



Comparison of solutions resulted from direct problems formulated as FRE

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ABSTRACT

In this paper, we investigate direct solution of FRE and compare their results with expected real consequences. We give an applied example formulated by a FRE problem and show that FRE defined by maximum t-conorm and an arbitrary t-norm can yield different interpretations for our example. A necessary condition and a sufficient condition are presented that guarantee FRE defined by maximum t-conorm and minimum t-norm attain the same solutions as human mind does. Also, we present a t-conorm and use it instead of maximum t-conorm in FRE to obtain solutions with highest similarity with real ones. Finally, we show that under some conditions, FRE defined by any t-conorm and any t-norm may find a solution which is not reasonable.

Keyword: Fuzzy relation, composition of fuzzy relations, Fuzzy relational equations, Fuzzy operators.

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1 Introduction

Fuzzy relational equations (FRE) are a generalization of Boolean relation equations. At first, Sanchez [39] investigated and developed this theory and its applications. Generally, fuzzy set theory has a number of properties that make it suitable for formulating the uncertain information upon which many applied concepts such as medical diagnosis and treatment are usually based. Because of this fact the theory of FRE was originally applied in problems of the medical diagnosis [39]. Later, The concept of FRE is used in many problems such as system analysis[38], decision making[1], fuzzy controller [10], fuzzy modeling[42], fuzzy analysis, especially fuzzy arithmetic[10]. In [21] it has been established that the majority of fuzzy inference systems can be implemented using FRE. Pedrycz [30] categorized and extended two ways of the generalizations of FRE in terms of sets under discussion and various operations which are taken into account. Since then , many theoretical improvements have been investigated and many applications have been presented. For instance, we can refer to [3,5,7,11-14,16,17,22,25,31-33,35,37,41,43]. Klement et al. [18-20] presented the basic analytical and algebraic properties of triangular norms and important classes of fuzzy operators generalization such as Archimedean, strict and nilpotent t-norms. Pedrycz and Vasilakos [34] converted a highly dimensional relational equation into a serious of single input FRE. In [36] author demonstrate how problems of interpolation and approximation of fuzzy functions are connected with solvability of systems of FRE. He focused on the problem of approximate solvability of a system of FRE. In [40], FRE has been extended to the setting of interval-valued FRE with a max-t-norm composition and three types of solution sets have been proposed. Markovskii showed that solving max-product FRE is closely related to the covering problem which is an NP-hard problem [24].

In [2] Chen and Wang designed an algorithm for obtaining the logical representation of all minimal solutions. They showed that a polynomial time algorithm to find all minimal solutions of FRE with max-min composition may not exist.

An interesting application of the fuzzy relations theory is in the field of image processing [9] and [6]. Also, the various types of FRE with continuous triangular norms were used for compression / decompression of images [4,15,28,23]. In [4] the image was divided in blocks and then the FRE with a t-norm were utilized for compression each block where results were obtained using the Lukasiewicz t-norm . In [26] any monochromatic image is interpreted as a fuzzy relation in which the entries are the normalized values of the pixels. This method is based essentially on the fact that the reconstructed images is obtained as the greatest or the smallest solution of a system of fuzzy equations. Also, authors showed processes for coding/decoding color images in the RGB and YUV spaces by using FRE of max-t type where t is the Yager t-norm. In [27] authors used particular FRE of compression / decompression of color images in the RGB and YUV spaces .Nobuhara et al. [29] formulated and solved a problem of image reconstruction using eigen fuzzy sets. They proposed two algorithms of generating

eigen fuzzy sets used in the reconstruction process. Di Nola and Russo [8] focused on the algebraic structures of such these problems by considering the Lukasiewicz transform as a residuated map.

In this paper, we firstly focus on a direct problem defined by the most familiar form of fuzzy relational equations which is formulated as follows:

$$x \circ A = b \quad (1)$$

Where $I = \{1, 2, \dots, n\}$, and $J = \{1, 2, \dots, m\}$. $A = (a_{ij})_{n \times m}$ is a fuzzy matrix in which $0 \leq a_{ij} \leq 1, \forall i \in I$ and $\forall j \in J$. x is a given n -dimensional fuzzy vector, and b is an unknown m -dimensional fuzzy vector in which $0 \leq x_i \leq 1, \forall i \in I$, and $0 \leq b_j \leq 1, \forall j \in J$, respectively. Also, " \circ " is an arbitrary t-norm, and a_j is the j th column of matrix A . Equivalently, Problem(1) can be written with more details as follows:

$$x \circ a_j = \max_{i=1}^n \{t(x_i, a_{ij})\} = b_j, \forall j \in J \quad (2)$$

We present an applied example that is formulated by (2) as a direct problem in which x and A are given and b is unknown. We present a necessary condition and a sufficient condition in which direct solution of Problem (2) defined by minimum t-norm leads to a solution coinciding with consequence resulted from human mind. Also, we show that direct Problem (2) defined by each t-norm cannot yield a rational vector b for our example. In latter case, provided with some assumptions hold, we present a t-conorm (instead of maximum operator in (2)) by which we can solve direct Problem (2) (in which $t(a, b) = \min(a, b)$) with acceptable results and we find a vector b that is exactly the same as rational consequences which are obtained by human mind. Also, we investigate conditions in which direct Problem (2) defined by each t-norm and each t-conorm cannot yield a reasonable solution.

In section 2, we give an example to prove that direct Problem (2) may result in a vector b which is too farther from the expected rational conclusions. In section 3, we present a t-conorm which is constructed by the convex combination of drastic sum and maximum operators. Then, we solve direct Problem (2) defined by minimum t-norm in which foregoing t-conorm takes the place of maximum operator, and we show that vector b obtained by the new problem has the highest similarity with the expected rational conclusions. Finally, conditions are presented in which any kind of Problem (2) (defined by arbitrary t-conorm and t-norm) does not result in a rational solution.

2. An applied example: pea or melon?

Consider a basket including several peas, strawberries, bananas, and melons. Also, consider a man who is aware of those four types of contents deciding to distinguish those objects only by touching them with closed eyes. By this assumption that the contents in the basket are just those four foregoing kinds, one will able to evaluate them in his mind by considering some properties (for instance, their weights). When one touches an object with sufficient large value of weight (for example he assigns 1 to the weight of that object from the interval $[0,1]$) he may argue that what is in

his hands is melon. On the other hand, when one touches a melon, he can feel that what is in his hands has large value of weight among other contents included in the basket. Then, we can define a fuzzy relation in which, membership value of being melon and simultaneously having large value of weight, is too high. By similar reasons, we may make a table based on four objects; pea, strawberry, banana, melon, and three properties; long, Heavy, and voluminous as follows:

	Pea	Strawberry	Banana	Melon
Long	0	0.3	0.98	0.7
Heavy	0.001	0.01	0.1	0.99
Voluminous	0.004	0.042	0.3	1

Table 1. Fuzzy relation between four objects; pea, strawberry, banana, melon, and three properties; long, heavy, voluminous.

By our assumption, an object selected from basket is melon iff its length = 0.7 or its weight = 0.99 or its volume = 1. Similar statements are true for being banana, being strawberry, and being pea. Furthermore, table 1 can be shown by a fuzzy relational matrix as follows:

$$\begin{array}{l} \text{Length} \\ \text{Weight} \\ \text{Volume} \end{array} \begin{array}{c} P \quad S \quad B \quad M \\ \left[\begin{array}{cccc} 0 & 0.3 & 0.98 & 0.7 \\ 0.001 & 0.01 & 0.1 & 0.99 \\ 0.004 & 0.042 & 0.3 & 1 \end{array} \right] = A \end{array}$$

where $P, S, B,$ and M denote pea, strawberry, banana, and melon, respectively.

Now, suppose we select an object x from the basket (with closed eyes) and we evaluate its properties as; $length(x) = x_1$, $weight(x) = x_2$, and $volume(x) = x_3$, where $x_1, x_2, x_3 \in [0, 1]$. We show these pieces of information by a 3-dimensional vector as shown below:

$$x = \begin{array}{c} L \quad W \quad V \\ \left[\begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \right] \end{array}$$

where $L, W,$ and V denote length, weight, and volume, respectively.

Then, by the composition of fuzzy relations, we can find the kind of this object by observing vector b which is attained by solving direct Problem (2) as follows:

$$\left[\begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \right] \circ \begin{array}{c} P \quad S \quad B \quad M \\ \left[\begin{array}{cccc} 0 & 0.3 & 0.98 & 0.7 \\ 0.001 & 0.01 & 0.1 & 0.99 \\ 0.004 & 0.042 & 0.3 & 1 \end{array} \right] \end{array} = \left[\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \right]$$

Vector b determines the membership value of each contents in the basket to which object x belongs. Precisely, b_1 is the degree of which x is pea, b_2 is the degree of which x is strawberry, b_3 is the degree of which x is banana, and b_4 is the degree of which x is melon.

In a certain experiment, we select an object \hat{x} from the basket and find that $\hat{x} = \begin{bmatrix} 0.004 & 0.002 & 0.003 \end{bmatrix}$. By comparing three properties of object \hat{x} with that of pea, strawberry, banana, and melon, it is easy to see that \hat{x} should be a pea. Therefore, we must have the same conclusion by finding vector b through solving Direct problem (2), too. At first, we solve problem $\hat{x} \circ A = b$ in which " \circ " is minimum t-norm and

$$A = \begin{array}{c} \begin{matrix} P & S & B & M \end{matrix} \\ \left[\begin{array}{cccc} 0 & 0.3 & 0.98 & 0.7 \\ 0.001 & 0.01 & 0.1 & 0.99 \\ 0.004 & 0.042 & 0.3 & 1 \end{array} \right] \end{array}$$

Easy calculations show that the result is the vector below:

$$b = \begin{array}{c} \begin{matrix} P & S & B & M \end{matrix} \\ \left[\begin{array}{cccc} 0.003 & 0.004 & 0.004 & 0.004 \end{array} \right] \end{array}$$

According to the vector b above, \hat{x} is a pea with membership value 0.003, and it is a melon with membership value 0.004 (?!). If we solve problem above with product t-norm, we will attain solutions as objectionable as previous one:

$$b = \begin{array}{c} \begin{matrix} P & S & B & M \end{matrix} \\ \left[\begin{array}{cccc} 0.000012 & 0.0012 & 0.00392 & 0.003 \end{array} \right] \end{array}$$

The result will be too wonderful if we solve problem above with Lukasiewicz t-norm or drastic product. In both cases, we have:

$$b = \begin{array}{c} \begin{matrix} P & S & B & M \end{matrix} \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0.003 \end{array} \right] \end{array}$$

More wonderful result is attained when we try to distinguish an object $\hat{x} = \begin{bmatrix} 0 & 0.001 & 0.004 \end{bmatrix}$ via direct solution of Problem (2). By considering the components of table 1, it is readily attained that \hat{x} is exactly a pea. But, when Problem (2) is defined by minimum, product, Lukasiewicz, and drastic product, yields vectors b_1 , b_2 , b_3 , and b_4 below, respectively:

$$b_1 = \begin{array}{c} \begin{matrix} P & S & B & M \end{matrix} \\ \left[\begin{array}{cccc} 0.004 & 0.004 & 0.004 & 0.004 \end{array} \right] \end{array}$$

$$b_2 = \begin{array}{c} \begin{matrix} P & S & B & M \end{matrix} \\ \left[\begin{array}{cccc} 16 \times 10^{-6} & 168 \times 10^{-6} & 12 \times 10^{-4} & 4 \times 10^{-3} \end{array} \right] \end{array}$$

$$b_3 = b_4 = \begin{matrix} P & S & B & M \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0.004 \end{array} \right] \end{matrix}$$

Actually, we expect $b_1 \geq b_2 \geq b_3 \geq b_4$ by a rational justification in all cases above while results attained from Problem (2) have inverse ordering in all cases.

In a general case, suppose $\ddot{x} = \begin{bmatrix} \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{bmatrix}$ is a selected object from the basket. Since $a_{i1} < a_{i4}$ for each $i \in \{1, 2, 3\}$, then by the non-decreasing property of t-norms we have $t(\ddot{x}_i, a_{i1}) \leq t(\ddot{x}_i, a_{i4})$ for each $i \in \{1, 2, 3\}$ and for each t-norm. Therefore, $\max_{i=1}^3 \{t(\ddot{x}_i, a_{i1})\} \leq \max_{i=1}^3 \{t(\ddot{x}_i, a_{i4})\}$ which implies $b_1 \leq b_4$. Since latter inequality is satisfied for each $\ddot{x} \in [0, 1]^3$ and for each t-norm, then problem (2) defined by each t-norm cannot truly distinguish between a pea and a melon.

3. Modification of Problem (2)

In this section, we investigate how human mind can solve example above with true solutions while Problem (2) does not. Also, we try to modify Problem (2) in a way which leads us to the same consequences as made by human mind. The process which is used in the mind for solving example above is based on the comparison, and comparison process itself is done by a difference operation. Actually, if we want to formulate the solution process of mind for our example with highest similarity with direct problem (2), we have:

$$1 - \min_{i=1}^n \{|x_i - a_{ij}|\} = b_j, \forall j \in J \quad (3)$$

or equivalently;

$$\max_{i=1}^n \{1 - |x_i - a_{ij}|\} = b_j, \forall j \in J \quad (4)$$

in which x_i and a_{ij} are given $\forall i \in I$ and $\forall j \in J$, and b_j is unknown $\forall j \in J$.

If we solve example above for $\dot{x} = \begin{bmatrix} 0.004 & 0.002 & 0.003 \end{bmatrix}$ and $\ddot{x} = \begin{bmatrix} 0 & 0.001 & 0.004 \end{bmatrix}$ by direct problem (3) or (4), we will have reasonable vectors \dot{b} , and \ddot{b} , respectively as follows:

$$\dot{b} = \begin{matrix} P & S & B & M \\ \left[\begin{array}{cccc} 0.999 & 0.992 & 0.902 & 0.304 \end{array} \right] \end{matrix}$$

$$\ddot{b} = \begin{matrix} P & S & B & M \\ \left[\begin{array}{cccc} 1 & 0.991 & 0.974 & 0.3 \end{array} \right] \end{matrix}$$

Here the question is whether there exists a t-norm by which vector b resulted from Problem (2) is the same as one attained from Problem (4), or else by which t-norm, Problem (2) yields a vector b with the most accurate approximation to vector b in (4). Lemma 1 and its corollary below show that not only vector b resulted from (2) defined by an arbitrary t-norm is not generally equal to its corresponding vector in (4), but also under some conditions, the amount of the accuracy for vector b in (2) as an

approximation solution for vector b in (4) is bounded above. However, Problem (2) is not generally equal to Problem (4), and also under some conditions, the accuracy of Problem (2) (defined by each t-norm) to approximate Problem (4) can not be desirably increased.

Lemma 1. Given Problem (2). Then,

$\max_{i=1}^n \{t(x_i, a_{ij})\} \leq \max_{i=1}^n \{\min(x_i, a_{ij})\} \leq \max_{i=1}^n \{1 - |x_i - a_{ij}|\}, \forall j \in J$, where t is an arbitrary t-norm.

Proof. The first inequality is well known. For the second, fix a $j_0 \in J$ and suppose $x_i \leq a_{ij_0}$ for some $i \in I$ (otherwise, if $x_i > a_{ij_0}$, result is similarly attained). In this case, if $x_i \leq 2a_{ij_0} - 1$, then $x_i \leq 1 - (a_{ij_0} - x_i) \leq a_{ij_0}$. Otherwise, if $x_i > 2a_{ij_0} - 1$, then $x_i \leq a_{ij_0} < 1 - (a_{ij_0} - x_i)$. Therefore, in both cases, we have:

$$\min(x_i, a_{ij_0}) \leq 1 - |x_i - a_{ij_0}| \quad (5)$$

Now, since last inequality holds for each $i \in I$, we have $\max_{i=1}^n \{\min(x_i, a_{ij_0})\} \leq \max_{i=1}^n \{1 - |x_i - a_{ij_0}|\}$. Since $j_0 \in J$ was arbitrary, then latter inequality is satisfied for each $j \in J$.

Corollary 1. Given Problem (2) defined by minimum t-norm. Consider a fixed $j_0 \in J$ and suppose $\min(x_i, a_{ij_0}) > 2 \max(x_i, a_{ij_0}) - 1, \forall i \in I$. Then,

$$\max_{i=1}^n \{t(x_i, a_{ij_0})\} \leq \max_{i=1}^n \{\min(x_i, a_{ij_0})\} < \max_{i=1}^n \{1 - |x_i - a_{ij_0}|\}.$$

Corollary 1 shows that under some conditions, Problem (2) defined by each t-norm may not even a good approximation of real solution resulted from Problem (4). Lemma 2 part (a) below gives a sufficient condition by which Problem (2) defined by minimum t-norm finds rational vector b , exactly. Also, a necessary condition has been given in part (b) when Problem (2) defined by minimum t-norm provides a reasonable vector b the same as we expect. Moreover,

Lemma 2. Given Problem (2).

a) If $\max(x_i, a_{ij}) = 1, \forall i \in I$ and $\forall j \in J$, then $\max_{i=1}^n \{\min(x_i, a_{ij})\} = \max_{i=1}^n \{1 - |x_i - a_{ij}|\}, \forall j \in J$.

b) Suppose $\max_{i=1}^n \{\min(x_i, a_{ij})\} = \max_{i=1}^n \{1 - |x_i - a_{ij}|\}, \forall j \in J$. Then, for each $j \in J$ there exist at least one $i_j \in I$ such that $\max(x_{i_j}, a_{ij}) = 1$.

Proof. a) If $\max(x_i, a_{ij}) = 1$, then $\min(x_i, a_{ij}) = 1 - |x_i - a_{ij}|$. Therefore, if $\max(x_i, a_{ij}) = 1, \forall i \in I$, we have $\min(x_i, a_{ij}) = 1 - |x_i - a_{ij}|, \forall i \in I$ which implies part (a). (b) Consider a fixed $j_0 \in J$ and suppose $\max_{i=1}^n \{\min(x_i, a_{ij_0})\} = \max_{i=1}^n \{1 - |x_i - a_{ij_0}|\}$. Let

$$\min(x_{i_1}, a_{i_1 j_0}) = \max_{i=1}^n \{\min(x_i, a_{ij_0})\} \text{ and } 1 - |x_{i_2} - a_{i_2 j_0}| = \max_{i=1}^n \{1 - |x_i - a_{ij_0}|\}$$

Then, by assumption of part (b), we have:

$$\min(x_{i_1}, a_{i_1 j_0}) = 1 - |x_{i_2} - a_{i_2 j_0}| \quad (*)$$

Now, by (5), we have:

$$\min(x_{i_1}, a_{i_1 j_0}) \leq 1 - |x_{i_1} - a_{i_1 j_0}| \leq \max_{i=1}^n \{1 - |x_i - a_{i j_0}|\} = 1 - |x_{i_2} - a_{i_2 j_0}|$$

Which together with (*) imply $\min(x_{i_1}, a_{i_1 j_0}) = 1 - |x_{i_1} - a_{i_1 j_0}|$. Since $j_0 \in J$ was arbitrary, last equality completes the proof.

Lemma 3. Consider Problem (2) and Suppose $i_1, i_2 \in I$ and $j \in J$.

a) If we have $\min(x_{i_1}, a_{i_1 j}) = \min(x_{i_2}, a_{i_2 j}) = 0$ or, if $\min(x_{i_1}, a_{i_1 j}) = 0$ and $\min(x_{i_2}, a_{i_2 j}) \neq 0$, then:

$$S_D\{\min(x_{i_1}, a_{i_1 j}), \min(x_{i_2}, a_{i_2 j})\} \leq \max\{1 - |x_{i_1} - a_{i_1 j}|, 1 - |x_{i_2} - a_{i_2 j}|\}$$

b) If $\min(x_{i_1}, a_{i_1 j}) \neq 0$ and $\min(x_{i_2}, a_{i_2 j}) \neq 0$, then:

$$\max\{1 - |x_{i_1} - a_{i_1 j}|, 1 - |x_{i_2} - a_{i_2 j}|\} \leq S_D\{\min(x_{i_1}, a_{i_1 j}), \min(x_{i_2}, a_{i_2 j})\}$$

Proof. a) Suppose $\min(x_{i_1}, a_{i_1 j}) = \min(x_{i_2}, a_{i_2 j}) = 0$. In this case, we have

$S_D\{\min(x_{i_1}, a_{i_1 j}), \min(x_{i_2}, a_{i_2 j})\} = 0$. Now, by noting that $0 \leq 1 - |x_i - a_{ij}| \leq 1, \forall i \in I$ and $\forall j \in J$, the result is attained. In another case, suppose $\min(x_{i_1}, a_{i_1 j}) = 0$ and $\min(x_{i_2}, a_{i_2 j}) \neq 0$. Then, by the definition of drastic sum and Lemma 1, we have:

$$S_D\{\min(x_{i_1}, a_{i_1 j}), \min(x_{i_2}, a_{i_2 j})\} = \min(x_{i_2}, a_{i_2 j}) \leq 1 - |x_{i_2} - a_{i_2 j}| \leq \max\{1 - |x_{i_1} - a_{i_1 j}|, 1 - |x_{i_2} - a_{i_2 j}|\}$$

that implies statement. (b) Suppose $\min(x_{i_1}, a_{i_1 j}) \neq 0$ and $\min(x_{i_2}, a_{i_2 j}) \neq 0$. Then, by the definition of drastic sum, we have $S_D\{\min(x_{i_1}, a_{i_1 j}), \min(x_{i_2}, a_{i_2 j})\} = 1$. This fact together with $0 \leq \max\{1 - |x_{i_1} - a_{i_1 j}|, 1 - |x_{i_2} - a_{i_2 j}|\} \leq 1$ prove part b.

Lemmas 1 and 3 lead us to a theorem below by which we will be able to introduce and use a new t-conorm instead of maximum operator in (2), for compensating the difference between solution attained from (2) and real solution resulted from (4).

Theorem 1. Consider Problem (2) and suppose for each $j \in J$ there exist at least two $i_1, i_2 \in I$ such that $\min(x_{i_1}, a_{i_1 j}) \neq 0$ and $\min(x_{i_2}, a_{i_2 j}) \neq 0$. Then, $\max_{i=1}^n \{\min(x_i, a_{ij})\} \leq \max_{i=1}^n \{1 - |x_i - a_{ij}|\} \leq S_D\{\min(x_{i_1}, a_{i_1 j}), \min(x_{i_2}, a_{i_2 j})\}$.

Proof. First inequality is the same which was proved in Lemma 1. Also, second inequality is obtained by repeating Lemma 3.

Now, let $i_1, i_2 \in I$ such that $\min(x_{i_1}, a_{i_1 j}) \neq 0$ and $\min(x_{i_2}, a_{i_2 j}) \neq 0$. By Theorem 1,

$$\max_{i=1}^{i_2} \{\min(x_i, a_{ij})\} \leq \max_{i=1}^{i_2} \{1 - |x_i - a_{ij}|\} \leq S_D\{\min(x_{i_1}, a_{i_1 j}), \min(x_{i_2}, a_{i_2 j})\}.$$

Then, for some $\lambda \in [0, 1]$, we have: $\lambda \max_{i=1}^{i_2} \{\min(x_i, a_{ij})\} + (1 - \lambda) S_D\{\min(x_{i_1}, a_{i_1 j}), \min(x_{i_2}, a_{i_2 j})\} = \max_{i=1}^{i_2} \{1 - |x_i - a_{ij}|\}$ in which :

$$\lambda = \frac{\max_{i=i_1}^{i_2} \{1 - |x_i - a_{ij}|\} - \max_{i=i_1}^{i_2} \{\min(x_i, a_{ij})\}}{S_D \{\min(x_i, a_{ij})\} - \max_{i=i_1}^{i_2} \{\min(x_i, a_{ij})\}} \quad (*)$$

Now, by statements above, we define $S_{a,b}(a, b) : [0, 1]^2 \rightarrow [0, 1]$ such that $S_{a,b}(a, b) = \lambda \max\{\min(a, b)\} + (1 - \lambda)S_D(a, b)$ in which λ is found by (*). It is easy to verify that $S_{a,b}(a, b)$ is really a t-conorm. Also, by extension of $S_{a,b}(a, b)$ for n arguments and usage of it instead of maximum operator in Problem (2), we have:

$$S_M^n \{\min(x_i, a_{ij})\} = \lambda \max_{i=1}^n \{\min(x_i, a_{ij})\} + (1 - \lambda) S_D^n \{\min(x_i, a_{ij})\}$$

in which $M = \{\min(x_i, a_{ij}), \forall i \in I\}$ and λ is found by equality below:

$$\lambda = \frac{\max_{i=1}^n \{1 - |x_i - a_{ij}|\} - \max_{i=1}^n \{\min(x_i, a_{ij})\}}{S_D^n \{\min(x_i, a_{ij})\} - \max_{i=1}^n \{\min(x_i, a_{ij})\}} \quad (**)$$

However, if assumption in Theorem 1 hold, statements above show that there exists t-conorm $S_{a,b}(a, b)$ such that $S_M^n \{\min(x_i, a_{ij})\} = \max_{i=1}^n \{1 - |x_i - a_{ij}|\}, \forall j \in J$. Now, we have proved theorem below.

Theorem 2. Consider Problem (2) and suppose for each $j \in J$ there exist at least two $i_1, i_2 \in I$ such that $\min(x_{i_1}, a_{i_1j}) \neq 0$ and $\min(x_{i_2}, a_{i_2j}) \neq 0$. Define $S_{a,b}(a, b) : [0, 1]^2 \rightarrow [0, 1]$ such that

$S_{a,b}(a, b) = \lambda \max\{\min(a, b)\} + (1 - \lambda)S_D(a, b)$ and

$$\lambda = \frac{\max_{i=i_1}^{i_2} \{1 - |x_i - a_{ij}|\} - \max_{i=i_1}^{i_2} \{\min(x_i, a_{ij})\}}{S_D \{\min(x_i, a_{ij})\} - \max_{i=i_1}^{i_2} \{\min(x_i, a_{ij})\}}$$

Then, $S_{a,b}(a, b)$ is a t-conorm and $S_M^n \{\min(x_i, a_{ij})\} = \max_{i=1}^n \{1 - |x_i - a_{ij}|\}, \forall j \in J$, in which $M = \{\min(x_i, a_{ij}), \forall i \in I\}$ and λ is found by

$$\lambda = \frac{\max_{i=1}^n \{1 - |x_i - a_{ij}|\} - \max_{i=1}^n \{\min(x_i, a_{ij})\}}{S_D^n \{\min(x_i, a_{ij})\} - \max_{i=1}^n \{\min(x_i, a_{ij})\}}.$$

Theorem 3. Consider Problem (2) and Suppose for each $j \in J$ there exists at most one $i_j \in I$ such that $\min(x_{i_j}, a_{i_j j}) \neq 0$. Then, for each t-conorm S and each t-norm T , we have

$$\bigvee_{i=1}^n \{t(x_i, a_{ij})\} \leq \bigwedge_{i=1}^n \{1 - |x_i - a_{ij}|\}, \forall j \in J.$$

Proof. Since for each $a, b \in [0, 1]$, we have $T(a, b) \leq \min(a, b)$ and $S(a, b) \leq S_D(a, b)$ for each t-norm T and each t-conorm S , it is sufficient to prove $\bigvee_{i=1}^n \{\min(x_i, a_{ij})\} < \bigwedge_{i=1}^n \{1 - |x_i - a_{ij}|\}$. At first, suppose that for a fixed $j \in J$, we have $\min(x_i, a_{ij}) = 0, \forall i \in I$. Then, $\bigvee_{i=1}^n \{\min(x_i, a_{ij})\} = 0$ which implies inequality. In another case, suppose for a fixed $j \in J$ there exists exactly one $i_j \in I$ such that $\min(x_{i_j}, a_{i_j j}) \neq 0$. In this case, $\bigvee_{i=1}^n \{\min(x_i, a_{ij})\} = \min(x_{i_j}, a_{i_j j})$. Now, by (5) we have:

$$\bigvee_{i=1}^n \{\min(x_i, a_{ij})\} = \min(x_{i_j}, a_{i_j j}) \leq 1 - |x_{i_j} - a_{i_j j}| \leq \bigwedge_{i=1}^n \{1 - |x_i - a_{ij}|\}$$

that requires the inequality.

Corollary 2. Consider Problem (2) and consider a fixed $j \in J$. If two following conditions hold, then $\bigvee_{i=1}^n \{T(x_i, a_{ij})\} < \bigwedge_{i=1}^n \{1 - |x_i - a_{ij}|\}$.

- (a) For each $j \in J$ there exists $i \in I$ such that $\max(x_i, a_{ij}) < 1$.
 (b) $\min(x_i, a_{ij}) = 0, \forall i \in I$.

Proof. Since $\bigvee_{i=1}^n \{T(x_i, a_{ij})\} \leq \bigvee_{i=1}^n \{\min(x_i, a_{ij})\}$ for each t-norm T and each t-conorm S , it is sufficient to prove $\bigvee_{i=1}^n \{\min(x_i, a_{ij})\} < \bigwedge_{i=1}^n \{1 - |x_i - a_{ij}|\}$. By assumption (b), $\bigvee_{i=1}^n \{\min(x_i, a_{ij})\} = 0$. Also, from Theorem 3, we have $\bigvee_{i=1}^n \{t(x_i, a_{ij})\} \leq \bigwedge_{i=1}^n \{1 - |x_i - a_{ij}|\}$. Then, to prove the corollary, it is sufficient to show $\bigvee_{i=1}^n \{t(x_i, a_{ij})\} \neq \bigwedge_{i=1}^n \{1 - |x_i - a_{ij}|\}$. By contrary, suppose $\bigwedge_{i=1}^n \{1 - |x_i - a_{ij}|\} = \bigvee_{i=1}^n \{t(x_i, a_{ij})\} = 0$. Therefore, $1 - |x_i - a_{ij}| = 0, \forall i \in I$, which implies $\min(x_i, a_{ij}) = 1 - |x_i - a_{ij}|, \forall i \in I$. Thus, $\max(x_i, a_{ij}) = 1, \forall i \in I$, which contradicts assumption (a). Then, the proof is complete.

Briefly, the amount of the accuracy of vector b resulted from the direct solution of FRI problem $x \circ A = b$ modeling an applied topic can be categorized as follows:

- If $\max\{x_i, a_{ij}\} = 1, \forall i \in I$ and $\forall j \in J$, Problem (2) defined by minimum t-norm yields exact solution (see Lemma 2 part (b)). In this case, Problem (2) is formulated as $x \circ a_j = \bigwedge_{i=1}^n \{\min(x_i, a_{ij})\} = b_j, \forall j \in J$.

- If for each $j \in J$ there exist at least two $i_1, i_2 \in I$ such that $\min(x_{i_1}, a_{i_1j}) \neq 0$ and $\min(x_{i_2}, a_{i_2j}) \neq 0$, then Problem (2) defined by minimum t-norm and $S_{a,b}$ t-conorm given in Theorem 2, yields exact solution (See Theorems 1 and 2). Here, Problem (2) is formulated as: $x \circ a_j = S_M^n \{\min(x_i, a_{ij})\} = b_j, \forall j \in J$.
- However, there are conditions in which we cannot obtain reasonable solutions;
 - If necessary condition stated in Lemma 2 part (b) does not hold, then Problem (2) defined by minimum t-norm fails to provide a rational solution for vector b .
 - If conditions expressed in Corollary 1 hold, we have:

$$\max_{i=1}^n \{T(x_i, a_{ij})\} < \max_{i=1}^n \{1 - |x_i - a_{ij}|\}, \forall j \in J$$

which implies that Problem (2) defined by any t-norm does not yield a justifiable solution b .

- If conditions in Corollary 2 hold, we have:

$$\sum_{i=1}^n \{T(x_i, a_{ij})\} < \max_{i=1}^n \{1 - |x_i - a_{ij}|\}, \forall j \in J$$

which requires Problem (2) defined by each t-norm and each t-conorm will attain a solution farther than one we expect. Also, in this case, the best solution of modified Problem (2) is attained by problem below:

$$S_D^n \{\min(x_i, a_{ij})\} = b_j, \forall j \in J$$

Then, the accuracy of each modification of Problem (2) is bounded above.

Therefore, Problem (2) may not guarantee that vector b is the same as we expect. Moreover, in a worse case, it may exist an example for which the results of Problem (2) in direct solution is not even a good approximation for expected rational consequences.

Conclusion

In this paper, we investigated direct solution of FRE and compared their results with expected real consequences. It was shown that FRE defined by maximum t-conorm and an arbitrary t-norm yields different interpretations. A necessary condition and a sufficient condition were presented that guarantee FRE defined by maximum t-conorm and minimum t-norm attain the same solutions as human mind does. Also, we presented a t-conorm and use it instead of maximum t-conorm in FRE to obtain solutions with highest similarity with real ones. Moreover, we showed that under some conditions, FRE defined by any t-conorm and any t-norm may find a solution which is not reasonable.

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