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4-total mean cordial labeling of union of some graphs with the complete bipartite graph $K_{2,n}$

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ABSTRACT

Let G be a graph. Let $f : V(G) \rightarrow \{0, 1, 2, \ldots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$, for all $i, j \in \{0, 1, 2, \ldots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \ldots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph. In this paper, we investigate the 4-total mean cordial labeling of some graphs derived from the complete bipartite graph $K_{2,n}$.

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1 Introduction

In this paper we consider simple, finite and undirected graphs only. Cordial labeling was introduced by Cahit [1] and cordial relation labeling technique was studied in [2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 19, 20, 21, 22, 23]. The notation of k-total mean cordial labeling has been introduced in [14]. Also in [14, 15, 16, 17, 18] investigate the 4-total mean cordial labeling behaviour of several graphs like cycle, complete graph, star, bistar, comb and crown. In this paper we examine the 4-total mean cordial labeling of union of some graphs with the complete bipartite graph $K_{2,n}$. Let x be any real number. Then $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms are not defined here follow from Harary[6] and Gallian[3]. .

2 k-total mean cordial graph

Definition 2.1. Let G be a graph. Let $f : V(G) \rightarrow \{0, 1, 2, \ldots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$, for all $i, j \in \{0, 1, 2, \ldots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \ldots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph.

3 Preliminaries

Definition 3.1. The *union* of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 3.2. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their *join* $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1 \text{ and } v \in V_2\}$.

Definition 3.3. Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The *corona* of G_1 with G_2 is the graph $G_1 \odot G_2$ obtained by taking one copy of G_1 , p_1 copies of G_2 and joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 where $1 \leq i \leq p_1$.

Definition 3.4. The *complement* $\overline{G}$ of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Definition 3.5. The complete bipartite graph $K_{1,n}$ is called a *Star*.

Definition 3.6. The *Bistar* $B_{m,n}$ is the graph obtained by joining the two central vertices of $K_{1,m}$ and $K_{1,n}$.

Definition 3.7. The graph $F_n = P_n + K_1$ is called a *Fan graph* where P_n is a path.

Definition 3.8. The graph $W_n = C_n + K_1$ is called a *wheel.*

Definition 3.9. The graph $L_n = P_n + K_2$ is called a *ladder.*

Notation 1 We denote the vertex set and edge set of the complete bipartite graph $K_{2,n}$ by $V(K_{2,n}) = \{u, v, u_i : 1 \le i \le n\}$ and $E(K_{2,n}) = \{uu_i, vu_i : 1 \le i \le n\}$ respectively.

4 Main results

Theorem 4.1 The graph $K_{2,n} \cup P_n$ is a 4-total mean cordial for all values of n.

Proof. Let P_n be the path $v_1\ v_2\ \ldots\ v_n$. Take the vertex set and edge set of $K_{2,n}$ as in Notation 1. Clearly $|V(K_{2,n} \cup P_n)| + |E(K_{2,n} \cup P_n)| = 5n + 1$.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \ge 2$.
Subcase 1. r is even.
Assign the labels 1, 3 to the vertices u, v respectively. We now assign the label 0 to the $4r$ vertices $u_1, u_2, \ldots, u_{4r}$. Now we assign the label 0 to the $\frac{r+2}{2}$ vertices $v_1, v_2, \ldots, v_{\frac{r+2}{2}}$. Next we assign the label 3 to the $\frac{5r}{2}$ vertices $v_{\frac{r+4}{2}}, v_{\frac{r+6}{2}}, \ldots, v_{3r+1}$. Then we assign the label 1 to the $\frac{r}{2}$ vertices $v_{3r+2}, v_{3r+3}, \ldots, v_{\frac{7r+2}{2}}$. Finally we assign the label 2 to the $\frac{r-2}{2}$ vertices $v_{\frac{7r+4}{2}}, v_{\frac{7r+6}{2}}, \ldots, v_{4r}$.

Subcase 2. r is odd.
Assign the labels 0, 3 to the vertices u, v respectively. Consider the vertices $u_1, u_2, \ldots, u_{4r}$. Assign the label 2 to the $4r$ vertices $u_1, u_2, \ldots, u_{4r}$. Now we assign the label 0 to the $\frac{5r+1}{2}$ vertices $v_1, v_2, \ldots, v_{\frac{5r+1}{2}}$. Then we assign the label 3 to the $\frac{r-1}{2}$ vertices $v_{\frac{5r+3}{2}}, v_{\frac{5r+5}{2}}, \ldots, v_{3r}$. Next we assign the label 2 to the $\frac{r-1}{2}$ vertices $v_{3r+1}, v_{3r+2}, \ldots, v_{\frac{7r-1}{2}}$. Finally we assign the label 1 to the $\frac{r+1}{2}$ vertices $v_{\frac{7r+1}{2}}, v_{\frac{7r+3}{2}}, \ldots, v_{4r}$.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r + 1$, $r \ge 2$.
Subcase 1. r is even.
As in Subcase 1 of Case 1, assign the label to the vertices u_i, v_i $(1 \le i \le 4r)$. Now we assign the labels 0, 3 respectively to the vertices u_{4r+1}, v_{4r+1}.

Subcase 2. r is odd.
Label the vertices u_i, v_i $(1 \le i \le 4r)$ as in Subcase 2 of Case 1. Next we assign the labels

3, 0 to the vertices u_{4r+1}, v_{4r+1} respectively.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r + 2$, $r \geq 2$.
Subcase 1. r is even.
In this case, assign the label to the vertices u_i, v_i $(1 \leq i \leq 4r)$ as in Subcase 1 of Case 1. Now we assign the labels 0, 3, 0, 3 to the vertices u_{4r+1}, u_{4r+2}, v_{4r+1}, v_{4r+2}.

Subcase 2. r is odd.
Label the vertices u_i, v_i $(1 \leq i \leq 4r)$ as in Subcase 2 of Case 1. Next we assign the labels 0, 3, 1, 3 to the vertices u_{4r+1}, u_{4r+2}, v_{4r+1}, v_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r + 3$, $r \geq 2$.
Subcase 1. r is even.
We assign the label to the vertices u_i, v_i $(1 \leq i \leq 4r)$ as in Subcase 1 of Case 1. Now we assign the labels 0, 0, 3, 0, 2, 3 to the vertices u_{4r+1}, u_{4r+2}, u_{4r+3}, v_{4r+1}, v_{4r+2}, v_{4r+3}.

Subcase 2. r is odd.
As in Subcase 2 of Case 1, assign the label to the vertices u_i, v_i $(1 \leq i \leq 4r)$. Finally we assign the labels 1, 2, 3, 0, 0, 3 to the vertices u_{4r+1}, u_{4r+2}, u_{4r+3}, v_{4r+1}, v_{4r+2}, v_{4r+3}.

The table 1, shows that this vertex labeling f is a 4-total mean cordial labeling.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$4r$	$5r+1$	$5r$	$5r$	$5r$
$4r+1$	$5r+2$	$5r+1$	$5r+1$	$5r+2$
$4r+2$	$5r+3$	$5r+2$	$5r+2$	$5r+3$
$4r+3$	$5r+4$	$5r+4$	$5r+4$	$5r+4$

Table 1:

Case 5. $n \in \{1, 2, 3, 4, 5, 6, 7\}$.
Table 2 gives a 4-total mean cordial labeling for this case.

□

Corollary 4.1.1 If $n \equiv 0, 3 \pmod 4$ or $n \equiv 1 \pmod 8$, then graph $K_{2,n} \cup C_n$ is a 4-total mean cordial.

Proof. Obviously the vertex labeling of Theorem **??** is also a 4 - total mean cordial labeling of $K_{2,n} \cup C_n$.

□

n	u	v	u_1	u_2	u_3	u_4	u_5	u_6	u_7	v_1	v_2	v_3	v_4	v_5	v_6	v_7
1	0	3	2							0						
2	1	3	1	1						0	0					
3	1	3	0	0	1					0	0	3				
4	0	3	2	2	2	2				0	0	0	2			
5	0	3	2	2	2	2	3			0	0	0	2	0		
6	0	3	2	2	2	2	2	3		0	0	0	0	2	1	
7	0	3	2	2	2	2	2	2	2	0	0	0	0	2	0	3

Table 2:

Theorem 4.2. The graph $K_{2,n} \cup \overline{K_n}$ is 4-total mean cordial for all values of n.

Proof. Take the vertex set and edge set of $K_{2,n}$ as in Notation 1. Let $v_1, v_2, \ldots, v_n$ be the vertices of $\overline{K_n}$. Note that $\left|V\left(K_{2,n} \cup \overline{K_n}\right)\right| + \left|E\left(K_{2,n} \cup \overline{K_n}\right)\right| = 4n+2$. Assign the labels 1, 3 to the vertices u, v respectively. Consider the vertices $u_1, u_2, \ldots, u_n$. Now we assign the label 0 to the n vertices $u_1, u_2, \ldots, u_n$. Finally we assign the label 3 to the n vertices $v_1, v_2, \ldots, v_n$. Clearly $t_{mf}(0) = t_{mf}(2) = n$; $t_{mf}(1) = t_{mf}(3) = n+1$.

□

Theorem 4.3. The graph $K_{2,n} \cup K_{1,n}$ is 4-total mean cordial for all values of n.

Proof. Take the vertex set and edge set of $K_{2,n}$ as in Notation 1. Let the vertex set of $K_{1,n}$ be, $V(K_{1,n}) = \{w, v_i : 1 \leq i \leq n\}$ and the edge set of $K_{1,n}$ be, $E(K_{1,n}) = \{wv_i : 1 \leq i \leq n\}$. Clearly $|V(K_{2,n} \cup K_{1,n})| + |E(K_{2,n} \cup K_{1,n})| = 5n+3$. Assign the labels 0, 3, 1 to the vertices u, v, w respectively.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the $2r$ vertices $u_1, u_2, \ldots, u_{2r}$. Next we assign the label 3 to the $2r$ vertices $u_{2r+1}, u_{2r+2}, \ldots, u_{4r}$. Now we assign the label 0 to the r vertices $v_1, v_2, \ldots, v_r$. Then we assign the label 1 to the $2r$ vertices $v_{r+1}, v_{r+2}, \ldots, v_{3r}$. Now we assign the label 2 to the vertex v_{3r+1}. Finally we assign the label 3 to the $r-1$ vertices $v_{3r+2}, v_{3r+3}, \ldots, v_{4r}$.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r+1$, $r \in \mathbb{N}$. Label the vertices u_i, v_i $(1 \leq i \leq 4r)$ as in Case 1. Now we assign the labels 3, 0 respectively to the vertices u_{4r+1}, v_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r+2$, $r \in \mathbb{N}$. In this case, we assign the label to the vertices u_i, v_i $(1 \leq i \leq 4r+1)$ as in Case 2. Next we assign the labels 3, 0 to the vertices u_{4r+2}, v_{4r+2} respectively.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r+3$, $r \in \mathbb{N}$. As in case 3, assign the label to the vertices u_i, v_i $(1 \le i \le 4r+2)$. Finally we assign the labels 2, 0 respectively to the vertices u_{4r+3}, v_{4r+3}.

Thus this vertex labeling f is a 4-total mean cordial labeling follows from the Table 3.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$5r+1$	$5r+1$	$5r+1$	$5r$
$n = 4r+1$	$5r+2$	$5r+2$	$5r+2$	$5r+2$
$n = 4r+2$	$5r+3$	$5r+3$	$5r+3$	$5r+4$
$n = 4r+3$	$5r+4$	$5r+5$	$5r+4$	$5r+5$

Table 3:

Case 5. $n = 1, 2, 3$.
Table 4 gives a 4-total mean cordial labeling for this case.

□

n	u	v	w	u_1	u_2	u_3	v_1	v_2	v_3
1	0	3	2	2			0		
2	0	3	2	1	3		0	0	
3	0	3	2	2	2	3	0	0	0

Table 4:

Theorem 4.4. The graph $K_{2,n} \cup B_{n,n}$ is 4-total mean cordial for all values of n.

Proof. Take the vertex set and edge set of $K_{2,n}$ as in Notation 1. Let $V(B_{n,n}) = \{x, y, x_i, y_i : 1 \le i \le n\}$ and $E(B_{n,n}) = \{xy, xx_i, yy_i : 1 \le i \le n\}$. Note that $|V(K_{2,n} \cup B_{n,n})| + |E(K_{2,n} \cup B_{n,n})| = 7n+5$. Assign the labels 1, 3, 0, 3 to the vertices u, v, x, y respectively.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$.
Subcase 1. r is odd.
Assign the label 0 to the $4r$ vertices $u_1, u_2, \ldots, u_{4r}$. Now we assign the label 0 to the $\frac{3r+1}{2}$ vertices $x_1, x_2, \ldots, x_{\frac{3r+1}{2}}$. Next we assign the label 1 to the $\frac{r+1}{2}$ vertices $x_{\frac{3r+3}{2}}, x_{\frac{3r+5}{2}}, \ldots, x_{2r+1}$. Now we assign the label 2 to the $2r-1$ vertices $x_{2r+2}, x_{2r+3}, \ldots, x_{4r}$. Next we assign the label 2 to the $r+1$ vertices $y_1, y_2, \ldots, y_{r+1}$. Finally we assign the label 3 to the $3r-1$ vertices $y_{r+2}, y_{r+3}, \ldots, y_{4r}$.

SubCase 2. r is even.
We assign the label 0 to the $4r$ vertices $u_1, u_2, \ldots, u_{4r}$. Now we assign the label 0 to the

$\frac{3r}{2}$ vertices x_1, x_2, …, $x_{\frac{3r}{2}}$. We now assign the label 1 to the $\frac{r+2}{2}$ vertices $x_{\frac{3r+2}{2}}$, $x_{\frac{3r+4}{2}}$, …, x_{2r+1}. Next we assign the label 2 to the $2r-1$ vertices x_{2r+2}, x_{2r+3}, …, x_{4r}. Now we assign the label 2 to the $r+1$ vertices y_1, y_2, …, y_{r+1}. Finally we assign the label 3 to the $3r-1$ vertices y_{r+2}, y_{r+3}, …, y_{4r}.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r+1$, $r \in \mathbb{N}$.
Subcase 1. r is odd.
As in Subcase 1 of Case 1, assign the label to the vertices u_i, x_i, y_i $(1 \le i \le 4r)$. Now we assign the labels 0, 2, 3 respectively to the vertices u_{4r+1}, x_{4r+1}, y_{4r+1}.

Subcase 2. r is even.
Label the vertices u_i, x_i, y_i $(1 \le i \le 4r)$ as in Subcase 2 of Case 1. Next we assign the labels 3, 0, 1 to the vertices u_{4r+1}, x_{4r+1}, y_{4r+1} respectively.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r+2$, $r \in \mathbb{N}$.
Subcase 1. r is odd.
In this case, assign the label to the vertices u_i, x_i, y_i $(1 \le i \le 4r+1)$ as in Subcase 1 of Case 2. Finally we assign the labels 0, 2, 3 to the vertices u_{4r+2}, x_{4r+2}, y_{4r+2}.

Subcase 2. r is even.
Label the vertices u_i, x_i, y_i $(1 \le i \le 4r+1)$ as in Subcase 2 of Case 2. Next we assign the labels 0, 2, 3 to the vertices u_{4r+2}, x_{4r+2}, y_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r+3$, $r \in \mathbb{N}$.
Subcase 1. r is odd.
We assign the label to the vertices u_i, x_i, y_i $(1 \le i \le 4r+2)$ as in Subcase 1 of Case 3. Now we assign the labels 3, 0, 1 to the vertices u_{4r+3}, x_{4r+3}, y_{4r+3}.

Subcase 2. r is even.
As in Subcase 2 of Case 3, assign the label to the vertices u_i, x_i, y_i $(1 \le i \le 4r+2)$. Finally we assign the labels 3, 0, 1 to the vertices u_{4r+3}, x_{4r+3}, y_{4r+3}.

The table 5, shows that this vertex labeling f is a 4-total mean cordial labeling.

Case 5. $n = 1, 2, 3$.
Table 6 gives a 4-total mean cordial labeling for this case.

□

Theorem 4.5. The graph $K_{2,n} \cup W_n$ is 4-total mean cordial for all $n \ge 3$.

n	Nature of r	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$4r$	r is odd	$7r+2$	$7r+1$	$7r+1$	$7r+1$
$4r$	r is even	$7r+1$	$7r+2$	$7r+1$	$7r+1$
$4r+1$	r is odd	$7r+3$	$7r+3$	$7r+3$	$7r+3$
$4r+1$	r is even	$7r+3$	$7r+3$	$7r+3$	$7r+3$
$4r+2$	r is odd	$7r+4$	$7r+5$	$7r+5$	$7r+5$
$4r+2$	r is even	$7r+4$	$7r+5$	$7r+5$	$7r+5$
$4r+3$	r is odd	$7r+6$	$7r+6$	$7r+7$	$7r+7$
$4r+3$	r is even	$7r+6$	$7r+6$	$7r+7$	$7r+7$

Table 5:

n	u	v	x	y	u_1	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3
1	1	3	0	1	3			0			2		
2	1	3	0	3	0	0		0	2		2	3	
3	1	3	0	3	0	0	0	0	2	2	2	3	3

Table 6:

Proof. Take the vertex set and edge set of $K_{2,n}$ as in Notation 1. Let the vertex set of W_n be, $V(W_n) = \{w, w_i : 1 \leq i \leq n\}$ and the edge set of W_n be, $E(W_n) = \{ww_i : 1 \leq i \leq n\} \cup \{w_iw_{i+1} : 1 \leq i \leq n-1\} \cup \{w_nw_1\}$. Clearly $|V(K_{2,n} \cup W_n)| + |E(K_{2,n} \cup W_n)| = 6n+3$. Assign the labels 0, 2, 0 to the vertices u, v, w respectively.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the $3r-1$ vertices $u_1, u_2, \ldots, u_{3r-1}$. Next we we assign the label 1 to the $r+1$ vertices $u_{3r}, u_{3r+1}, \ldots, u_{4r}$. Now we assign the label 3 to the $3r$ vertices $w_1, w_2, \ldots, w_{3r}$. Finally we assign the label 2 to the r vertices $w_{3r+1}, w_{3r+2}, \ldots, w_{4r}$.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r+1$, $r \in \mathbb{N}$. Label the vertices u_i, w_i $(1 \leq i \leq 4r)$ as in Case 1. Now we assign the labels 0, 3 to the vertices u_{4r+1}, w_{4r+1} respectively.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r+2$, $r \in \mathbb{N}$. In this case, we assign the label to the vertices u_i, w_i $(1 \leq i \leq 4r)$ as in Case 1. Next we assign the labels 0, 0, 2, 3 respectively to the vertices u_{4r+1}, u_{4r+2}, w_{4r+1}, w_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r+3$, $r \in \mathbb{N}$. As in case 1, assign the label to the vertices u_i, w_i $(1 \leq i \leq 4r)$. Finally we assign the labels 0, 0, 0, 2, 3, 3 to the vertices u_{4r+1}, u_{4r+2}, u_{4r+3}, w_{4r+1}, w_{4r+2}, w_{4r+3} respectively.

Thus this vertex labeling f is a 4-total mean cordial labeling follows from the Table 7.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$6r$	$6r + 1$	$6r + 1$	$6r + 1$
$n = 4r + 1$	$6r + 2$	$6r + 2$	$6r + 2$	$6r + 3$
$n = 4r + 2$	$6r + 4$	$6r + 4$	$6r + 4$	$6r + 3$
$n = 4r + 3$	$6r + 6$	$6r + 5$	$6r + 5$	$6r + 5$

Table 7:

Case 5. $n = 3$.
Table 8 gives a 4-total mean cordial labeling for this case.

□

n	u	v	w	u_1	u_2	u_3	w_1	w_2	w_3
3	0	2	0	0	0	1	2	3	3

Table 8:

Theorem 4.6. The graph $K_{2,n} \cup F_n$ is 4-total mean cordial for all $n \geq 2$.

Proof. Take the vertex set and edge set of $K_{2,n}$ as in Notation 1.
Let $V(F_n) = \{w, w_i : 1 \leq i \leq n\}$ and $E(F_n) = \{ww_i : 1 \leq i \leq n\} \cup \{w_i w_{i+1} : 1 \leq i \leq n-1\}$.
Note that $|V(K_{2,n} \cup F_n)| + |E(K_{2,n} \cup F_n)| = 6n + 2$. Assign the labels 0, 2, 0 to the vertices u, v, w respectively.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the $3r - 1$ vertices $u_1, u_2, \ldots, u_{3r-1}$. Next we we assign the label 1 to the $r+1$ vertices $u_{3r}, u_{3r+1}, \ldots, u_{4r}$. Now we assign the label 2 to the r vertices $w_1, w_2, \ldots, w_r$. Finally we assign the label 3 to the $3r$ vertices $w_{r+1}, w_{r+2}, \ldots, w_{4r}$.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r + 1$, $r \in \mathbb{N}$. We assign the label to the vertices u_i, w_i $(1 \leq i \leq 4r)$ as in Case 1. Next we assign the labels 0, 3 respectively to the vertices u_{4r+1}, w_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r + 2$, $r \in \mathbb{N}$. Label the vertices u_i, w_i $(1 \leq i \leq 4r + 1)$ as in Case 2. Now we assign the labels 0, 3 to the vertices u_{4r+2}, w_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r + 3$, $r \in \mathbb{N}$. Now we assign the label 0 to the $3r - 1$ vertices $u_1, u_2, \ldots, u_{3r-1}$. Next we we assign the label 1 to the $r + 1$ vertices $u_{3r}, u_{3r+1}, \ldots, u_{4r}$. Now we assign the

labels 1, 3, 3 respectively to the vertices u_{4r+1}, u_{4r+2}, u_{4r+3}. Consider we assign the label 3 to the $3r+1$ vertices w_1, w_2, …, w_{3r+1}. Now we assign the label 1 to the vertex w_{3r+2}. Next we assign the label 2 to the $r-2$ vertices w_{3r+3}, w_{3r+4}, …, w_{4r}. Finally we assign the labels 2, 0, 0 to the vertices w_{4r+1}, w_{4r+2}, w_{4r+3}.

From the Table 9, this vertex labeling f is a 4-total mean cordial labeling.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n=4r$	$6r$	$6r+1$	$6r+1$	$6r$
$n=4r+1$	$6r+2$	$6r+2$	$6r+2$	$6r+2$
$n=4r+2$	$6r+4$	$6r+3$	$6r+3$	$6r+4$
$n=4r+3$	$6r+5$	$6r+5$	$6r+5$	$6r+5$

Table 9:

Case 5. $n=2,3$.
Table 10 gives a 4-total mean cordial labeling for this case.

□

n	u	v	w	u_1	u_2	u_3	w_1	w_2	w_3
2	0	2	0	0	1		3	3	
3	0	2	1	0	0	3	1	3	3

Table 10:

Theorem 4.7. The graph $K_{2,n} \cup H_n$ is 4-total mean cordial for all $n \geq 3$.

Proof. Take the vertex set and edge set of $K_{2,n}$ as in Notation 1.
Let the vertex set of H_n be, $V(H_n) = \{w, w_i, v_i : 1 \leq i \leq n\}$ and the edge set of H_n be, $E(H_n) = \{ww_i, w_iv_i : 1 \leq i \leq n\} \cup \{w_iw_{i+1} : 1 \leq i \leq n-1\} \cup \{w_nw_1\}$. Clearly $|V(K_{2,n} \cup H_n)| + |E(K_{2,n} \cup H_n)| = 8n+3$. Assign the labels 1, 3, 2 to the vertices u, v, w respectively. Assign the label 3 to the n vertices u_1, u_2, …, u_n. Next we we assign the label 0 to the n vertices w_1, w_2, …, w_n. Finally we assign the label 2 to the n vertices v_1, v_2, …, v_n. Obiviously $t_{mf}(0) = 2n$; $t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = 2n+1$. □

Theorem 4.8 The graph $K_{2,n} \cup L_n$ is 4-total mean cordial for all $n \geq 2$.

Proof. Take the vertex set and edge set of $K_{2,n}$ as in Notation 1. Let $V(L_n) = \{v_i, w_i : 1 \leq i \leq n\}$ and $E(L_n) = \{v_iw_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1}, w_iw_{i+1} : 1 \leq i \leq n-1\}$.
Obviously $|V(K_{2,n} \cup L_n)| + |E(K_{2,n} \cup L_n)| = 8n$. Assign the labels 0, 3 to the vertices u, v respectively. Assign the label 1 to the n vertices u_1, u_2, …, u_n. Next we we assign the label 0 to the n vertices v_1, v_2, …, v_n. Finally we assign the label 3 to the n vertices w_1, w_2, …, w_n. Clearly $t_{mf}(0) = 2n = t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = 2n$. □

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