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Some new results on the number of fair dominating sets

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ABSTRACT

Let $G = (V, E)$ be a simple graph. A dominating set of G is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. The cardinality of a smallest dominating set of G, denoted by $\gamma(G)$, is the domination number of G. For $k \geq 1$, a k-fair dominating set (kFD-set) in G, is a dominating set S such that $|N(v) \cap D| = k$ for every vertex $v \in V \setminus D$. A fair dominating set in G is a kFD-set for some integer $k \geq 1$. Let $\mathcal{D}_f(G, i)$ be the family of the fair dominating sets of a graph G with cardinality i and let $d_f(G, i) = |\mathcal{D}_f(G, i)|$. The fair domination polynomial of G is $D_f(G, x) = \sum_{i=1}^{|V(G)|} d_f(G, i)x^i$. In this paper,

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ABSTRACT continued: after computation of the fair domination number of power of cycle, we count the number of the fair dominating sets of certain graphs such as cubic graphs of order 10, power of paths and power of cycles. As a consequence, all cubic graphs of order 10 and especially the Petersen graph are determined uniquely by their fair domination polynomial.

1 Introduction and definition

Let $G = (V, E)$ be a simple graph with n vertices. A set $D \subseteq V(G)$ is a dominating set, if every vertex in $V(G)\backslash D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. Dominating sets are of practical interest in several areas. There are different kinds of dominating sets and domination numbers which studied well in the literature. Most of the papers published in the domination theory, try to determine exact expressions for domination number $\gamma(G)$, which is the minimum cardinality of a dominating set of graph ([9]). The distance between two vertices u and v denoted by $d(u, v)$ is the number of edges in a shortest path (also called a graph geodesic) connecting them. Let $S \subseteq V$ be any subset of vertices of G. The induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S.

For $k \geq 1$, a k-fair dominating set (kFD-set) in G, is a dominating set D such that $|N(v) \cap D| = k$ for every vertex $v \in V \setminus D$. The k-fair domination number of G, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality $fd_k(G)$ is called a $fd_k(G)$-set. A fair dominating set, abbreviated FD-set, in G is a kFD-set for some integer $k \geq 1$. The fair domination number, denoted by $\gamma_f(G)$, of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. An FD-set of G of cardinality $\gamma_f(G)$ is called a $\gamma_f(G)$-set.

By convention, if $G = \overline{K_n}$, we define $\gamma_f(G) = n$. By the definition, it is easy to see that for any graph G of order n, $\gamma(G) \leq \gamma_f(G) \leq n$ and $\gamma_f(G) = n$ if and only if $G = \overline{K_n}$. Caro, Hansberg and Henning in [8] showed that for a disconnected graph G (without isolated vertices) of order $n \geq 3$, $\gamma_f(G) \leq n - 2$, and they constructed an infinite family of graphs achieving equality in this bound.

Regarding to enumerative side of dominating sets, Alikhani and Peng in [2], have introduced the domination polynomial of a graph. The domination polynomial of graph G is the generating function for the number of dominating sets of G, i.e., $D(G, x) = \sum_{i=1}^{|V(G)|} d(G, i)x^i$ (see [1, 2]). This polynomial and its roots have been actively studied in recent years (see for example [7, 10]). Let $\mathcal{D}_f(G, i)$ be the family of the fair dominating sets of a graph G with cardinality i and let $d_f(G, i) = |\mathcal{D}_f(G, i)|$. It is natural to count the number of the fair dominating sets of some specific graphs. The fair domination polynomial of G is $D_f(G, x) = \sum_{i=1}^{|V(G)|} d_f(G, i)x^i$. Recently Alikhani and Safazadeh [4] have counted the number of fair dominating sets of some specific graphs such as cycles, paths and some cactus graphs.

The following theorem is an easy result in the number of the fair dominating sets of graphs:

Theorem 1.1 Let G be a graph with $|V(G)| = n$. Then

(i) If G is connected, then $d_f(G,n) = 1$ and $d_f(G,n-1) = n$.

(ii) If $i < \gamma_f(G)$ or $i > n$, then $d_f(G,i) = 0$.

(iii) $D_f(G,x)$ has no constant term.

(iv) Zero is a root of $D_f(G,x)$, with multiplicity $\gamma_f(G)$.

Similar to the study of dominating equivalent of graphs ([1, 3]), two graphs G and H are said to be *fair dominating equivalent*, or simply $\mathcal{D}_f$-equivalent, written $G \sim_f H$, if $D_f(G,x) = D_f(H,x)$. It is evident that the relation $\sim_f$ of being $\mathcal{D}_f$-equivalence is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the *$\mathcal{D}_f$-equivalence classes*. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} : H \sim_f G\}.$$

We call $[G]$ the equivalence class determined by G. A graph G is said to be *fair dominating unique*, or simply *$\mathcal{D}_f$-unique*, if $[G] = \{G\}$. In this case, we say the graph G is completely determined by its fair domination polynomial.

In Section 2, we consider the cubic graphs of order 10 and study their fair domination polynomials. As a consequence, we show that all cubic graphs of order 10 and especially the Petersen graph are determined uniquely by their fair domination polynomial. In Section 3, we consider the power of cycle graph and path graph and study the number of their fair dominating sets.

2 Fair domination polynomial of cubic graphs of order 10

Authors in [3] have studied the domination polynomials of cubic graphs of order 10 and as a consequence, they have shown that the Petersen graph is determined uniquely by its domination polynomial.
In this section, we study some coefficients of the fair domination polynomial of regular graphs and then compute the fair domination polynomial of cubic graphs of order 10. As usual $\mathrm{Aut}(G)$ denotes the automorphism group of G. A vertex-transitive graph, is a graph such that every pair of vertices is equivalent under some elements of its automorphism group. More explicitly, a vertex-transitive graph is a graph whose automorphism group is transitive. For every vertex v of graph G, we denote the family of all fair dominating sets of G with cardinality i which contain v by $\mathcal{D}_f^v(G,i)$ and $d_f^v(G,i) = |\mathcal{D}_f^v(G,i)|$.

Lemma 2.1 Let $G = (V,E)$ be a vertex transitive graph of order n and $v \in V$. For any $1 \le i \le n$, we have $d_f(G,i) = \frac{n}{i} d_f^v(G,i)$.

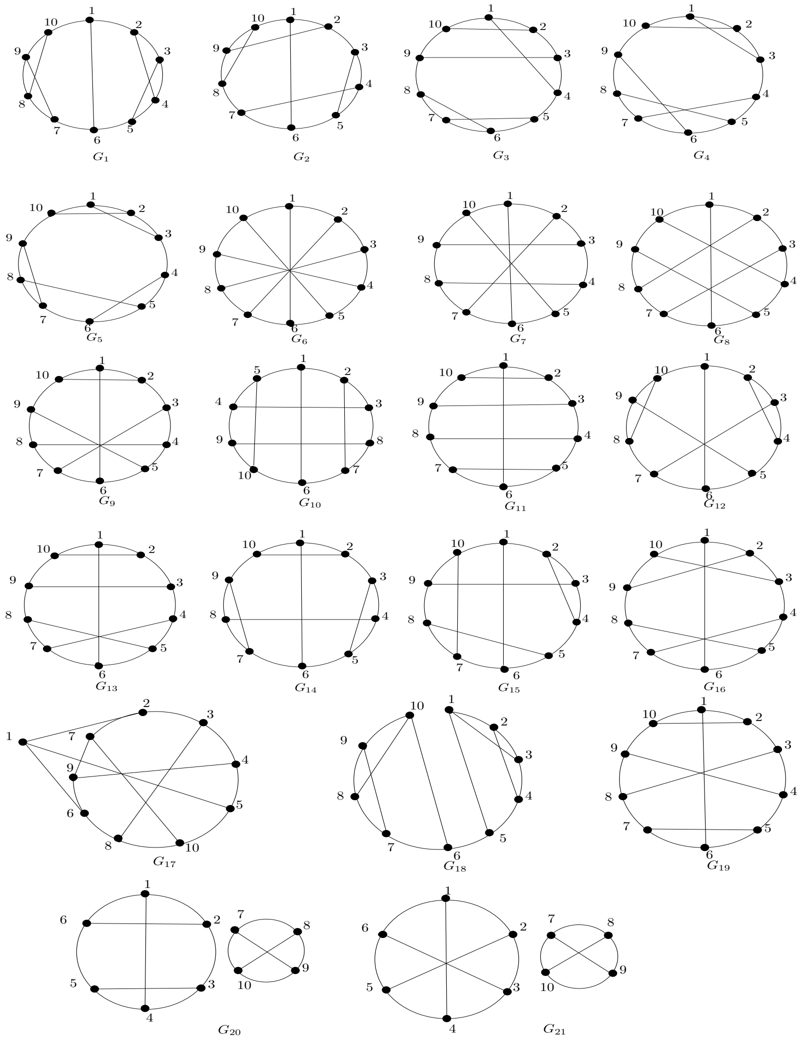

Figure 1: Cubic graphs of order 10.

Proof. If D is a fair dominating set of vertex transitive graph G of size i and $\theta \in \mathrm{Aut}(G)$, then $\theta(D)$ is also a fair dominating set of G with size i. Also, since G is a vertex transitive graph, so for every vertices v and u, $d_f^v(G,i) = d_f^u(G,i)$. If D is a fair dominating set of size i, then there are exactly i vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_i}$ such that D counted in $d_f^{v_{j_k}}(G,i)$, for any $1 \leq k \leq i$. Therefore $d_f(G,i) = \frac{n}{i} d_f^v(G,i)$ and the proof is complete. □

Lemma 2.2 If $G = (V,E)$ is a k-regular graph of order n, then $d_f(G, n-2) = \binom{n}{2}$.

Proof. Let G be a k-regular graph of order n with vertex set V. Each vertex of G is adjacent to k vertices and we have two cases for two arbitrary vertices v and u of G:
Case 1. If u and v are adjacent, then both have exactly $k-1$ neighbors in $V \setminus \{u,v\}$.
Case 2. If u and v are not adjacent, then they have exactly k neighbors in $V \setminus \{u,v\}$.
Therefore for every vertices v and u of G, $V \setminus \{u,v\}$ is a fair dominating set. So $d_f(G, n-2) = \binom{n}{2}$ and the proof is complete. □

k	1	2	3	4	5	6	7	8	9
G_1	0	0	0	5	4	40	30	45	10
G_2	0	0	2	5	2	38	30	45	10
G_3	0	0	4	4	1	43	30	45	10
G_4	0	0	14	0	0	50	30	45	10
G_5	0	0	4	6	0	44	30	45	10
G_6	0	0	0	10	2	40	30	45	10
G_7	0	0	2	6	6	32	30	45	10
G_8	0	0	0	9	8	28	30	45	10
G_9	0	0	1	5	5	33	30	45	10
G_{10}	0	0	10	0	2	40	30	45	10
G_{11}	0	0	6	2	0	42	30	45	10

k	1	2	3	4	5	6	7	8	9
G_{12}	0	0	0	7	6	30	30	45	10
G_{13}	0	0	5	4	2	41	30	45	10
G_{14}	0	0	3	3	3	35	30	45	10
G_{15}	0	0	6	1	3	37	30	45	10
G_{16}	0	0	0	13	2	44	30	45	10
G_{17}	0	0	0	15	12	20	30	45	10
G_{18}	0	0	8	5	0	56	30	45	10
G_{19}	0	0	0	4	4	34	30	45	10
G_{20}	0	0	12	9	0	58	30	45	10
G_{21}	0	0	36	1	0	72	30	45	10

Table 1: The number of the fair dominating sets of cubic graphs of order 10.

Now we consider exactly 21 cubic graphs of order 10 given in Figure 1 (see [3]). There are just two non-connected cubic graphs of order 10. Note that the graph G_{17} is the Petersen graph. Similar to [3], we compute the fair domination polynomial of these graphs. Using Matlab we computed the coefficients of the fair domination polynomial of these graphs (see Table 1).

The following theorem gives the fair domination polynomial of the Petersen graph.

Theorem 2.3 The fair domination polynomial of the Petersen graph P is

$$D_f(P,x) = x^{10} + 10x^9 + 45x^8 + 30x^7 + 20x^6 + 12x^5 + 15x^4.$$

Proof. The fair domination number of the Petersen graph is $\gamma_f(P) = 4$. Since the Petersen graph P is a 3-regular graph of order 10, by Lemma 2.2, we have $d_f(P,8) = \binom{10}{2} = 45$. On the other hand, since P is a vertex transitive graph, using Lemma 2.1, for every vertex v of P and $i = 4,5,6,7$ we have $d_f(P,i) = \frac{n}{i} d_f^v(P,i)$. Let v be vertex with label 1 in G_{17}.
There are have exactly 21 fair dominating sets with cardinality seven and contain vertex 1 in the graph G_{17} as follows:
$\{1,2,3,4,6,7,10\}, \{1,2,3,4,6,9,10\}, \{1,2,3,4,8,9,10\}, \{1,2,3,5,6,9,10\}, \{1,2,3,5,7,8,9\}, \{1,2,3,5,8,9,10\}, \{1,2,4,5,6,7,8\}, \{1,2,4,5,7,8,9\}, \{1,2,4,5,8,9,10\}, \{1,2,4,6,7,8,10\}, \{1,2,4,6,8,9,10\}, \{1,2,4,7,8,9,10\}, \{1,3,4,5,6,7,8\}, \{1,3,4,5,6,7,10\}, \{1,3,4,5,7,8,9\}, \{1,3,4,6,7,8,10\}, \{1,3,4,6,7,9,10\}, \{1,3,4,7,8,9,10\}, \{1,3,5,6,7,8,9\}, \{1,3,5,6,7,9,10\}, \{1,3,5,7,8,9,10\}$.
There are 12 fair dominating sets with cardinality 6 which contain vertex 1 in the graph G_{17} as follows:
$\{1,2,3,4,7,8\}, \{1,2,3,5,6,7\}, \{1,2,3,7,9,10\}, \{1,2,4,5,6,10\}, \{1,2,4,8,9,10\},$

$\{1,2,5,6,8,9\}, \{1,3,4,5,9,10\}, \{1,3,4,6,7,10\}, \{1,3,5,7,8,9\}, \{1,3,6,8,9,10\},$
$\{1,4,5,7,8,10\}, \{1,4,6,7,8,9\}$.
There are six fair dominating sets with cardinality 5 and contain vertex 1 in the graph G_{17} as follows:
$\{1,2,3,4,5\}, \{1,2,3,6,8\}, \{1,2,5,7,10\}, \{1,2,6,7,9\}, \{1,4,5,6,9\}, \{1,5,6,8,10\}$.
There are six fair dominating sets with cardinality 4 and contain vertex 1 in graph G_{17} as follows:
$\{1,2,3,7\}, \{1,2,5,6\}, \{1,3,9,10\}, \{1,4,5,10\}, \{1,4,7,8\}, \{1,6,8,9\}$.
So we have the result. □

A finite sequence of real numbers $(a_0, a_1, a_2, \ldots, a_n)$ is said to be

1. *unimodal* if $a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$ for some $k \in \{0,1,2,\ldots,n\}$;

2. *logarithmically-concave* (or simply *log-concave*), if the inequality $a_k^2 \geq a_{k-1}a_{k+1}$ is valid for every $k \in \{1,2,\ldots,n-1\}$.

Hence, a polynomial $\sum_{k=0}^{n} a_k x^k$ is said to be unimodal (or log-concave) if the coefficient sequence $\{a_k\}$ is unimodal (or log-concave). It is well-known that any log-concave polynomial with positive coefficients is also unimodal, and that the sequence of binomial coefficients $\{\binom{n}{k}\}$ is log-concave. The unimodality of various families of graphs has been the focus of a large amount of study. It is conjectured that the domination polynomial of a graph is unimodal (see [2]). This conjecture is still open. Most of the sequences of the number of some kind of dominating sets look unimodal, but here by Theorem 2.3 we have the following corollary:

Corollary 2.4

(i) The fair domination polynomials of cubic graphs of order 10 (especially the Petersen graph) are not unimodal and log-concave.

(ii) All cubic graphs of order 10 and especially the Petersen graph are determined uniquely by their fair domination polynomial.

3 The number of fair dominating sets for power graphs

In this section, we count the number of the fair dominating sets of power of cycles and paths. First, we recall the definition of graph power and some of its properties.

Definition 3.1 The k-th power G^k of an undirected graph G is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in G is at most k. It is easy to see that if a graph has the diameter d, then its d-th power is the complete graph.

Here we consider the power of cycles. Note that the number of the fair dominating sets of cycles has been investigated in [4]. Let $C_n, n \geq 3$, be the cycle with n vertices $V(C_n) = \{1, 2, ..., n\}$ and $E(C_n) = \{\{1, 2\}, \{2, 3\}, ..., \{n-1, n\}, \{n, 1\}\}$. Let $f\mathcal{C}_n^i$ be the family of the fair dominating sets of C_n with cardinality i. A simple path is a path where all its internal vertices have degree two.

Lemma 3.2 The following properties hold for cycles:

(i) ([8]) $\gamma_f(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$, unless $n \equiv 2 \ (mod\, 3)$ and $n \geq 5$ in which case $\gamma_f(C_n) = \gamma(C_n) + 1 = \lceil \frac{n}{3} \rceil + 1$.

(ii) $f\mathcal{C}_j^i = \emptyset$, if and only if $i > j$ or $i < \lceil \frac{j}{3} \rceil$. (by (i) above).

(iii) If a graph G contains a simple path of length $3k-1$, then every fair dominating set of G must contain at least k vertices of the path.

In the following theorem, we obtain the fair domination number of power of cycles for some cases.

Theorem 3.3 The fair domination number of power of cycle C_n^m $(m \geq 2)$ is

$$\gamma_f(C_n^m) = \begin{cases} 1 & m \geq \lfloor \frac{n}{2} \rfloor, \\ 2 & n \text{ is even and } m = \frac{n}{2} - 1, \\ 2\left\lceil \frac{n-m}{3} \right\rceil - 1 & n \text{ is odd and } m = \lfloor \frac{n}{2} \rfloor - 1, \\ 2\left\lceil \frac{n-m}{4} \right\rceil - 1 & n \text{ is even and } m = \lfloor \frac{n}{2} \rfloor - 2. \end{cases}$$

Proof. For $m \geq \lfloor \frac{n}{2} \rfloor$, the graph C_n^m is isomorphic to the complete graph K_n, and so simply we have the result. For $m = \frac{n}{2} - 1$, when n is even, the degree of each vertex of C_n^m is $n-2$. We choose two vertices that are not connected to each other, so we have the smallest fair dominating set in this case.

Now, suppose that n is odd and $m = \lfloor \frac{n}{2} \rfloor - 1$. To prove that $\gamma_f(C_n^m) = 2\left\lceil \frac{n-m}{3} \right\rceil - 1$, we consider three cases $n = 6k+1, n = 6k+3$ and $n = 6k+5$ for some $k \in \mathbb{N}$. It is notable that the value of m for $n = 6k+1, n = 6k+3$ and $n = 6k+5$ is $m = 3k-1, m = 3k$ and $m = 3k+1$, respectively. Consider the set D as a subset of vertices of C_n^m as follows:

$$D := \bigcup_{j=1}^{k+1} \{m - (3j-4)\} \bigcup_{j=1}^{k} \{n - (3j-1)\} \cup \{1, m+3\}.$$

We only prove the case $n = 6k+3$. Analogously, we can prove the cases $n = 6k+1$ and $n = 6k+5$.

For $n = 6k+3$, we have $D = \{1, 4, 7, \ldots, n-2\}$ and $|D| = 2k+1 = 2\left\lceil \frac{n-m}{3} \right\rceil - 1$. It is easy to verify that every vertex of C_n^m which is outside D is adjacent to $2k$ vertices of D. Then, the set D is a fair dominating set of C_n^m. Hence, $\gamma_f(C_n^m) \leq 2k+1 = 2\left\lceil \frac{n-m}{3} \right\rceil - 1$. Now, we show that $\gamma_f(C_n^m) \geq 2k+1$. Suppose by contradiction that $\gamma_f(C_n^m) < 2k+1$. Let S be a γ_f-set of C_n^m. Let $r > 0$ be an integer such that for every vertex $u \in V \setminus S$, we have $|N(u) \cap S| = r$. Since C_n^m is a $2m$-regular graph, the number of the edges of the induced subgraph $G[S-V]$ is $\frac{(2m-r)|V-S|}{2}$, and the number of the edges C_n^m whose one endpoint is in S and the other endpoint is in $V \setminus S$ is equal to $r \times |V-S|$. Hence, since the number of the edges of C_n^m is mn, the number of the edges of the induced subgraph of $G[S]$ is equal to $mn - \left(r|V-S| + \frac{(2m-r)|V-S|}{2}\right)$. If $s := |S|$, then,

$$mn - \left(r|V-S| + \frac{(2m-r)|V-S|}{2}\right) = \frac{2ms - rn + rs}{2}.$$

It is notable that $|S| \leq 2k$ and $r \leq |S|$, so $r \leq s$. It is clear that the number of the edges of $G[S]$ is at most $\binom{s}{2} = \frac{s(s-1)}{2}$. So, we should have

$$\frac{2ms - rn + rs}{2} \leq \frac{s(s-1)}{2}.$$

Therefore, we have

$$r \geq \frac{(2m+1)s - s^2}{n-s}. \tag{1}$$

For $n = 6k+3$ and $m = 3k$,

$$r \geq \frac{(1+6k)s - s^2}{6k+3-s}. \tag{2}$$

Now, we consider two cases as follows.

- $r \leq s-1$**.** From Equation (2), we have

  $$\frac{(1+6k)s - s^2}{6k+3-s} \leq s-1. \tag{3}$$

  So, by applying some simple algebraic computations on inequality 3, we have $s \geq 2k+1$ which contradicts that $s \leq 2k$.

- $r = s$**.** Assume that S is an independent set. Since C_n^m is a $2m$-regular graph, every vertex of S should be adjacent to $2m$ vertices of $V \setminus S$. On the other hand, since $r = s$, every vertex of $V \setminus S$ is adjacent to all vertices of S and so $2m = |V-S| = n-s$. Therefore, $6k = 6k+3-s$. Hence, $s = 3$. Let u and v be two vertices of S. Without loss of generality, assume that $u = c_1$. In the graph C_n^m, the vertex c_1 is adjacent to the vertices $c_2, c_3, \ldots, c_{m+1}$, $c_{n-m+1}, c_{n-m+2}, \ldots, c_n$. Since the vertex u is not adjacent to v, we have $v \in \{c_{m+2}, c_{m+3}, \ldots, c_{n-m}\}$. On the other hand, since $r = s$, we have $N(v) = N(u)$. It means that v is adjacent to the vertices $c_2, c_3, \ldots, c_{m+1}$, $c_{n-m+1}, c_{n-m+2}, \ldots, c_n$ but this is a contradiction, because it is easy to verify that in the graph C_n^m, every vertex of $\{c_{m+2}, c_{m+3}, \ldots, c_{n-m}\}$ is not adjacent to either a vertex of $\{c_2, c_3, \ldots, c_{m+1}\}$ or $\{c_{n-m+1}, c_{n-m+2}, \ldots, c_n\}$.

  Now, we assume that S is not an independent set. Let u and v be two adjacent vertices of S. The number of the triangle uvw where w is a vertex of $V \setminus S$ is $2(n-s) = 12k+6-2s$. Suppose, without loss of generality, that $u := c_1$. In the graph C_n^m, the number of triangles c_1xy is maximized when $x = c_2$ and $y \in N(c_1) \cap N(c_2)$. Then, the number of the triangles c_1c_2y with $y \in N(c_1) \cap N(c_2)$ is $2(m-1) = 6k-2$. Since $s \leq 2k$, we have $12k+6-2s \geq 8k+6$. Since $6k-2 < 8k+6$, this is a contradiction.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
C_6^2	0	3	8	15	6												
C_7^2	0	0	7	7	21	7											
C_8^2	0	0	8	6	8	28	8										
C_9^2	0	0	3	9	18	12	36	9									
C_{10}^2	0	5	0	20	2	40	20	45	10								
C_{11}^2	0	0	0	11	22	0	66	33	55	11							
C_{12}^2	0	0	0	3	24	24	0	99	52	66	12						
C_{13}^2	0	0	0	0	26	13	52	0	143	78	78	13					
C_{14}^2	0	0	0	0	14	35	2	119	0	203	112	91	14				
C_{15}^2	0	0	5	0	3	55	15	0	235	3	285	155	105	15			
C_{16}^2	0	0	0	0	0	40	64	6	0	408	16	396	208	120	16		
C_{17}^2	0	0	0	0	0	17	85	51	34	0	646	51	544	272	136	17	
C_{18}^2	0	0	0	0	0	3	90	63	62	144	0	960	126	738	348	153	18

Table 2: The coefficients of x^k in the fair domination polynomial of C_n^2.

Hence, we have $\gamma_f(C_n^m) \geq 2k+1$.

Now, suppose that n is even and $m = \lfloor \frac{n}{2} \rfloor - 2$. We consider four cases as follows: $\{n = 8k, m = 4k-2\}$, $\{n = 8k+2, m = 4k-1\}$, $\{n = 8k+4, m = 4k\}$ and $\{n = 8k+6, m = 4k+1\}$. The proof of these cases is similar to the case $n = 6k+3$ mentioned above just we use the set D as follows.

$$D := \bigcup_{j=1}^{k} \{m - (4j-5)\} \bigcup_{j=1}^{k-1} \{n - (4j-1)\} \cup \{1, m+4\}, \text{for } n = 8k \text{ or } n = 8k+2.$$

$$D := \bigcup_{j=1}^{k} \{m - (4j-5)\} \bigcup_{j=1}^{k-1} \{n - (4j-1)\} \cup \{1, m+5\}, \text{for } n = 8k+4$$

$$D := \bigcup_{j=1}^{k} \{m - (4j-5)\} \bigcup_{j=1}^{k} \{n - (4j-1)\} \cup \{1, 3, m+4\}, \text{for } n = 8k+6.$$

. □

The number of fair dominating sets of C_n^2 for $6 \leq n \leq 18$ has shown in Table 2.

From Table 2 we have the following conjecture:

Conjecture 3.4

(i) For every $n \geq 7$,

$$d_f(C_n^2, n-3) = 7\binom{n-7}{0} + \binom{n-7}{1} + 3\binom{n-7}{2} + \binom{n-7}{3}.$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
C_7^3	7	21	35	35	21	7											
C_8^3	0	4	0	22	32	28	8										
C_9^3	0	0	3	0	27	30	36	9									
C_{10}^3	0	0	10	0	2	40	30	45	10								
C_{11}^3	0	0	11	0	0	33	44	33	55	11							
C_{12}^3	0	0	4	3	0	12	48	75	40	66	12						
C_{13}^3	0	0	0	13	13	0	78	39	130	52	78	13					
C_{14}^3	0	7	0	35	0	105	2	203	14	203	70	91	14				
C_{15}^3	0	0	0	15	3	0	75	60	280	3	285	95	105	15			
C_{16}^3	0	0	0	4	16	0	0	38	160	352	0	380	128	120	16		
C_{17}^3	0	0	0	0	34	0	0	0	102	221	493	0	493	170	136	17	
C_{18}^3	0	0	0	0	36	3	0	0	8	315	162	780	0	630	222	153	18

Table 3: The coefficients of x^k in the fair domination polynomial of C_n^3.

(ii) For every $n \geq 11$,

$$d_f(C_n^2, n-4) = 11+55\binom{n-11}{0}+33\binom{n-11}{1}+11\binom{n-11}{2}+5\binom{n-11}{3}+\binom{n-11}{4}.$$

(iii) For every $n \geq 15$,

$$d_f(C_n^2, n-5) = 3\binom{n-15}{0}+13\binom{n-15}{1}+22\binom{n-15}{2}+18\binom{n-15}{3}+7\binom{n-15}{4}+\binom{n-15}{5}.$$

(iv) For every $n \geq 14$, $d_f(C_n^2, n-6) =$
$119\binom{n-14}{0} + 116\binom{n-14}{1} + 57\binom{n-14}{2} + 8\binom{n-14}{3} + 3\binom{n-14}{4} + 4\binom{n-14}{5} + \binom{n-14}{6}$.

Also, we have shown the number of the fair dominating sets of C_n^3 for $7 \leq n \leq 18$ in Table 3.

Finally, using Matlab we obtained the number of the fair dominating sets of P_n^2 for $4 \leq n \leq 20$ and have shown them in Table 4. Using Tables 2,3 and 4 we have the following result.

Theorem 3.5 The fair domination polynomial of power of paths and cycles is not unimodal.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
P_4^2	2	2	4																
P_5^2	1	0	2	5															
P_6^2	0	1	2	3	6														
P_7^2	0	2	1	2	5	7													
P_8^2	0	3	0	5	4	8	8												
P_9^2	0	2	0	4	5	9	12	9											
P_{10}^2	0	1	0	1	6	8	14	17	10										
P_{11}^2	0	0	1	0	6	4	16	20	23	11									
P_{12}^2	0	0	2	0	4	6	6	28	28	30	12								
P_{13}^2	0	0	3	0	1	12	4	15	44	39	38	13							
P_{14}^2	0	0	2	0	0	11	12	8	32	65	54	47	14						
P_{15}^2	0	0	1	0	0	5	18	13	22	57	93	74	57	15					
P_{16}^2	0	0	0	1	0	1	20	13	22	53	90	131	100	68	16				
P_{17}^2	0	0	0	2	0	0	15	22	11	44	109	132	183	133	80	17			
P_{18}^2	0	0	0	3	0	0	6	37	12	17	92	198	186	254	174	93	18		
P_{19}^2	0	0	0	2	0	0	1	36	36	11	37	182	328	258	350	224	107	19	
P_{20}^2	0	0	0	1	0	0	0	21	60	29	22	94	330	508	358	478	284	122	20

Table 4: The coefficients of x^k in the fair domination polynomial of P_n^2.

4 Conclusion

In this paper, we studied the number of fair dominating sets of some specific graphs. We obtained results for the cubic graphs of order 10. As a consequence, we observed that all cubic graphs of order 10 and especially the Petersen graph are determined uniquely by their fair domination polynomials. Moreover, we studied the number of the fair dominating sets of power of cycles and paths. There are many open problems in the study of the number of the fair dominating sets of power of a graph that we state and close the paper with some of them:

Problem 1: What is the closed formula for the fair domination number of power of a graph?

For any $n \in \mathbb{N}$, the n-subdivision of G is a simple graph $G^{\frac{1}{n}}$ which is constructed by replacing each edge of G with a path of length n. The fractional power of G, is m^{th} power of the n-subdivision of G, i.e., $(G^{\frac{1}{n}})^m$ or n-subdivision of m-th power of G, i.e., $(G^m)^{\frac{1}{n}}$ (see [5]).

Problem 2: What is the fair domination number of fractional power of a graph?

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