rainbow-colored if none pair of its edges (vertices) are colored with the same color. Subject to a graph T, an edge (vertex) coloring of a graph G is called a T-rainbow edge (vertex) coloring (T-rec(T-rvc)), if every copy of T in G is rainbow-colored. It is interesting to note that every proper edge coloring of a graph G corresponds to a $K_{1,l}$-rec ($l \leq \delta(G)$) of G and also every proper vertex coloring of a graph corresponds to a P_2-rvc of the graph.

Generalizing this concept for digraphs, an edge coloring of a digraph D is a P_3-rec, if for every directed path P of D with length 2, edges of P are colored with different colors. In other words, an edge coloring of a digraph D is a P_3-rec of D, if no vertices have an incoming edge and an outgoing edge with the same color. The minimum number of colors needed for a P_3-rec of D is called P_3-rec number of D and it is denoted with $\chi'_{\vec{P_3}}(D)$. In [17], P_3-rec number of a digraph is called dichoromatic index of the digraph. It is interesting to note that for every digraph D, $\chi'_{\vec{P_k}}(D) = \chi'_{\vec{C_k}}(D)$, for any natural number $k \geq 3$.

For a digraph D, by $\chi(D)$ and $\chi'(D)$, we mean the vertex coloring number and the edge coloring number of the underlaying graph of D. The maximum in-degree of vertices of D is denoted with $\Delta_{in}(D)$ and the maximum out-degree of vertices of D is denoted with $\Delta_{out}(D)$.

In a digraph, a vertex that has no incoming edges or has no outgoing edges is called flexible. For a digraph D, the set of all flexible vertices is denoted by $F(D)$.

Let D be a digraph with at least one edge. If each vertex of D is flexible, then D is a bipartite digraph such that all edges start from the same part. This type of digraphs is called one-way bipartite digraphs. It is interesting to note that in a P_3-rec of a digraph, each color class is an one-way bipartite digraph.

Proposition 1. *For a digraph D with at least one edge, $\chi'_{\vec{P_3}}(D) = 1$ iff D is an one-way bipartite digraph (All vertices of D are flexible).*

There are some upper bounds for P_3-rec number of a digraph D. It is easy to find out that $\chi'_{\vec{P_3}}(D) \leq min\{\chi(D), \chi'(D)\}$.

Theorem 1. *[17] For every digraph D, $\lceil log_2\chi(D)\rceil \leq \chi'_{\vec{P_3}}(D) \leq 2\lceil log_2\chi(D)\rceil$.*

For a digraph D, the clique number of the underlying graph of D is denoted by $\omega(D)$. The following theorem proves an interesting result about digraphs D that $\omega(D) = \chi(D)$.

Theorem 2. *For any digraph D that $\omega(D) = \chi(D)$, there is a 2-approximation algorithm for finding a P_3-rec coloring of D.*

Proof. There is a polynomial time algorithm for finding a coloring of D with $\chi(D)$ colors [16].

let $P = \{V_1, ..., V_{\chi(D)}\}$ be the partition of vertex set of D into color classes. For each edge (x, y), there are distinct sets V_i and V_j in P such that $x \in V_i$ and $y \in V_j$. We find the binary representation of i and j using $\lceil log_2\chi(D)\rceil$ bits and denote them by $b(i)$ and $b(j)$, respectively. Suppose that $k-1$ first bits of $b(i)$ and $b(j)$ are the same and k'th bit of $b(i)$ and $b(j)$ are different. If k'th bit of $b(i)$ is *zero* then color (x, y) by color 2k-1 and otherwise color (x, y) by color 2k. The obtaining coloring is a P_3-rec and the number of used colors is at most $2\lceil log_2\chi(D)\rceil$. By Theorem 1, the number of used colors is at most two times $\chi'_{\vec{P_3}}(D)$.

□

2. Acyclic digraphs and transitive orientation digraphs

We can find some more upper bound for $\chi'_{\vec{P_3}}(D)$ when D is an acyclic digraph [1].

A sequence $P_1, \ldots, P_k$ of subsets of vertex set of a digraph is called an acyclic partition if $\{P_1, \ldots, P_k\}$ is a partition of vertex set of D and for each edge (x, y), if P_i contains x and P_j contains y, then $i < j$. It is obvious that every digraph with an acyclic partition is an acyclic digraph and vise versa.

For an acyclic digraph D, the minimum k such that D has an acyclic partition into k sets is denoted by $AP(D)$. By definition each set P_i is independent, therefore $\chi(D) \leq AP(D)$. Theorem 3 proves that $\lceil log_2 AP(D) \rceil$ colors are enough to find a P_3-rec for an acyclic digraph D.

Theorem 3. *For every acyclic digraph D,*

$$\chi'_{\vec{P_3}}(D) \leq \min\{\Delta_{in}(D) + 1, \Delta_{out}(D) + 1, \lceil log_2 AP(D) \rceil\}.$$

Proof. If D is an acyclic digraph. Find a topological ordering [2] of vertices of D. Color the outgoing edges of vertices based on their order in the topological ordering. Color the outgoing arcs of v with a color which is not used in the coloring of incoming arcs of v. It is a P_3-rec of D and the number of used colors is not more than $\Delta_{in}(D)+1$. A similar approach can be used to prove that $\chi'_{\vec{P_3}}(D) \leq \Delta_{out}(D) + 1$.

Let $P_1, \ldots, P_{AP(D)}$ be an acyclic partition of D. For each edge (x, y), there are distinct sets P_i and P_j such that $i < j$, $x \in P_i$ and $y \in P_j$. We find the binary representation of i and j using $\lceil log_2 AP(D) \rceil$ bits and denote them by $b(i)$ and $b(j)$, respectively. Suppose that $k - 1$ first bits of $b(i)$ and $b(j)$ are the same and k'th bit of $b(i)$ and $b(j)$ are different. Since $i < j$, the k'th bit of $b(i)$ is *zero* and the k'th bit of $b(j)$ is one. Color (x, y) with color k. The obtaining coloring is a P_3-rec and the number of used colors is at most $\lceil log_2 AP(D) \rceil$.

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[1] A digraph with no directed cycles is called an acyclic digraph

[2] An ordering of vertices such that for each edge (x, y), x appears before y in the ordering. Every acyclic digraph has a topological ordering

$\square$

A digraph is a transitive orientation digraph if for every three vertices x, y and z, if (x, y) and (y, z) are edges of D, then (x, z) is an edge of D.

Theorem 4. *Let D be a transitive orientation digraph, then $\chi'_{\vec{P_3}}(D) = \lceil log_2\chi(D)\rceil$.*

Proof. According to Theorem 1, it is sufficient to prove that $\chi'_{\vec{p_3}}(D) \leq \lceil log_2\chi(D)\rceil$ colors. Since D is a transitive orientation digraph, the underlying graph of D is a comparability graph[3]. Hence $\chi(D) = \omega(D)$.

For completing the proof, we use the idea mentioned in [21]. For each vertex x, let $\Delta(x)$ be the maximum number of vertices in a directed path that ends in x. Now define $N_i = \{x|\Delta(x) = i\}$. We can obtain the following.

- N_1 is not empty
- If N_{i+1} is not empty, then N_i is not empty.

Let $\omega \in N$ be the maximal number such that N_ω is not empty. Then $\omega = \omega(D) = \chi(D)$ [21]. The sets $N_1, ..., N_\omega$ form an acyclic partition of D. Hence $\chi(D) = AP(D)$. Since D is an acyclic digraph, by Theorem 3 $\chi'_{\vec{p_3}}(D) \leq \lceil log_2\chi(D)\rceil$.

$\square$

3. Characterization of graphs with P_3-rec number 2

In [17], it is proved that if D is bipartite then $\chi'_{\vec{p_3}}(D) \leq 2$. In this section, we characterize digraphs whose P_3-rec number is 2.

Theorem 5. *Let D be a digraph with at least one edge. Then $\chi'_{\vec{P_3}}(D) = 2$ if and only if digraph $D\backslash F(D)$ is K_1 or it is bipartite.*

[3] A graph is a comparability graph if it has an orientation such that the obtained digraph is a transitive orientation digraph.

Proof. If $D \backslash F(D)$ is K_1 or it is a bipartite digraph, then it has at least a vertex which is not flexible. So $\chi'_{\vec{p_3}}(D) \geq 2$.

If $D \backslash F(D)$ is K_1, then it has exactly one vertex v which is not flexible. Color every outgoing edge from v with color 1, and color every incoming edge to v with color 2. Color other edges of D arbitrarily with colors 1 and 2.

If $D \backslash F(D)$ is a bipartite digraph with parts A and B. For each vertex v in A, color every outgoing edge from v with color 1 and color every incoming edge to v with color 2 and for each vertex u in B, color every outgoing edge from u with color 2 and color every incoming edge to u with color 1. Color other edges of D arbitrarily with colors 1 and 2. The obtained coloring is a P_3-rec of D.

Suppose that $\chi'_{\vec{P_3}}(D) = 2$ and $D \backslash F(D)$ is not K_1. If $D \backslash F(D)$ has no edges then obviously, it is a bipartite digraph. Suppose $D \backslash F(D)$ has some edge (x, y). Find a P_3-rec of D with colors 1 and 2. Partition the vertex set of D into two subsets A and B. Set A contains all the vertices with at least one outgoing edge colored by the same color used for (x, y) and B contains the other vertices. Define $A' = A \backslash F(D)$ and $B' = B \backslash F(D)$. It is easy to find out that $x \in A'$, $y \in B'$ and every edge of $D \backslash F(D)$ has one end in A' and one end in B'. Therefore, $D \backslash F(D)$ is a bipartite digraph. □

Corollary 1. *For a digraph D, if $\chi(D \backslash F(D)) = 3$, then $\chi'_{\vec{p_3}}(D) = 3$.*

Proof. By Theorem 3 $\chi'_{\vec{p_3}}(D) \leq 3$ and by Proposition 1 and Theorem 5, $\chi'_{\vec{p_3}}(D) \geq 3$. □

Note that the opposite of Corollary 1 is not true.

4. Planar graphs

By the obtained results, we can see that characterizing digraphs with P_3-rec number 1 and 2 can be done in linear time. Theorem 6 proves that characterizing digraphs with P_3-rec number 3 is NP-complete even for planar graphs.

Theorem 6. *Determining that a planar digraph has a P_3-rec with three colors is an NP-complete problem.*

Proof. It is easy to see that the problem is NP. It is well known that the 3-colorability problem for planar graphs is an NP-complete problem [8]. In order to complete the proof, we reduce the 3-colorability problem for planar graphs.

Let G be a graph. Construct digraph $\overleftrightarrow{G}$ by replacing each edge xy of G with two edges (x, y) and (y, x). It is obvious that $\overleftrightarrow{G}$ is a planar digraph.

Since G is the underlying graph of $\overleftrightarrow{G}$, if G has a vertex coloring with three colors, then $\chi'_{\vec{p_3}}(\overleftrightarrow{G}) \leq 3$.

Conversely, suppose there is a P_3-rec of $\overleftrightarrow{G}$ with three colors B, R, G. For each vertex v in $\overleftrightarrow{G}$, incoming edges of v or outgoing edges v have the same color. Therefore, we can define six following subsets.

$IB = \{v|$ all incoming arcs of v are colored with color $B\}$

$IR = \{v|$ all incoming arcs of v are colored with color $R\}$

$IG = \{v|$ all incoming arcs of v are colored with color $G\}$

$OB = \{v|$ all outgoing arcs of v are colored with color $B\}$

$OR = \{v|$ all outgoing arcs of v are colored with color $R\}$

$OG = \{v|$ all outgoing arcs of v are colored with color $G\}$

Define three subsets $X = \{IB \cup OG\}$, $Y = \{(IR \setminus OG) \cup OB\}$, $Z = \{(IG \setminus OB) \cup (OR \setminus IB)\}$. We can obtained the followings:

- Every pair of subsets do not intersect.
- Each subset is an independent set of G.
- Each vertex of G is in exactly one of the three subsets.

This proves that $\chi(G) \leq 3$.

□

The proof of Theorem 6, shows that for every planar or none planar graph G, if $\chi'_{\vec{P_3}}(\overleftrightarrow{G}) = 3$ then $\chi(G) = 3$ and so we can obtain the following corollary.

Corollary 2. *For a graph G, if $2 \leq \chi(G) \leq 3$, then $\chi(G) = \chi'_{\vec{P_3}}(\overleftrightarrow{G})$. Moreover if $\chi(G) = 4$, then $\chi'_{\vec{P_3}}(\overleftrightarrow{G}) = 4$.*

Corollary 3. *For every planar graph G with at least one edge, $\chi(G) = \chi'_{\vec{P_3}}(\overleftrightarrow{G})$.*

It is interesting to note that the Corollary is not true for none planar graphs. For graph K_7, $\chi(K_7) = 7$ but, $\chi'_{\vec{p_3}}((\overleftrightarrow{K_7}) \leq 2\lceil log_2 7\rceil = 6$.

5. Conclusion

In this paper, template-driven rainbow edge coloring has been studied where the host is a digraph and the template is a directed path with length 2. As mentioned interestingly in [4], the application of this concept in efficient design of parallel algorithms on multiprocessor architectures is the main motivation for studying this type of coloring problems. Some combinatorial and algorithmic results are presented in this paper. The results show that the problem is hard enough to worth extra work.

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