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An iterative method and maximal solution of Coupled algebraic Riccati equations

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ABSTRACT

Coupled Riccati equation has widely been applied to various engineering areas such as jump linear quadratic problem, particle transport theory, and Wiener–Hopf decomposition of Markov chains. In this paper, we consider an iterative method for computing Hermitian solution of the Coupled Algebraic Riccati Equations (CARE) which is usually encountered in control theory. We show some properties of this iterative method. Furthermore, it will also be demonstrated that the maximal solution can be obtained numerically via a certain linear or quadratic inequalities optimization problem. Numerical examples are presented and the results are compared.

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1 Introduction:

Coupled Riccati equation has widely been applied to various engineering areas such as jump linear quadratic problem, particle transport theory, and Wiener–Hopf decomposition of Markov chains [1, 14, 17]. In this paper, we consider the solution of CARE of the optimal control for jump linear systems. This problem was investigated in [8, 9].
Consider the following CARE:

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$$\begin{cases} \mathcal{R}_1(X_1, ..., X_N) = 0 \\ \mathcal{R}_2(X_1, ..., X_N) = 0 \\ \vdots \\ \mathcal{R}_N(X_1, ..., X_N) = 0 \end{cases}$$

for $k = 1, 2, ..., N$ where

$$\mathcal{R}_k(X_1, ..., X_N) = D_k X_k + X_k D_k - X_k S_k X_k + Q_k + \sum_{j=1, j\neq k}^{N} \lambda_{kj} X_j. \tag{1}$$

where λ_{kj} are positive real constants and $D_k, S_k, Q_k \in \mathbb{R}^{n\times n}$ are constant matrices.

For example, coupled Riccati equation (1) arises in the optimal control of the following jump linear system

$$dx(t) = A(r)x(t) + B(r)u(t), \qquad x(t_0) = x_0,$$

where $x(t)$ is an n-dimensional vector of the states of the system, $u(t)$ is a control input of dimension m, A and B are mode-dependent matrices of appropriate dimension and r is a Markovian random process representing the mode of the system and takes on values in a discrete set $\Psi = \{1, 2, ..., N\}$. The stationary transition probabilities of the modes of the system are determined by the transition rate matrix given by

$$\Pi = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{N1} & \lambda_{N2} & \dots & \lambda_{NN} \end{pmatrix},$$

where the entries λ_{ij} have properties $\lambda_{ij} \geq 0$, $i \neq j$ and $\lambda_{ii} = -\sum_{j\neq i} \lambda_{ij}$.
The performance of the given linear dynamic system is evaluated by the following criterion

$$J = \mathbf{E}\left\{\int_0^\infty \left[x^T(t)Q(r)x(t) + u^T(t)R(r)u(t)\right] dt|_{t_0, x(t_0), r(t_0)}\right\},$$

where $Q(r) \geq 0$, $R(r) > 0$ for every r. The optimal feedback controls of the mentioned problem are given by

$$u_{opt}(t) = -R_k^{-1} B_k^T X_k x(t), \qquad k = 1, 2, ..., N,$$

where the subscript k shows that the system is in mode $r = k$ and $A(r) = A_r$, $B(r) = B_r$, $Q(r) = Q_r$, $R(r) = R_r$. In $u_{opt}(t)$, the matrices $X_k (k = 1, 2, ..., N)$ are the positive semidefinite solutions of a set of the coupled algebraic Riccati equations:

$$\left(A_k + \frac{1}{2}\lambda_{kk}\right)^T X_k + X_k \left(A_k + \frac{1}{2}\lambda_{kk}\right)^T - X_k B_k R_k^{-1} B_k^T X_k + Q_k + \sum_{j=1, j\neq k}^{N} \lambda_{kj} X_j = 0,$$

and $k = 1, 2, ..., N$ (See [10, 17]).
Therefore, considering important applications of coupled algebraic Riccati equation (1), a surging number of researchers have been interested in studying this equation in recent years.
For example, some studies focused on iterative methods to solve algebraic Riccati equation. In particular, Newton's method and the fixed point iteration were used to find the minimal positive solution for the non-symmetric algebraic Riccati equation [12]. In another study, the linearized implicit iteration method was utilized for computing its minimal nonnegative solution [18]. Furthermore, the numerical solution of the projected non-symmetric algebraic Riccati equations via Newton's method [6], the matrix bounds and iterative algorithms for the coupled algebraic Riccati equation [15, 16] and the modified alternately linearized implicit iteration methods were also applied for solving this problem [11].
Some other studies attempted to demonstrate the upper or the lower solution bounds of CARE. For instance, while the lower matrix bound of the solution of the unified coupled Riccati equation was examined in [13], the upper solution bounds of the discrete algebraic Riccati matrix equation[5], the improved upper solution bounds of the continuous coupled algebraic Riccati matrix equation [20] and two new upper bounds of the solution for the continuous algebraic Riccati equation and their application were also investigated [19].
Another group of studies examined the largescale non-symmetric Riccati equations from diverse perspectives. More specifically, Krylov subspace-based methods [4], low-rank Newton-ADI methods [2] and low-rank ADI-type algorithm [3] could be mentioned as examples of such studies.
This paper is organized as follows: the next section is devoted to the statement of the numerical method based on Newton's method to solve Problem (1). Also, the convergence of this method will be proved. The aim of Section 3 is to express the problem as an equivalent optimization problem. Some numerical simulations are done in Section 4 and we conclude with some remarks in Section 5.

2 Remodified Newton's iteration method

The classical approach in iterative solution to a system of equations indicates the use of the already computed approximations to obtain the current approximation value. In [10], the following iterative approximztion has been introduced:

$$\begin{aligned}\left(D_k - S_k X_k^{(i)}\right)^T X_k^{(i+1)} + X_k^{(i+1)} \left(D_k - S_k X_k^{(i)}\right) + \sum_{j=1}^{k-1} \lambda_{kj} X_j^{(i+1)} \\ + \sum_{j=k+1}^{N} \lambda_{kj} X_j^{(i)} + X_k^{(i)} S_k X_k^{(i)} + Q_k = 0. \\ for \quad k = 1, 2, ..., N, \qquad i = 0, 1, 2, ...\end{aligned}$$

First, we define some key terms.The notation $\mathcal{H}^n$ indicates the linear space of Hermitian matrices of size n over the field of real numbers. For any $A, B \in \mathcal{H}^n$, we write $A > B$

(or $A \geq B$) if $A - B$ is positive definite (or $A - B$ is positive semidefinite). We use some properties of positive definite and positive semidefinite matrices. So, if $A > 0$ and $B > 0$, then $A + B > 0$, if $A \geq 0$ and $B \geq 0$, then $A + B \geq 0$ and if $A > 0$ and $B \geq 0$, then $A + B > 0$. The spectrum of any complex matrix A will be demonstrated by $\sigma(A)$. A matrix A is asymptotically stable if all the eigenvalues of A lie in the open left-half plane and A is stable if all eigenvalues of A lie in the closed left-half plane. For a linear operator $\mathcal{L}$ on $\mathcal{H}^n$, let $\rho(\mathcal{L}) = \max\{|\lambda| \ : \lambda \in \sigma(\mathcal{L})\}$ be the spectral radius and $\beta(\mathcal{L}) = \max\{Re(\lambda) \ : \lambda \in \sigma(\mathcal{L})\}$. $\mathcal{L}$ is called asymptotically stable if the eigenvalues to $\mathcal{L}$ lie in the open left-half plane and stable, if the eigenvalues to $\mathcal{L}$ lie in the closed left-half plane.
On the basis of the iterative method introduced in [10], we present the following remodified Newton's iteration method (RMNM).
By introducing parameter $0 \leq \omega \leq 1$, for $l = 0, 1, ...$, we have the following iterative furmula:

$$\begin{aligned}\left(D_k - S_k X_k^{(l)}\right)^T X_k^{(l+1)} + X_k^{(l+1)}\left(D_k - S_k X_k^{(l)}\right) + \sum_{j=1}^{k-1} \lambda_{kj}\left(\omega X_j^{(l+1)} + (1-\omega) X_j^{(l)}\right) \\ + \sum_{j=k+1}^{N} \lambda_{kj} X_j^{(l)} + X_k^{(l)} S_k X_k^{(l)} + Q_k = 0.\end{aligned} \tag{2}$$

When $\omega = 0$ and $\omega = 1$, RMNM is equivalent to Newton's method and method in [10], respectively.
Theorem 2.1 Suppose that there exist symmetric matrices $\widetilde{X}_k, X_k^{(0)}, \quad k = 1, ..., N$, where $\mathcal{R}_k(\widetilde{X}_1, ..., \widetilde{X}_N) \geq 0$; $X_k^{(0)} \geq \widetilde{X}_k$; $\mathcal{R}_k(X_1^{(0)}, ..., X_N^{(0)}) \leq 0$ and $D_k - S_k X_k^{(0)}$ is asymptotically stable for all $k = 1, ..., N$. Then, the matrix sequences $\{X_1^{(l)}\}_{l=1}^{\infty}, ..., \{X_N^{(l)}\}_{l=1}^{\infty}$ defined by (2) have properties:

(i) For $k = 1, ..., N$ we have $X_k^{(l)} \geq X_k^{(l+1)}$, $X_k^{(l)} \geq \widetilde{X}_k$ and $\mathcal{R}_k(X_1^{(l)}, ..., X_N^{(l)}) \leq \omega \sum_{j=1}^{k-1} \lambda_{kj}\left(X_j^{(l)} - X_j^{(l+1)}\right)$ where $l = 0, 1, 2, ...$;

(ii) $D_k - S_k X_k^{(l)}$ is asymptotically stable for $k = 1, ..., N$ and $l = 0, 1, 2, ...$;

(iii) The sequences $\{X_1^{(l)}\}, ..., \{X_N^{(l)}\}$ converge to the solution $X_1^+, ..., X_N^+$ of the equations $\mathcal{R}_k(X_1, ..., X_N) = 0$ and $X_k^+ \geq \widetilde{X}_k$ for $k = 1, ..., N$;

(iv) The matrix $D_k - S_k X_k^+$ for $k = 1, ..., N$ are stable. In addition if $\mathcal{R}_k(\widetilde{X}_1, ..., \widetilde{X}_N) > 0$ for $k = 1, ..., N$, then the matrices $D_k - S_k X_k^+$ are asymptotically stable.

Proof: Let $l = 0$. According to the theorem conditions, $X_k^{(0)} \geq \widetilde{X}_k$, $D_k - S_k X_k^{(0)}$ is asymptotically stable and $\mathcal{R}_k(X_1^{(0)}, ..., X_N^{(0)}) \leq 0$ for $k = 1, ..., N$. We will prove inequalities $X_k^{(0)} \geq X_k^{(1)}$ for $k = 1, ..., N$. From iteration (2) for $l = 0$ we get

$$\begin{aligned}\left(D_k - S_k X_k^{(0)}\right)^T X_k^{(1)} + X_k^{(1)}\left(D_k - S_k X_k^{(0)}\right) = -\sum_{j=1}^{k-1} \lambda_{kj}\left(\omega X_j^{(1)} + (1-\omega) X_j^{(0)}\right) \\ - \sum_{j=k+1}^{N} \lambda_{kj} X_j^{(0)} - X_k^{(0)} S_k X_k^{(0)} - Q_k\end{aligned}$$

and thus $X_k^{(1)}$ is the unique solution of the last equation because each matrix $D_k - S_k X_k^{(0)}$ is asymptotically stable. We get the equality

$$\begin{aligned}\left(D_k - S_k X_k^{(0)}\right)^T \left(X_k^{(1)} - X_k^{(0)}\right) + \left(X_k^{(1)} - X_k^{(0)}\right)\left(D_k - S_k X_k^{(0)}\right) \\ = -\mathcal{R}_k\left(X_1^{(0)}, ..., X_N^{(0)}\right) - \omega \sum_{j=1}^{k-1} \lambda_{kj}\left(X_j^{(1)} - X_j^{(0)}\right)\end{aligned} \tag{3}$$

for $k = 1, ..., N$. In the last equation, for $k = 1$, the right-hand side $\left(-\mathcal{R}_1\left(X_1^{(0)}, ..., X_N^{(0)}\right)\right)$ is positive semidefinite and its solution $X_1^{(1)} - X_1^{(0)}$ is negative semidefinite, which means $X_1^{(0)} \geq X_1^{(1)}$. Consider (3) where $k = 2$, for the right-hand side, $-\mathcal{R}_2\left(X_1^{(0)}, ..., X_N^{(0)}\right) - \omega\lambda_{21}\left(X_1^{(1)} - X_1^{(0)}\right) \geq 0$ and matrix $\left(D_2 - S_2 X_2^{(0)}\right)$ is asymptotically stable. Then, solution $X_2^{(1)} - X_2^{(0)}$ is negative semidefinite, or $X_2^{(0)} \geq X_2^{(1)}$. Following similar arguments, it is proved that $X_k^{(0)} \geq X_k^{(1)}$ for $k = 3, ..., N$.

Now, assume that there exists a natural number $l = r - 1$ and the matrix sequences $\{X_1^{(l)}\}_0^r, ..., \{X_N^{(l)}\}_0^r$ are computed and properties (i) and (ii) are observed, i.e. for $k = 1, ..., N$, $X_k^{(r-1)} \geq X_k^{(r)}$, $X_k^{(r-1)} \geq \widetilde{X}_k$ and $\mathcal{R}_k(X_1^{(r-1)}, ..., X_N^{(r-1)}) \leq \omega \sum_{j=1}^{k-1} \lambda_{kj}\left(X_j^{(r-1)} - X_j^{(r)}\right)$ and $D_k - S_k X_k^{(r-1)}$ are asymptotically stable. We will show that for $k = 1, ..., N$, the following statements are true:

$X_k^{(r)} \geq \widetilde{X}_k$ and $D_k - S_k X_k^{(r)}$ are asymptotically stable, we will show how to compute each $X_k^{(r+1)}$ and that inequalities $X_k^{(r)} \geq X_k^{(r+1)}$ hold, and finally, we will prove inequalities $\mathcal{R}_k(X_1^{(r)}, ..., X_N^{(r)}) \leq \omega \sum_{j=1}^{k-1} \lambda_{kj}\left(X_j^{(r)} - X_j^{(r+1)}\right)$.

We start with inequalities $X_k^{(r)} \geq \widetilde{X}_k$ for $k = 1, ..., N$. Using $\mathcal{R}_k(\widetilde{X}_1, ..., \widetilde{X}_N)$ and the inequality $\mathcal{R}_k(\widetilde{X}_1, ..., \widetilde{X}_N) \geq 0$, we get

$$\begin{aligned}&\left(D_k - S_k X_k^{(r-1)}\right)^T \left(X_k^{(r)} - \widetilde{X}_k\right) + \left(X_k^{(r)} - \widetilde{X}_k\right)\left(D_k - S_k X_k^{(r-1)}\right) \\ &= -X_k^{(r-1)} S_k X_k^{(r-1)} - Q_k - \sum_{j=1}^{k-1} \lambda_{kj}\left(\omega X_j^{(r)} + (1-\omega) X_j^{(r-1)}\right) - \sum_{j=k+1}^{N} \lambda_{kj} X_j^{(r-1)} \\ &- \left(D_k - S_k X_k^{(r-1)}\right)^T \widetilde{X}_k - \widetilde{X}_k\left(D_k - S_k X_k^{(r-1)}\right) \\ &= -X_k^{(r-1)} S_k X_k^{(r-1)} - Q_k - \sum_{j=1}^{k-1} \lambda_{kj}\left(\omega X_j^{(r)} + (1-\omega) X_j^{(r-1)}\right) - \sum_{j=k+1}^{N} \lambda_{kj} X_j^{(r-1)} \\ &- D_k^T \widetilde{X}_k - \widetilde{X}_k D_k + X_k^{(r-1)} S_k \widetilde{X}_k + \widetilde{X}_k S_k X_k^{(r-1)} \\ &= -X_k^{(r-1)} S_k X_k^{(r-1)} - \sum_{j=1}^{k-1} \lambda_{kj}\left(\omega X_j^{(r)} + (1-\omega) X_j^{(r-1)}\right) - \sum_{j=k+1}^{N} \lambda_{kj} X_j^{(r-1)} \\ &+ X_k^{(r-1)} S_k \widetilde{X}_k + \widetilde{X}_k S_k X_k^{(r-1)} + \sum_{j \neq k} \lambda_{kj} \widetilde{X}_k - \widetilde{X}_k S_k \widetilde{X}_k - \mathcal{R}_k\left(\widetilde{X}_1, ..., \widetilde{X}_N\right).\end{aligned}$$

We obtain equality

$$\begin{aligned}&\left(D_k - S_kX_k^{(r-1)}\right)^T\left(X_k^{(r)} - \widetilde{X}_k\right) + \left(X_k^{(r)} - \widetilde{X}_k\right)\left(D_k - S_kX_k^{(r-1)}\right)\\ &= -\left(X_k^{(r-1)} - \widetilde{X}_k\right)S_k\left(X_k^{(r-1)} - \widetilde{X}_k\right) - \sum_{j=1}^{k-1}\lambda_{kj}\left(\omega X_j^{(r)} + (1-\omega)X_j^{(r-1)} - \widetilde{X}_j\right)\\ &-\sum_{j=k+1}^{N}\lambda_{kj}\left(X_j^{(r-1)} - \widetilde{X}_j\right) - \mathcal{R}_k\left(\widetilde{X}_1,...,\widetilde{X}_N\right).\end{aligned} \tag{4}$$

Thus, for $k = 1$, for the right-hand side (4), we have

$$-\left(X_1^{(r-1)} - \widetilde{X}_1\right)S_1\left(X_1^{(r-1)} - \widetilde{X}_1\right) - \sum_{j=2}^{N}\lambda_{kj}\left(X_j^{(r-1)} - \widetilde{X}_j\right) - \mathcal{R}_1\left(\widetilde{X}_1,...,\widetilde{X}_N\right) \leq 0$$

and thus solution $\left(X_1^{(r)} - \widetilde{X}_1\right)$ to (4) is a positive semidefinite matrix or $X_1^r \geq \widetilde{X}_1$. We know $X_1^{(r-1)} \geq X_1^{(r)}$; so, $\omega X_1^{(r)} + (1-\omega)X_1^{(r-1)} \geq \widetilde{X}_1$. For $k = 2$, the right-hand side (4) is

$$\begin{aligned}&-\left(X_2^{(r-1)} - \widetilde{X}_2\right)S_2\left(X_2^{(r-1)} - \widetilde{X}_2\right) - \lambda_{21}\left(\omega X_1^{(r)} + (1-\omega)X_1^{(r-1)} - \widetilde{X}_1\right)\\ &-\sum_{j=3}^{N}\lambda_{kj}\left(X_j^{(r-1)} - \widetilde{X}_j\right) - \mathcal{R}_2\left(\widetilde{X}_1,...,\widetilde{X}_N\right) \leq 0\end{aligned}$$

and hence, $X_2^{(r)} - \widetilde{X}_2 \geq 0$. Inequalities $X_k^{(r)} - \widetilde{X}_k \geq 0$ for $k = 3,...,N$ are established in a similar way.
We will prove that all matrices $D_k - S_kX_k^{(r)}, (k = 1,...,N)$ are asymptotically stable. Writing

$$D_k - S_kX_k^{(r)} = D_k - S_kX_k^{(r-1)} + S_k\left(X_k^{(r-1)} - X_k^{(r)}\right)$$

we compute,

$$\begin{aligned}&\left(D_k - S_kX_k^{(r)}\right)^T\left(X_k^{(r)} - \widetilde{X}_k\right) + \left(X_k^{(r)} - \widetilde{X}_k\right)\left(D_k - S_kX_k^{(r)}\right)\\ &= \left(D_k - S_kX_k^{(r-1)}\right)^T\left(X_k^{(r)} - \widetilde{X}_k\right) + \left(X_k^{(r)} - \widetilde{X}_k\right)\left(D_k - S_kX_k^{(r-1)}\right)\\ &+\left(X_k^{(r-1)} - X_k^{(r)}\right)S_k\left(X_k^{(r)} - \widetilde{X}_k\right) + \left(X_k^{(r)} - \widetilde{X}_k\right)S_k\left(X_k^{(r-1)} - X_k^{(r)}\right)\\ &\overset{(4)}{\leq} -\sum_{j=1}^{k-1}\lambda_{kj}\left(\omega X_j^{(r)} + (1-\omega)X_j^{(r-1)} - \widetilde{X}_j\right) - \sum_{j=k+1}^{N}\lambda_{kj}\left(X_j^{(r-1)} - \widetilde{X}_j\right)\\ &-\left(X_k^{(r-1)} - \widetilde{X}_k\right)S_k\left(X_k^{(r-1)} - \widetilde{X}_k\right) + \left(X_k^{(r-1)} - X_k^{(r)}\right)S_k\left(X_k^{(r)} - \widetilde{X}_k\right)\\ &+\left(X_k^{(r)} - \widetilde{X}_k\right)S_k\left(X_k^{(r-1)} - X_k^{(r)}\right)\\ &= -\sum_{j=1}^{k-1}\lambda_{kj}\left(\omega X_j^{(r)} + (1-\omega)X_j^{(r-1)} - \widetilde{X}_j\right) - \sum_{j=k+1}^{N}\lambda_{kj}\left(X_j^{(r-1)} - \widetilde{X}_j\right)\\ &-\left(X_k^{(r)} - \widetilde{X}_k\right)S_k\left(X_k^{(r)} - \widetilde{X}_k\right) - \left(X_k^{(r-1)} - X_k^{(r)}\right)S_k\left(X_k^{(r-1)} - X_k^{(r)}\right)\\ &\leq -\left(X_k^{(r-1)} - X_k^{(r)}\right)S_k\left(X_k^{(r-1)} - X_k^{(r)}\right)\end{aligned}$$

Conclusively,

$$\left(D_k - S_k X_k^{(r)}\right)^T \left(X_k^{(r)} - \widetilde{X}_k\right) + \left(X_k^{(r)} - \widetilde{X}_k\right)\left(D_k - S_k X_k^{(r)}\right) \leq -\left(X_k^{(r-1)} - X_k^{(r)}\right) S_k \left(X_k^{(r-1)} - X_k^{(r)}\right).$$

Let us assume that there is a number k so that $D_k - S_k X_k^{(r)}$ is not asymptotically stable. Thus, there exists an eigenvalue λ of $D_k - S_k X_k^{(r)}$ with $Re(\lambda) \geq 0$ and a nonzero eigenvector x with $\left(D_k - S_k X_k^{(r)}\right) x = \lambda x$. Through the last inequality, we get

$$0 \leq 2Re(\lambda) x^T \left(X_k^{(r)} - \widetilde{X}_k\right) x \leq -x^T \left(X_k^{(r-1)} - X_k^{(r)}\right) S_k \left(X_k^{(r-1)} - X_k^{(r)}\right) x \leq 0.$$

Hence,

$$x^T \left(X_k^{(r-1)} - X_k^{(r)}\right) S_k \left(X_k^{(r-1)} - X_k^{(r)}\right) x = 0$$
$$S_k X_k^{(r-1)} x = S_k X_k^{(r)} x.$$

Since

$$\left(D_k - S_k X_k^{(r-1)}\right) x = D_k x - S_k X_k^{(r-1)} x = D_k x - S_k X_k^{(r)} x = \left(D_k - S_k X_k^{(r)}\right) x = \lambda x,$$

λ is an eigenvalue of $D_k - S_k X_k^{(r-1)}$, which is contradictory to the c-stability of this matrix. Our assumption is not true and hence, $D_k - S_k X_k^{(r)}$ is asymptotically stable for $k = 1, ..., N$. Further, we will compute matrices $X_k^{(r+1)}$ and will prove $X_k^{(r)} \geq X_k^{(r+1)}$ for $k = 1, ..., N$. By iteration (2), for $i = r$ and for each $k = 1, ..., N$, we obtain:

$$\left(D_k - S_k X_k^{(r)}\right)^T X_k^{(r+1)} + X_k^{(r+1)} \left(D_k - S_k X_k^{(r)}\right) + \sum_{j=1}^{k-1} \lambda_{kj} \left(\omega X_j^{(r+1)} + (1-\omega) X_j^{(r)}\right)$$
$$+ \sum_{j=k+1}^{N} \lambda_{kj} X_j^{(r)} + X_k^{(r)} S_k X_k^{(r)} + Q_k = 0.$$

Since $D_k - S_k X_k^{(r)}$ is asymptotically stable, $X_k^{(r+1)}$ is the unique solution of the last equation.

Let us consider

$$\begin{aligned}
&\left(D_k - S_k X_k^{(r)}\right)^T \left(X_k^{(r)} - X_k^{(r+1)}\right) + \left(X_k^{(r)} - X_k^{(r+1)}\right)\left(D_k - S_k X_k^{(r)}\right)\\
&= \left(D_k - S_k X_k^{(r)}\right)^T X_k^{(r)} + X_k^{(r)}\left(D_k - S_k X_k^{(r)}\right) - \left(D_k - S_k X_k^{(r)}\right)^T X_k^{(r+1)} - X_k^{(r+1)}\left(D_k - S_k X_k^{(r)}\right)\\
&\overset{(2)}{=} \left(D_k - S_k X_k^{(r-1)} + S_k\left(X_k^{(r-1)} - X_k^{(r)}\right)\right)^T X_k^{(r)} + X_k^{(r)}\left(D_k - S_k X_k^{(r-1)} + S_k\left(X_k^{(r-1)} - X_k^{(r)}\right)\right)\\
&+ \sum_{j=1}^{k-1}\lambda_{kj}\left(\omega X_j^{(r+1)} + (1-\omega)X_j^{(r)}\right) + \sum_{j=k+1}^{N}\lambda_{kj}X_j^{(r)} + X_k^{(r)}S_kX_k^{(r)} + Q_k\\
&= \left(D_k - S_k X_k^{(r-1)}\right)^T X_k^{(r)} + X_k^{(r)}\left(D_k - S_k X_k^{(r-1)}\right) + \left(X_k^{(r-1)} - X_k^{(r)}\right)S_kX_k^{(r)}\\
&+ X_k^{(r)}S_k\left(X_k^{(r-1)} - X_k^{(r)}\right) + \sum_{j=1}^{k-1}\lambda_{kj}\left(\omega X_j^{(r+1)} + (1-\omega)X_j^{(r)}\right) + \sum_{j=k+1}^{N}\lambda_{kj}X_j^{(r)} + X_k^{(r)}S_kX_k^{(r)} + Q_k\\
&\overset{(2)}{=} -\omega\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r)} - X_j^{(r+1)}\right) - (1-\omega)\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r-1)} - X_j^{(r)}\right) - \sum_{j=k+1}^{N}\lambda_{kj}\left(X_j^{(r-1)} - X_j^{(r)}\right)\\
&- X_k^{(r-1)}S_kX_k^{(r-1)} + X_k^{(r-1)}S_kX_k^{(r)} - X_k^{(r)}S_kX_k^{(r)} + X_k^{(r)}S_kX_k^{(r-1)} - X_k^{(r)}S_kX_k^{(r)} + X_k^{(r)}S_kX_k^{(r)}\\
&= -\omega\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r)} - X_j^{(r+1)}\right) - (1-\omega)\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r-1)} - X_j^{(r)}\right) - \sum_{j=k+1}^{N}\lambda_{kj}\left(X_j^{(r-1)} - X_j^{(r)}\right)\\
&- \left(X_k^{(r-1)} - X_k^{(r)}\right)S_k\left(X_k^{(r-1)} - X_k^{(r)}\right).
\end{aligned}$$

After these transformations, we obtain:

$$\begin{aligned}
&\left(D_k - S_k X_k^{(r)}\right)^T \left(X_k^{(r)} - X_k^{(r+1)}\right) + \left(X_k^{(r)} - X_k^{(r+1)}\right)\left(D_k - S_k X_k^{(r)}\right)\\
&= -\omega\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r)} - X_j^{(r+1)}\right) - (1-\omega)\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r-1)} - X_j^{(r)}\right) - \sum_{j=k+1}^{N}\lambda_{kj}\left(X_j^{(r-1)} - X_j^{(r)}\right) \qquad (5)\\
&- \left(X_k^{(r-1)} - X_k^{(r)}\right)S_k\left(X_k^{(r-1)} - X_k^{(r)}\right).
\end{aligned}$$

Let $k = 1$. The right-hand side

$$-\sum_{j=2}^{N}\lambda_{1j}\left(X_j^{(r-1)} - X_j^{(r)}\right) - \left(X_1^{(r-1)} - X_1^{(r)}\right)S_1\left(X_1^{(r-1)} - X_1^{(r)}\right)$$

is a negative semidefinite matrix and hence, matrix $X_1^{(r)} - X_1^{(r+1)}$ is a positive semidefinite one. Consider (5) for $k = 2$. We know $X_1^{(r)} - X_1^{(r+1)} \geq 0$. After analogous considerations, we get $X_2^{(r)} - X_2^{(r+1)} \geq 0$. In a similar way, it is proved that $X_k^{(r)} - X_k^{(r+1)} \geq 0$ for $k = 3, ..., N$. We continue with the proof of the fact $\mathcal{R}_k(X_1^{(r)}, ..., X_N^{(r)}) \leq \omega\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r)} - X_j^{(r+1)}\right)$ where $k = 1, ..., N$. Let us consider

$$\mathcal{R}_k(X_1^{(r)}, ..., X_N^{(r)}) = \left(D_k - S_k X_k^{(r)}\right)^T X_k^{(r)} + X_k^{(r)}\left(D_k - S_k X_k^{(r)}\right) + \sum_{j=1, j\neq k}^{N}\lambda_{kj}X_j^{(r)} + X_k^{(r)}S_kX_k^{(r)} + Q_k.$$

Using iteration (2) leads us to

$$\mathcal{R}_k(X_1^{(r)},...,X_N^{(r)}) = \left(D_k - S_kX_k^{(r)}\right)^T X_k^{(r)} + X_k^{(r)}\left(D_k - S_kX_k^{(r)}\right) - \left(D_k - S_kX_k^{(r)}\right)^T X_k^{(r+1)}$$
$$-X_k^{(r+1)}\left(D_k - S_kX_k^{(r)}\right) - \omega\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r+1)} - X_j^{(r)}\right)$$

or

$$\begin{aligned}&\left(D_k - S_kX_k^{(r)}\right)^T\left(X_k^{(r)} - X_k^{(r+1)}\right) + \left(X_k^{(r)} - X_k^{(r+1)}\right)\left(D_k - S_kX_k^{(r)}\right)\\ &= \mathcal{R}_k(X_1^{(r)},...,X_N^{(r)}) + \omega\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r+1)} - X_j^{(r)}\right)\end{aligned} \tag{6}$$

In (6), we put down $k = 1$ and thus

$$\mathcal{R}_1(X_1^{(r)},...,X_N^{(r)}) = \left(D_1 - S_1X_1^{(r)}\right)^T\left(X_1^{(r)} - X_1^{(r+1)}\right) + \left(X_1^{(r)} - X_1^{(r+1)}\right)\left(D_1 - S_1X_1^{(r)}\right).$$

Since $D_1 - S_1X_1^{(r)}$ is asymptotically stable and $X_1^{(r)} - X_1^{(r+1)} \geq 0$, we conclude that $\mathcal{R}_1(X_1^{(r)},...,X_N^{(r)}) \leq 0$. With $k = 2$ in (6), we find

$$\begin{aligned}&\left(D_2 - S_2X_2^{(r)}\right)^T\left(X_2^{(r)} - X_2^{(r+1)}\right) + \left(X_2^{(r)} - X_2^{(r+1)}\right)\left(D_2 - S_2X_2^{(r)}\right)\\ &= \mathcal{R}_2(X_1^{(r)},...,X_N^{(r)}) + \omega\lambda_{21}\left(X_1^{(r+1)} - X_1^{(r)}\right),\end{aligned}$$

which means

$$\begin{aligned}&\mathcal{R}_2(X_1^{(r)},...,X_N^{(r)}) + \omega\lambda_{21}\left(X_1^{(r+1)} - X_1^{(r)}\right) \leq 0\\ &\mathcal{R}_2(X_1^{(r)},...,X_N^{(r)}) \leq \omega\lambda_{21}\left(X_1^{(r)} - X_1^{(r+1)}\right).\end{aligned}$$

The following inequalities are proved in a similar way:

$$\mathcal{R}_k(X_1^{(r)},...,X_N^{(r)}) \leq \omega\sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^{(r)} - X_j^{(r+1)}\right), \qquad k = 3,...,N.$$

The induction process for proving (i) and (ii) is now complete. Matrix sequences $\{X_1^{(l)}\}_0^\infty,...,\{X_N^{(l)}\}_0^\infty$ converge and their limit matrices $X_1^+,...,X_N^+$ complete a solution to the system of Riccati equations $\mathcal{R}_k(X_1,...,X_N) = 0$ with $x_k^+ \geq \widetilde{X}_k$ for $k = 1,...,N$. Since all matrices $D_k - S_kX_k^{(l)}(k = 1,...,N; l = 1,...)$ are asymptotically stable, corresponding limit matrices $D_k - S_kX_k^+$ are stable.

Now, we are assuming that $\mathcal{R}_k(\widetilde{X}_1,...,\widetilde{X}_N) > 0$. Reaching the limit in (4) when $r \to \infty$, we get

$$(D_k - S_kX_k^+)^T\left(X_k^+ - \widetilde{X}_k\right) + \left(X_k^+ - \widetilde{X}_k\right)(D_k - S_kX_k^+)$$
$$= -\mathcal{R}_k(\widetilde{X}_1,...,\widetilde{X}_N) - \sum_{j=1}^{k-1}\lambda_{kj}\left(X_j^+ - \widetilde{X}_j\right) - \sum_{j=k+1}^{N}\lambda_{kj}\left(X_j^+ - \widetilde{X}_j\right) - \left(X_k^+ - \widetilde{X}_k\right)S_k\left(X_k^+ - \widetilde{X}_k\right).$$

We consider the last equality consecutively for $k = 1, ..., N$. We know that $X_k^+ - \widetilde{X}_k \geq 0$ and the right-hand sides will be negative definite because $\mathcal{R}_k(\widetilde{X}_1, ..., \widetilde{X}_N)$ are positive definite. Thus, it follows (from Lyapunov equation's properties) that matrices $D_k - S_k X_k^+$ are asymptotically stable for $k = 1, ..., N$. □

3 Maximal solution

In this section, we establish a link between the optimization problem and solution X^+. Consider the following optimization programing problem:

$$\begin{aligned} &\max \qquad tr\left(\sum_{i=1}^{N} X_i\right) \\ &S.\ to: \quad \mathcal{R}_i(X_1, ..., X_N) \geq 0, \qquad for \quad i = 1, ..., N, \end{aligned} \tag{7}$$

Lemma 3.1 The maximal solution $X^+ \in \mathcal{H}^n$ for CARE (2) is the unique solution of programming problem (7). Furthermore, since D_k and X_k are symmetric, (7) can be rewritten as follows:

$$\begin{aligned} &\max \quad tr\left(\sum_{i=1}^{N} X_i\right) \\ &\begin{pmatrix} D_k X_k + X_k D_k + Q_k + \sum\limits_{j=1, j\neq k}^{N} \lambda_{kj} X_j & X_k \\ X_k & S_k^{-1} \end{pmatrix} \geq 0 \end{aligned} \tag{8}$$

Proof: By Theorem **??**, $X^+ \in \mathcal{H}^n$, $\mathcal{R}(X^+) = 0$ (thus, constraints (7) are satisfied for X^+), and $X^+ \geq \widetilde{X}$ for any $\widetilde{X}$ satisfying constraints (7), which implies that

$$tr\left(\sum_{i=1}^{N} X_i^+\right) \geq tr\left(\sum_{i=1}^{N} \widetilde{X}_i\right)$$

and so, the optimal solution is given by X^+. Since D_k are symmetric, from Schur's complement $\widetilde{X} = \left(\widetilde{X}_1, ..., \widetilde{X}_N\right)$ satisfies constraints (8) if and only if $\widetilde{X} \in \mathcal{H}^n$ and $D_k X_k + X_k D_k + Q_k + \sum\limits_{j=1, j\neq k}^{N} \lambda_{kj} X_j - X_k S_k X_k \geq 0$. □

In [17], the following CARE is considered:

$$\mathcal{R}_k = X_k C_k X_k - X_k D_k - A_k X_k + B_k + \sum_{j\neq k} e_{kj} X_j = 0, \tag{9}$$

where $k \in \{1, 2, ..., m\}$, e_{kj} are positive real constants, and $A_k, B_k, C_k, D_k \in \mathbb{R}^{n\times n}$ are constant matrices.

To state the theorem presented in [17], we start with some definitions. Let $\mathbb{R}^{n\times n}$ represent the set of $n \times n$ real matrices. For $A \in \mathbb{R}^{n\times n}$, A^T is the transpose matrix of A. We write

$A > 0 (A \geq 0)$ if matrix A is positive (nonnegative), i.e., $a_{ij} > 0 (a_{ij} \geq 0)$ for all $i, j = 1, 2, ..., n$. If $A - B$ is positive (nonnegative), then we write $A > B (A \geq B)$. $\|.\|$ defines the matrix norm. A is called a Z-matrix if all its off-diagonal elements are non-positive. Obviously, any Z-matrix A can be written as $sI - B$ with $B \geq 0$. A Z-matrix A is called an M-matrix if $s > \rho(B)$, where $\rho(.)$ is the spectral radius. It is called a singular M-matrix if $s = \rho(B)$. The symbol $\otimes$ indicates the Kronecker product [17].

Theorem 3.1 For the coupled algebraic Riccati equation (9), let $B_k > 0, C_k > 0$ and $I \otimes A_k + D_k^T \otimes I$ is an M-matrix . If there exists a positive matrix group $X = (X_1, X_2, ..., X_m)$, such that:

$$\mathcal{R}_k(X) \leq 0,$$

then (9) has a minimal positive solution group $S = (S_1, S_2, ..., S_m)$, such that $S \leq X$. If $X = \left(X_1^{(0)}, X_2^{(0)}, ..., X_m^{(0)}\right) = (0, 0, ..., 0)$, then sequence $\{X_k^{(l)}\}$ defined by $X_k^{(l+1)} = X_k^{(l)} - \left(\mathcal{R}'_{X_k^{(l)}}\right)^{-1} \mathcal{R}\left(X_k^{(l)}\right)$ has the following relation:

$$X_k^{(0)} < X_k^{(1)} < ..., \qquad \lim_{l \to \infty} X_k^{(l)} = S_k.$$

Furthermore, for $k = 1, 2, ..., m$:

$$M_{S_k} = I \otimes (A_k - S_k C_k) + (D_k - C_k S_k)^T \otimes I$$

is either an M-matrix or a singular M-matrix. To see the details of the proof, refer to [17].

Lemma 3.2 Maximal solution S^* for CARE (9) is the unique solution of the following programming problem:

$$\begin{aligned} &\min \quad tr\left(\sum_{k=1}^{m} X_k\right) \\ &X_k C_k X_k - X_k D_k - A_k X_k + B_k + \sum_{j \neq k} e_{kj} X_j \leq 0 \\ &\qquad X_k \geq 0, \qquad k = 1, 2, ..., m. \end{aligned} \tag{10}$$

Proof: By Theorem 3.1, $\mathcal{R}(S) \leq 0$ (thus, constraints (10) are satisfied for S), and $S \leq X$ for any X satisfying the constraints, which implies that

$$tr\left(\sum_{k=1}^{m} S_k\right) \leq tr\left(\sum_{k=1}^{m} X_k\right)$$

and so, the optimal solution is given by S. □

4 Numerical examples

Now, to show the efficiency of our method, we solve some numerical examples. It is worth mentioning that these examples are taken from [10, 17].

Example 4.1 [17], Consider equations (9) with $m = n = 2$ and $[e_{ij}] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,

$$A_1 = \begin{pmatrix} 5 & -2 \\ -1 & 6 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} \quad D_1 = \begin{pmatrix} 5 & -1 \\ -2 & 4 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 4 & -6 \\ -1 & 3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix} \quad D_2 = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

Applying lemma 3.2, we have:

$$\begin{aligned} \min \quad & tr(X_1 + X_2) \\ S.\ to: \quad & X_1 C_1 X_1 - X_1 D_1 - A_1 X_1 + B_1 + e_{12} X_2 \leq 0; \\ & X_2 C_2 X_2 - X_2 D_2 - A_2 X_2 + B_2 + e_{21} X_1 \leq 0; \\ & X_1 \geq 0, \qquad X_2 \geq 0 \end{aligned}$$

where $X_1 = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad X_2 = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$.

This problem has no feasible solution; So, equations (9) have no positive solution in this case.

Example 4.2 [17], Consider equations (9) with $m = n = 2$ and $[e_{ij}] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$:

$$A_1 = \begin{pmatrix} 5 & -1 \\ -1 & 4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{36}{7} & 16 \\ 18 & 33 \end{pmatrix}, \quad C_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{7} \end{pmatrix} \quad D_1 = \begin{pmatrix} 8 & -2 \\ -1 & 6 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 9 & -1 \\ -2 & \frac{10}{3} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \frac{3}{20} & 9 \\ 1 & \frac{1}{2} \end{pmatrix}, \quad C_2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{5} \end{pmatrix} \quad D_2 = \begin{pmatrix} 10 & \frac{-1}{3} \\ -1 & 3 \end{pmatrix}$$

Applying lemma 3.2, the feasible solution is:

$$X_1 = \begin{pmatrix} 0.9332 & 2.6056 \\ 2.3697 & 5.1380 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0.1510 & 1.1799 \\ 0.4121 & 1.5206 \end{pmatrix}$$

Example 4.3 [17], Consider equations (9) with $m = n = 3$ and $[e_{ij}] = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}$:

$$A_1 = \begin{pmatrix} 11 & -1 & -2 \\ -3 & 8 & -2 \\ -1 & -2 & 9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 18 & -1 & -0.5 \\ -2 & 9 & -3 \\ -1 & -1 & 8 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 9 & -2 & -1 \\ -1 & 8 & -1 \\ -2 & -2 & 14 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 5 & 9 & 4 \\ 9 & 8 & 9 \\ 2 & 10 & 10 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 24 & 23 & 0.5 \\ 6 & 2 & 20 \\ 0.3 & 10 & 20 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 7 & 5 & 1.5 \\ 1.5 & 6 & 1 \\ 0.5 & 1.5 & 1.8 \end{pmatrix}$$

$$C_1=\begin{pmatrix}\frac{1}{12}&\frac{1}{12}&1\\\frac{1}{14}&\frac{1}{18}&\frac{1}{12}\\\frac{1}{13}&\frac{1}{14}&\frac{1}{15}\end{pmatrix},\quad C_2=\begin{pmatrix}\frac{1}{13}&\frac{11}{5}&\frac{1}{12}\\\frac{1}{14}&\frac{1}{16}&\frac{1}{13}\\\frac{1}{17}&\frac{1}{14}&\frac{1}{12}\end{pmatrix},\quad C_3=\begin{pmatrix}\frac{1}{12}&\frac{1}{13}&\frac{1}{14}\\\frac{1}{15}&\frac{1}{13}&\frac{1}{16}\\\frac{1}{19}&\frac{1}{18}&\frac{1}{17}\end{pmatrix}$$

$$D_1=\begin{pmatrix}9&-2&-2\\-1&7&-1\\-2&-3&10\end{pmatrix},\quad D_2=\begin{pmatrix}12&-1&-2\\-2&11&-3\\-1&-3&16\end{pmatrix},\quad D_3=\begin{pmatrix}10&-1&-4\\-2&14&-2\\-2&-1&12\end{pmatrix}$$

Applying lemma 3.2, the feasible solution is:

$$X_1=\begin{pmatrix}0.4467&0.8672&0.4218\\0.8823&1.2637&0.8941\\0.4083&1.1003&0.7835\end{pmatrix},\quad X_2=\begin{pmatrix}0.9471&0.9473&0.2847\\0.5750&0.5830&1.1437\\0.2590&0.8311&1.0648\end{pmatrix}\quad X_3=\begin{pmatrix}0.4543&0.2886&0.2102\\0.1749&0.3153&0.1446\\0.0967&0.1122&0.1276\end{pmatrix}$$

Example 4.4

[10], Consider equations (2) with matrix coeffients $D_k=A_k+\frac{1}{2}\lambda_{kk}I,\ S_k=B_kR_k^{-1}B_k$, where

$$A_1=\begin{pmatrix}-2.1051&-1.1648&0.9347&0.5194\\-0.0807&-2.8949&0.3835&0.8310\\0.6914&10.5940&-36.8199&3.8560\\1.0692&13.4230&22.1185&-13.1801\end{pmatrix},\quad B_1=\begin{pmatrix}0.7564\\0.9910\\9.8255\\7.2266\end{pmatrix}$$

$$A_2=\begin{pmatrix}-2.6430&-1.2497&0.5269&0.6539\\-0.7910&-2.8570&0.0920&0.4160\\21.0357&22.8659&-26.4655&-1.7214\\27.3096&7.8736&-3.8604&-29.5345\end{pmatrix},\quad B_2=\begin{pmatrix}0.3653\\0.2470\\7.5336\\6.5152\end{pmatrix}$$

$$Q_1=Q_2=\begin{pmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix},\quad \Pi=\begin{pmatrix}-2&2\\1.5&-1.5\end{pmatrix},\quad R_1=R_2=1.$$

Applying RMNM with $X^{(0)}=0$ and $\omega=0.7$, we compute the following two solutions to the given system after 4 iterations.

$$X_1=\begin{pmatrix}0.2408&0.0705&0.0393&0.0182\\0.0705&0.0308&0.0085&0.0064\\0.0393&0.0085&0.0157&0.0025\\0.0182&0.0064&0.0025&0.0016\end{pmatrix},\quad X_2=\begin{pmatrix}0.5026&0.1343&0.0518&0.0097\\0.1343&0.0485&0.0138&0.0026\\0.0518&0.0138&0.0193&0.0002\\0.0097&0.0026&0.0002&0.0003\end{pmatrix},$$

5 Conclusion

In this study, a new iterative method for computing a Hermitian solution to a system of coupled algebraic Riccati equations was presented. To do this, we compared the results from these experiments with those of other studies. Our new iterations method has properties which were proved in Theorem 2.1. Also, we established a link between the optimization problem and CARE solution. Finally, we offered some corresponding numerical examples to demonstrate the effectiveness of the derived iteration method.

6 Data availability

All data generated or analysed during this study are included in this published article and its supplementary information files.

7 Conflict of Interest

The authors declare that they have no conflict of interest.

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