Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

Approximating the Number of Lattice Points inside a Regular Polygon

Mahdi Imanparast *[1]

[1]Department of Computer Science, University of Bojnord.

ABSTRACT

We study the problem of counting the number of lattice points inside a regular polygon with n sides when its center is at the origin, and present an exact algorithm with $\mathcal{O}(k^2 \log n)$ time and two approximate answers for this problem, where k is the absolute value of side length of the minimum bounding box of the regular polygon. Numerical results show the efficiency of the approximations in calculating the answer to this problem.

Keyword: Guass's circle problem, lattice points, regular polygons, approximation algorithms.

ARTICLE INFO

Article history:
Research paper
Received 04, April 2022
Received in revised form 02, MAY 2022
Accepted 27 May 2022
Available online 01, June 2022

AMS subject Classification: 68U05, 68W25, 68Rxx.

1 Introduction

One of the oldest and most interesting problems in mathematics is the Gauss's circle problem, which is due to Carl Friedrich Gauss [6, 8]. This problem determines how many integer/ lattice points include inside a circle of radius R whose center is at the origin. Lattice points are points like (m, n) in the plane, where its both coordinates m and n are integer numbers. The Gauss's circle problem is interesting since it relates to the number of ways in which an integer can be expressed as the sum of two squares [2], and due to the structure of the problem, it has been also considered in number theory [3, 4]. This problem has also been studied in higher dimensions [1, 7]. An approximate answer to

*Corresponding author: M. Imanparast . Email: m.imanparast@ub.ac.ir

this problem is the area inside a circle, because on average, each unit square contains one lattice point. Therefore, the number of lattice points inside a circle is approximately equal to its area. Hence, one of the issues related to the Gauss's circle problem is finding a bound to determine the exact amount of error that expresses the difference between the number of lattice points inside the circle and the value is calculated by its area.
In this paper, we study a similar problem in regular polygons and provide two approximate solutions to it, and in several examples, the difference between these answers and the exact value is compared. The rest of the paper is organized as follows. Subsection 1.1 provides a background on how to accurately calculate the number of lattice points inside a circle. In section 2, to solving the problem of counting the number of lattice points inside a regular polygon, an algorithm to compute the exact solution and two approximate approaches are presented. Finally, section 3 includes the numerical results of the proposed approximate solutions on some regular polygons, and section 4 is conclusion.
1 INTRODUCTION

1.1 Background

In this subsection, we study a background of how to accurately calculate the number of lattice points inside a circle and present a relation that we will use it in the next sections. Let $\mathcal{N}(R)$ represents the number of lattice points inside a circle of radius R whose center is at origin in the two-dimensional space. As mentioned in the previous section, an approximation for the number of points inside the circle is area of the circle, such that $\mathcal{N}(R) = \pi R^2 + E(R)$, and calculating a bound for the error $E(R)$ is one of the topics that has been studied over the years [1]. But a solution can be provided as follows to calculate the exact number of integer points inside a circle. To this end, first consider the number of integer points within an interval containing the origin. It is clear that in the interval $[-R, R]$, for interval $[-R, 0)$, there are $\lfloor R \rfloor$ integer points, and likewise in the interval $(0, R]$, there are $\lfloor R \rfloor$ integer points.
Therefore, considering the origin, the number of integer points in the interval $[-R, R]$ will be:

$$\mathcal{N}_1(R) = 2\lfloor R \rfloor + 1. \tag{1}$$

According to the Figure 1, we can see that for an integer point x on the x-axis which is inside the circle, there is an interval of integer points as $[-\sqrt{R^2 - x^2}, \sqrt{R^2 - x^2}]$. Since we know the number of integer points within an interval according to Equation (1), so the number of integer points inside a circle of radius R in $\mathbb{R}^2$ would be as follows:

$$\mathcal{N}(R) = \sum_{x=-\lfloor R \rfloor}^{\lfloor R \rfloor} \mathcal{N}_1(\lfloor \sqrt{R^2 - x^2} \rfloor) = \sum_{x=-\lfloor R \rfloor}^{\lfloor R \rfloor} 2\lfloor \sqrt{R^2 - x^2} \rfloor + 1. \tag{2}$$

We will use this relation for present an approximation approach for the problem of lattice points inside regular polygons in the next section.

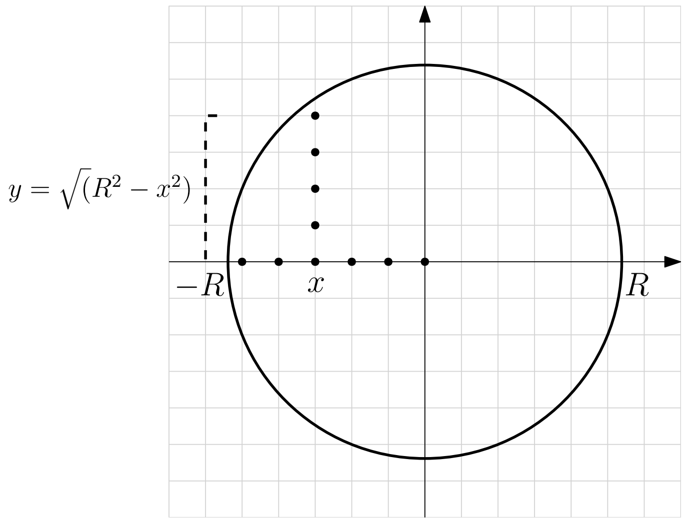

Figure 1: Calculating the exact value of the number of integer points inside a circle with radius R and the center located at the origin.

2 Main results

In this section, we present an exact algorithm and two approximate approaches for calculating the number of integer points inside a regular polygon. Consider a regular polygon in Figure 2 and its inscribed circle. Let we indicate the number of lattice points inside a regular polygons P with n sides of length s by $\mathcal{N}(P_{n,s})$.

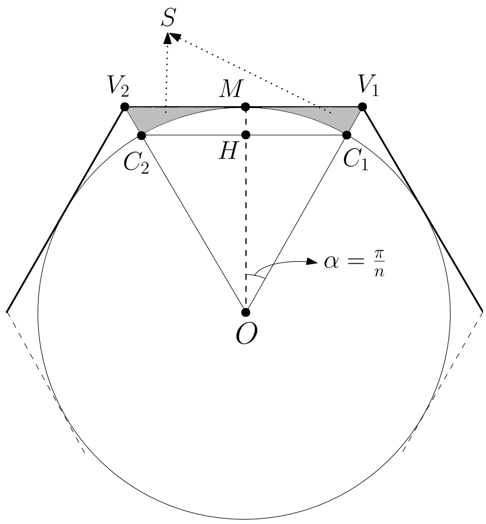

Figure 2: A regular polygon with n sides of side length $s = |V_1V_2|$, and its inscribed circle with radius $R = |OM|$.

2.1 An exact algorithm for regular polygons

In this subsection, we present an exact algorithm for calculating the number of lattice points inside a regular polygon P. To this end, we first compute the minimum bounding box B which includes the regular polygon. If the side length of the box B be K, then, number of lattice points inside B is k^2, when $k = \lfloor K \rfloor$. We can easily list these k^2 lattice points, because these points are integer points between the lowest and highest value of the two sides of box B in x and y coordinates. Then, the exact number of integer points inside a regular polygon with n sides can be calculated by doing inclusion-test in convex regular polygon for all the integer points inside box B. Since, we have k^2 integer points in box B, and we need $O(\log n)$ time for each point inside the box B to do the inclusion-test by the convex polygon [5], so we need $\mathcal{O}(k^2 \log n)$ time to do the inclusion-test for all k^2 points inside the box B. Therefore, the total time of the algorithm is $\mathcal{O}(k^2 \log n)$.
As can be seen, the time of calculation of the exact value $\mathcal{N}(P_{n,s})$ has a logarithmic relation to the value n and a quadratic relation to the value of k. Therefore, as the number of regular polygon sides n, as well as the size of the polygon increases, the running time of the algorithm will increase and it may not be efficient to use. Hence, finding the approximate solutions to this problem will be important.

2.2 Approximation 1: Use the regular polygon area

The idea we use in our approximations is that any integer point can be considered to correspond to a unit square with the center of the integer point. Therefore, we consider the number of unit squares in the area as an approximation of the number of integer points.
Consider a regular polygon with n sides such that the length of each side is s. We know that the area of this regular polygon is given as follows:

$$A = \frac{aP}{2}$$

where $P = ns$ is perimeter of the regular polygon, and $a = R = \frac{s}{2\tan\frac{\pi}{n}}$ are height of the regular polygon or radius of its inscribed circle (see Figure 2). Therefore, an approximation for the number of integer points inside a regular polygon with center in origin can be considered as follows:

$$\tilde{\mathcal{N}_1}(P_{n,s}) = \frac{nsR}{2}. \tag{3}$$

Therefore, we consider $\tilde{\mathcal{N}_1}(P_{n,s})$ as the first approximation for the number of integer points inside a regular polygon.

2.3 Approximation 2: Use the area of the inscribed circle

Consider a regular polygon and its inscribed circle as shown in Figure 2. It is clear that the difference between the integer points inside a regular polygon and its inscribed circle depends on the number of integer points inside the difference area between them. Let the length of each side of the regular polygon be s, and radius of its inscribed circle be R. Then, we have:

$$s = |V_1V_2| = 2R\tan\alpha, \qquad |C_1C_2| = 2R\sin\alpha$$
$$|OH| = R\cos\alpha, \qquad |OM| = R.$$

On the other hand, the area of the quadrilateral $\square(C_1V_1V_2C_2)$ can be calculated as follows:

$$Area(\square(C_1V_1V_2C_2)) = Area(\triangle(OV_1V_2)) - Area(\triangle(OC_1C_2)).$$

And for two triangles $\triangle(OV_1V_2)$ and $\triangle(OC_1C_2)$, we have:

$$Area(\triangle(OV_1V_2)) = \frac{1}{2}|V_1V_2||OM| = \frac{1}{2}(2R\tan\alpha)(R) = R^2\tan\alpha$$
$$Area(\triangle(OC_1C_2)) = \frac{1}{2}|C_1C_2||OH| = \frac{1}{2}(2R\sin\alpha)(R\cos\alpha) = R^2\sin\alpha\cos\alpha = R^2\frac{1}{2}\sin 2\alpha.$$

Therefore,

$$Area(\square(C_1V_1V_2C_2)) = Area(\triangle(OV_1V_2)) - Area(\triangle(OC_1C_2)).$$
$$= R^2\tan\alpha - R^2\sin\alpha\cos\alpha$$
$$= R^2(\tan\alpha - \sin\alpha\cos\alpha)$$
$$= R^2(\tan\alpha - \frac{1}{2}\sin 2\alpha).$$

Also, the area of a circular segment $O(C_1MC_2)$ with angle 2α is:

$$O(C_1MC_2) = \frac{1}{2}(2\alpha - \sin 2\alpha)R^2 = R^2(\alpha - \frac{1}{2}\sin 2\alpha)$$

Therefore, the area of the difference S (see Figure 2) between a regular polygon and its inscribed circle in the circular sector OV_1V_2 is:

$$Area(S) = Area(\square(C_1V_1V_2C_2)) - Area(O(C_1MC_2))$$
$$= R^2[(\tan\alpha - \frac{1}{2}\sin 2\alpha) - (\alpha - \frac{1}{2}\sin 2\alpha)]$$
$$= R^2[\tan\alpha - \frac{1}{2}\sin 2\alpha - \alpha + \frac{1}{2}\sin 2\alpha]$$
$$= R^2(\tan\alpha - \alpha)$$

On the other hand, according to Equation (2), the exact number of lattice points inside the inscribed circle is:

$$\mathcal{N}(R) = \sum_{x=-\lfloor R\rfloor}^{\lfloor R\rfloor} 2\lfloor\sqrt{R^2 - x^2}\rfloor + 1.$$

Therefore, the number of lattice points inside a regular polygon with n sides of length s, and center at origin can be written as follows:

$$\begin{aligned}\tilde{\mathcal{N}}_2(P_{n,s}) &\cong nArea(S) + N(R) \\ &= nR^2(\tan\alpha - \alpha) + \sum_{x=-\lfloor R\rfloor}^{\lfloor R\rfloor} 2\lfloor\sqrt{R^2-x^2}\rfloor + 1 \\ &= nR^2(\tan\frac{\pi}{n} - \frac{\pi}{n}) + \sum_{x=-\lfloor R\rfloor}^{\lfloor R\rfloor} 2\lfloor\sqrt{R^2-x^2}\rfloor + 1\end{aligned}$$

Finally, to avoid rounding error, we can write:

$$\tilde{\mathcal{N}}_2(P_{n,s}) \cong n\lfloor R^2(\tan\frac{\pi}{n} - \frac{\pi}{n}) + 0.5\rfloor + \sum_{x=-\lfloor R\rfloor}^{\lfloor R\rfloor} 2\lfloor\sqrt{R^2-x^2}\rfloor + 1. \tag{4}$$

Therefore, we consider $\tilde{\mathcal{N}}_2(P_{n,s})$ as the second approximation for the number of integer points inside a regular polygon.

3 Numerical results

In this section, we use two approximations in Equations (3) and (4) for several regular polygons and compare the results with the exact value of $\mathcal{N}(P_{n,s})$.

Example 1: Consider a regular polygon in Figure 3 with $n = 6, s = 4$. So, we have:

$$R = \frac{s}{2\tan\frac{\pi}{6}} = 3.46.$$

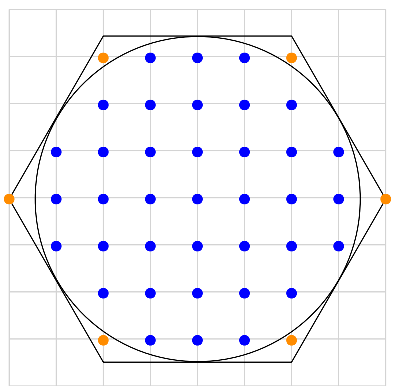

Figure 3: A regular polygon with $n = 6$ sides and $s = 4$.

As shown in Figure 3, the exact value of the number of integer points inside this regular polygon is:

$$\mathcal{N}(P_{6,4}) = 43.$$

Now, given Equations (3) and (4), we will have the following two approximations:

$$\tilde{\mathcal{N}_1}(P_{6,4}) = \frac{nsR}{2} = \frac{(6)(4)(3.46)}{2} = 41.52,$$

$$\tilde{\mathcal{N}_2}(P_{6,4}) = 6\lfloor (3.46)^2 (\tan\frac{\pi}{6} - \frac{\pi}{6}) + 0.5 \rfloor + \sum_{x=-\lfloor 3.46 \rfloor}^{\lfloor 3.46 \rfloor} (2\lfloor \sqrt{3.46^2 - x^2} \rfloor + 1)$$
$$= 6(1) + 37 = 6 + 37 = 43.$$

As can be seen, the second approximation $\tilde{\mathcal{N}_2}(P_{6,4})$ provides a better approximate answer which is equal to the exact value 43.

Example 2: Consider a regular polygon in Figure 4 with $n = 7, s = 4.81$. So, we have:

$$R = \frac{s}{2\tan\frac{\pi}{7}} = 5.$$

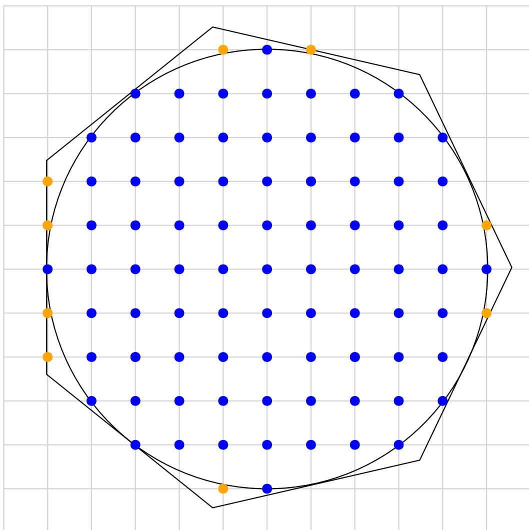

Figure 4: A regular polygon with $n = 7$ sides and $s = 4.81$.

As shown in Figure 4, the exact value of the number of integer points inside this regular polygon is:

$$\mathcal{N}(P_{7,4.81})) = 90.$$

Now, given Equations (3) and (4), we will have the following two approximations:

$$\tilde{\mathcal{N}}_1(P_{7,4.81}) = \frac{nsR}{2} = \frac{(7)(4.81)(5)}{2} = 84.17,$$

$$\tilde{\mathcal{N}}_2(P_{7,4.81}) = 7\lfloor (5)^2(\tan\frac{\pi}{7} - \frac{\pi}{7}) + 0.5\rfloor + \sum_{x=-\lfloor 5\rfloor}^{\lfloor 5\rfloor} (2\lfloor\sqrt{5^2 - x^2}\rfloor + 1)$$
$$= 7(\lfloor 1.319\rfloor) + 81 = 7(1) + 81 = 7 + 81 = 88.$$

As can be seen, the second approximation $\tilde{\mathcal{N}}_2(P_{7,4.81})$ provides a better approximate answer.

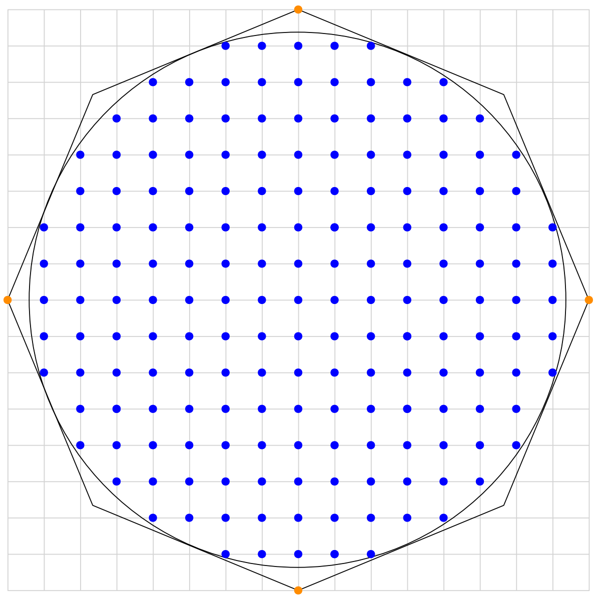

Figure 5: A regular polygon with $n = 8$ sides and $s = 6$.

Example 3: Consider a regular polygon in Figure 5 with $n = 8, s = 6$. So, we have:

$$R = \frac{s}{2\tan\frac{\pi}{8}} = 7.24.$$

As shown in Figure 5, the exact value of the number of integer points inside this regular polygon is:

$$\mathcal{N}(P_{8,6})) = 181.$$

Now, given Equations (3) and (4), we will have the following two approximations:

$$\tilde{\mathcal{N}}_1(P_{8,6}) = \frac{nsR}{2} = \frac{(8)(6)(7.24)}{2} = 173.76,$$

$$\tilde{\mathcal{N}}_2(P_{8,6}) = 8\lfloor(7.24)^2(\tan\frac{\pi}{8} - \frac{\pi}{8}) + 0.5\rfloor + \sum_{x=-\lfloor 7.24\rfloor}^{\lfloor 7.24\rfloor}(2\lfloor\sqrt{7.24^2 - x^2}\rfloor + 1)$$
$$= 8(1) + 177 = 8 + 177 = 185.$$

As can be seen, the second approximation $\tilde{\mathcal{N}}_2(P_{8,6})$ provides a better approximate answer.

Example 4: Consider a regular polygon in Figure 6 with $n = 12, s = 3.125$. So, we have:

$$R = \frac{s}{2\tan\frac{\pi}{12}} = 6,$$

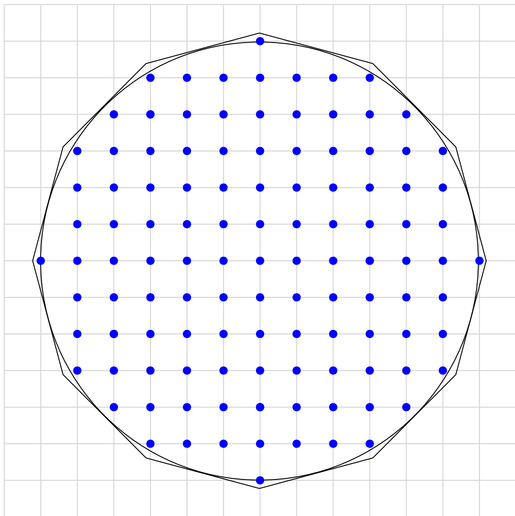

Figure 6: A regular polygon with $n = 12$ sides and $s = 3.215$.

As shown in Figure 6, the exact value of the number of integer points inside this regular polygon is:

$$\mathcal{N}(P_{12,3.215})) = 113.$$

Now, given Equations (3) and (4), we will have the following two approximations:

$$\tilde{\mathcal{N}}_1(P_{12,3.215}) = \frac{nsR}{2} = \frac{(12)(3.215)(6)}{2} = 115.74,$$

$$\tilde{\mathcal{N}}_2(P_{12,3.215}) = \lfloor 12(6)^2(\tan\frac{\pi}{12} - \frac{\pi}{12}) + 0.5\rfloor + \sum_{x=-\lfloor 6\rfloor}^{\lfloor 6\rfloor}(2\lfloor\sqrt{6^2 - x^2}\rfloor + 1)$$
$$= 12(\lfloor 0.72\rfloor) + 113 = 12(0) + 113 = 0 + 113 = 113.$$

As can be seen, the second approximation $\tilde{\mathcal{N}}_2(P_{12,3.215})$ provides a better approximate answer which is equal to the exact value 113.

The above examples show that the second approximation $\tilde{\mathcal{N}}_2(P_{n,s})$ gives better answer than the first approximation $\tilde{\mathcal{N}}_1(P_{n,s})$ for most cases.

4 Conclusion

In this paper, we study the problem of calculating $\mathcal{N}(P_{n,s})$, the number of integer points inside a regular polygon with n sides of length s and provide an exact algorithm and two approximate approaches to solve this problem. Numerical results show that the approximation $\tilde{\mathcal{N}}_2(P_{n,s})$, which is based on the calculation of the difference area and the number of lattice points inside the inscribed circle, despite the ease of calculating, provides near-optimal answer for this problem.

References

[1] Frasier, W., and Gotlieb, C. C., A Calculation of the Number of Lattice Points in the Circle and Sphere. Mathematics of Computation, 16(79), (1962), 282-290.

[2] Hardy, G. H., On the expression of a number as the sum of two squares. Quarterly Journal of Mathematics, 46, (1915), 263-283.

[3] Hirschhorn, M. D., Partial fractions and four classical theorems of number theory. The American Mathematical Monthly, 107(3), (2000), 260-264.

[4] Huxley, M. N., Integer points, exponential sums and the Riemann zeta function. In Surveys in Number Theory: Papers from the Millennial Conference on Number Theory (p. 109). AK Peters/CRC Press, (2002).

[5] Imanparast, and M., Kazemi Torbaghan, M., On Point-inclusion Test in Convex Polygons and Polyhedrons. Journal of Algorithms and Computation, 52(1), (2020), 197-207.

[6] Titchmarsh, E. C., The Lattice Points in a Circle. In Proceedings of the London Mathematical Society, 2(1), (1935), 96-115.

[7] Tsang, K., Counting Lattice Points in the Sphere. Bulletin of the London Mathematical Society, 32(6), (2000), 679-688.

[8] Wintner, A., On the Lattice Problem of Gauss. American Journal of Mathematics, 63(3), (1941), 619-627.